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ABELIAN GROUPS WITH THE DIRECT SUMMAND SUM PROPERTY

Abstract. If R is an associative ring, with unity, the R-module (the abelian group) M is said to have the direct summand sum property (in short D.S.S.P.) if the sum (that is the submodule (the subgroup) of M generated by the union) of any two direct summands of M is again a direct summand in M. The present work gives descriptions of some classes of abelian groups with this property.

1. Preliminaries

Let R be an associative ring, with unity. We say that an R-module M has the (strong) direct summand intersection property, in short (S.D.S.I.P.) D.S.I.P., if the intersection of (any family of) two direct summands of M is again a direct summand in M. The history of the modules with the direct summand intersection property begins with Kaplansky, which in [7] presents the following exercise (ex. 51, p. 49) for solving:

a) *Let M be a free module over a principal ideal ring, S a submodule of M, and T a direct summand of M. Prove that $S \cap T$ is a direct summand of S.*

b) *Let M be a free module over a principal ideal ring. Show that the intersection of any finite number of direct summands of M is again a direct summand of M.*

c) *Let M be a free module of countable rank over a principal ideal ring. Show that the intersection of any number of direct summands is a direct summand.*

This fact suggested to Fuchs in [4] to ask, in his turn, for the solution of the same exercise for free abelian groups (ex.14.4, p.76), countable free abelian groups or free abelian groups of the power of the continuum (ex.19.5, p.95). Moreover, Fuchs proposes the following open problem (problem 9): *"Characterize the groups in which the intersection of two direct summands* *is again a direct summand."* Solutions to this Kaplansky-Fuchs's problem were obtained in [1]–[3], [5], [6], [10]–[19] and [21].

In [14] we presented the concept of *"R-module (abelian group) with the direct summand sum property"* (in short D.S.S.P.) as being an R-module (abelian group) M, which has the property that the sum (that is the submodule (the subgroup) of M generated by the union) of any two direct summands of M is a direct summand too, and we presented the first characterizations of R-modules with this property. We also presented there the concept of *"R-module (abelian group) which has the strong direct summand sum property"*, in short S.D.S.S.P., as being an R-module (abelian group) M which has the property that the sum (that is the submodule (the subgroup) of M generated by the union) of any family of direct summands of M is again a direct summand of M, and we have proposed the following open problem for solving *"Characterize the R-modules (the abelian groups) in which the sum of two direct summands is again a direct summand"*. This problem is the dual of Kaplansky-Fuchs's problem, which we have presented above.

In [17] we presented other characterizations of these R-modules, results concerning certain classes of R-modules (injective or projective) over an associative ring R, with unity, as well as results concerning certain rings with this property.

In this work we will present the structure theorems for three classes of abelian groups with D.S.S.P. that is: the torsion groups, the torsion-free groups and the splitting mixed groups. In this context, throughout this paper by group we mean abelian group in additive notation.

The paper is structured in four sections: in this first section we will present the results obtained in [17] which we need here, as well as a few cases in which D.S.I.P. involves D.S.S.P., and in the sections 2, 3, respectively 4, we will describe the torsion groups, the torsion-free groups, respectively the splitting mixed groups, with D.S.S.P.

(1.1): *If an R-module has S.D.S.S.P., it also has D.S.S.P.; the converse is generally false.*

(1.2): *If the R-module M has D.S.S.P. (respectively S.D.S.S.P.), then every direct summand of M also has D.S.S.P. (respectively S.D.S.S.P.).*

(1.3.): *Let* $M = \bigoplus_{i\in I} M_i$ *be an R-module, where for every* $i \in I$, M_i *is fully invariant in M. Then M has D.S.S.P. (respectively S.D.S.S.P.) if and only if for every* $i \in I$, M_i *has D.S.S.P. (respectively S.D.S.S.P.).*

(1.4): *Let R be a principal ideal domain and let P be the set of all unassociated prime elements from R. If* $M = \bigoplus_{p\in P} M_p$ *is a torsion R-module, decom-*

posed according to [9, 6.11.3], *then* M *has D.S.S.P. (respectively S.D.S.S.P.) if and only if for every* $p \in P$, M_p *has D.S.S.P. (respectively S.D.S.S.P.).*

(1.5): *If the R-module* M *has D.S.S.P., then the following statements hold:*

1) *For every decomposition* $M = A \oplus B$ *and every homomorphism* $f : A \to B$, Imf *is a direct summand in* B;

2) *If* A *and* B *are indecomposable R-modules and* $A \oplus B$ *is a direct summand in* M, *then:*

either i) $Hom(A, B) = 0$ *or*

ii) *if* $0 \neq f \in Hom(A, B)$, *then* f *is epimorphism.*

(1.6): *The following statements are equivalent for a ring R:*

a) *All injective R-modules have D.S.S.P.*

b) R *is left hereditary.*

Next we are going to show that certain classes of abelian groups which have D.S.I.P. also have D.S.S.P.

PROPOSITION 1.7. *If* G *is an abelian group with D.S.I.P., then in any of the following cases,* G *(also) has D.S.S.P.:*

1) G *is a p-group which is either reduced or divisible;*

2) G *is a torsion group, whose p-components satisfy the conditions from point* 1);

3) G *is a divisible group;*

4) G *is a torsion-free group of the form*

$$G = \bigoplus_{i \in I} G_i, \tag{1}$$

where, for every $i \in I$, G_i *is reduced group of rank one and for every* $i_1, i_2 \in I$, $i_1 \neq i_2$, $t(G_{i_1})$ *and* $t(G_{i_2})$ *are incomparable types.*

5) G *is a splitting mixed group of the form*

$$G = D \oplus T, \tag{2}$$

where D *is divisible and torsion-free and* T *is reduced and torsion.*

Proof. 1) Let G be an abelian p-group with D.S.I.P., which satisfies the statement conditions. Then, according to [6, Theorem 2], we have two cases:

CASE 1: There is a $n \in N^*$ such that either $G = Z(p^n)$ or $G = Z(p^\infty)$. In this case G has in a trivial way D.S.S.P., since it is indecomposable.

CASE 2: There is a cardinal m_p such that $G = \oplus_{m_p} Z(p)$. Then G is an elementary p-group; so it has D.S.S.P. (In this case any subgroup of G is a direct summand in G.)

2) The statement follows from what has been proved at point 1) and from (1.4).

3) According to [10,4.4], a divisible group G has D.S.I.P. if and only if either $G = \bigoplus_{p \in P_1} Z(p^\infty)$ or $G = \bigoplus_{r_0} Q$, where P_1 is a subset of the set P of all prime numbers, and $r_0 = r_0(G)$ is the torsion-free rank of G. In the first case G satisfies the conditions from point 2), and in the second case G is a vector space over Q. In both cases G has D.S.S.P.

Otherwise: We can apply (2.1)2).

4) If G is a group as in the statement, then for every $i \in I$, G_i is an indecomposable and fully invariant direct summand in G. So, in this case, according to (1.3), G has D.S.S.P.

5) Let G be a group of the form (2). Then, according to what we proved in points 2), respectively 3), D and T have D.S.S.P. Since D and T are fully invariant in G, also in this case (1.3) completes the proof.

From (1.7) it follows that in certain cases D.S.I.P. involves D.S.S.P. This is not always true, that is there are abelian groups with D.S.I.P. and which do not have D.S.S.P. In this context, we have (also) the following result:

REMARK 1.8. *If G is a completely decomposable torsion-free group which satisfies* [6, Lemma 10], *as well, then G does not have D.S.S.P.*

Proof. Let G be a completely decomposable torsion-free group with D.S.I.P. and which satisfies [6, Lemma 10], as well. Then $G = H \oplus (\bigoplus_{i \in I} X_i)$, where: H is a finite rank completely decomposable torsion-free homogeneous group, for every $i \in I$, X_i has rank one, for every $i, j \in I$, $i \neq j$, $t(X_i)$ and $t(X_j)$ are incomparable types and for every $i \in I$, $t(H) < t(X_i)$. In these conditions each direct summand of rank one of H is isomorphic to a proper subgroup of X_i, for every $i \in I$, according to [4, §85, p. 112]. Therefore, if Y is a direct summand in H, for every $i \in I$, according to (1.5)1), $Y \oplus X_i$ does not have D.S.S.P. and according to (1.2), G doesn't have this property anymore.

2. Torsion groups

Before passing to determine the structure of the abelian groups with D.S.S.P. some remarks should be made:

REMARKS 2.1. 1) *If A is a subgroup of B, which is not a direct summand in B, then the group $G = A \oplus B$ does not have D.S.S.P. In particular, if B is indecomposable, then for any proper subgroup A of B, the group $G = A \oplus B$ does not have D.S.S.P.*

2) *Any (abelian) divisible group has D.S.S.P.*

Proof. 1) Let A, B and G be as in the statement and let $i : A \to B$ be the inclusion map. According to (1.5)1), $A = i(A)$ is a direct summand in B-contradiction to the hypothesis.

2) Since Z is a hereditary ring, the statement follows from (1.6).

REMARKS 2.2. 1) *For every* $m, n \in N^*$, *with the property that* $m + n \geq 3$, $Z(p^m) \oplus Z(p^n)$ *does not have D.S.S.P.*

2) *For every* $n \in N^*$, $Z(p^n) \oplus Z(p^\infty)$ *does not have D.S.S.P.*

Proof. 1) For every $m, n \in N^*$, with the property that $m + n \geq 3$, there are non-null homomorphisms from $Z(p^m)$ to $Z(p^n)$, or vice-versa, which are not epimorphisms. Now we can apply (1.5)2).

Otherwise: We can apply (2.1)1).

2) For every $n \in N^*$, there are non-null homomorphisms $f : Z(p^n) \to Z(p^\infty)$ which are not epimorphisms. Again we apply (1.5)2).

Otherwise: We can apply (2.1)1).

For p-groups with D.S.S.P. we have:

THEOREM 2.3. *The following statements are equivalent for a p-group G:*

1) *G has D.S.S.P.;*

2) *either* a) *G is indecomposable, or* b) *either $pG = 0$ or G is divisible.*

Proof. 1) implies 2)a) If G is indecomposable, then this gives the required result.

b) First we are going to show that for every $a \in G[p]$, either $h_p(a) = 0$ or $h_p(a) = \infty$. We suppose that there is $a \in G[p]$ such that $h_p(a) = k$, where $k \in N$ and $0 < k < \infty$. It follows that $a \in p^k G$; so there is a $b \in G$ such that $a = p^k b$ and $pa = 0$. Therefore $p^{k+1} b = 0$ and $o(b) = p^{k+1}$. According to [4, 27.5], $\langle b \rangle$ is a direct summand in G; so $G = \langle b \rangle \oplus G_1$ and $\langle a \rangle \subseteq \langle b \rangle$. Now, we suppose that for any $g \in G_1[p]$, $\langle g \rangle$ is a direct summand in G_1. Then $G_1[p]$ is a direct summand in G_1; therefore $G_1 = G_1[p] \oplus F_1$. But $F_1[p] = F_1 \cap G_1[p] = 0$. It follows that $F_1 = 0$, $G_1 = G_1[p]$ and, up to isomorphism, $G = Z(p^{k+1}) \oplus (\oplus_{m_p} Z(p))$, where m_p is any cardinal. We consider $S = Z(p^{k+1}) \oplus Z(p)$ a direct summand in G. Since $k \geq 1$, according to (2.2)1), S does not have D.S.S.P. and then G doesn't have this property anymore—contradiction to the hypothesis. It follows that there is a $g \in G_1[p]$ such that $\langle g \rangle$ is not a direct summand in G_1. We choose such a $g \in G_1[p]$. Then $o(b + g) = p^{k+1}$ and $\langle b + g \rangle$ is a direct summand in G; so $G = \langle b + g \rangle \oplus H$. It can be easily proved that $\langle b \rangle \cap \langle b + g \rangle = \langle pb \rangle$. In this case, according to the choice of g, $\langle b \rangle + \langle b + g \rangle = \langle b \rangle \oplus \langle g \rangle$ is not a direct summand in G; hence G does not have D.S.S.P. - again we obtain a contradiction to the hypothesis. So the initial supposition is false and for every $a \in G[p]$, either $h_p(a) = 0$ or $h_p(a) = \infty$.

Let U be the set of p-bounded direct summands of G and let $\{A_1, A_2, \ldots, A_n, \ldots\} \subseteq U$ be a totally ordered subset of U. Then $A = \langle \bigcup_{i \geq 1} A_i \rangle$ is a subgroup of G and, according to [8, p. 151], A is a direct summand in G. By Zorn's Lemma we obtain that $G = B \oplus C$, where B is a maximal element of U. Let $g \in C[p]$ be any element from $C[p]$. If $h_p(g) = 0$, then $C = \langle g \rangle \oplus F$ and $B \oplus \langle g \rangle \in U$, contradicting the maximality of B. So, from above proved facts it follows that, for every $g \in C[p]$ we have that $h_p(g) = \infty$. But, in this case, C is, up to isomorphism, a direct sum of copies of $Z(p^\infty)$. Therefore the direct summands of G are either p-bounded or isomorphic to $\oplus_{n_p} Z(p^\infty)$, where n_p is any cardinal. From (2.2)2) it follows that G is either reduced, in which case $pG = 0$, or divisible.

2) implies 1) We suppose that the p-group G satisfies one of the conditions a) or b). If G is indecomposable it has in a trivial way D.S.S.P. If $pG = 0$ then G is an elementary p-group and it has D.S.S.P. If G is divisible then (2.1)2) completes the proof.

From (2.3) we obtain the structure of a p-group with D.S.S.P.

COROLLARY 2.4. *Let G be an abelian p-group. Then G has D.S.S.P. if and only if either:*

(3) (i) $G = Z(p^n)$,

or

(4) ii) $G = (\oplus_{m_p} Z(p))$,

or

(5) iii) $G = (\oplus_{n_p} Z(p^\infty))$

where: $n \in N^*$, $n \geq 2$, and m_p and n_p are any cardinals.

From (1.4) and (2.4) we obtain the structure of a torsion group with D.S.S.P.

COROLLARY 2.5. *Let G be an abelian torsion group. Then G has D.S.S.P. if and only if:*

$$G = (\bigoplus_{p \in P_1} A_p) \oplus (\bigoplus_{p \in P_2} B_p) \oplus (\bigoplus_{p \in P_3} C_p), \tag{6}$$

where:

—P_1, P_2 and P_3 are subsets of the set P of all prime numbers, with the property that $P_1 \cap P_2 = P_1 \cap P_3 = P_2 \cap P_3 = \emptyset$,

—for every $p \in P_1$, A_p is a divisible p-group,

—for every $p \in P_2$, B_p is an elementary p-group,

—for every $p \in P_3$, C_p is a reduced indecomposable (non-elementary) p-group.

3. Torsion-free groups

Now we shall pass to torsion-free groups with D.S.S.P. We begin with:

LEMMA 3.1. *If H and K are reduced, indecomposable torsion-free groups, then the following statements are equivalent:*

1) *The group $G = H \oplus K$ has D.S.S.P.*
2) $Hom(H,K) = Hom(K,H) = 0$.

Proof. 1) implies 2) Suppose that G has D.S.S.P. and that there is a homomorphism $0 \neq f : H \to K$. Since K is reduced there is a prime number p such that $pK \neq K$. If p is such a number and g is the multiplication by p in K, then $g \cdot f : H \to K$ is not epimorphism. Now (1.5)2) shows that $Hom(H,K) = 0$. Analogously we obtain that $Hom(K,H) = 0$.

2) implies 1) If $Hom(H,K) = Hom(K,H) = 0$, then G is a direct sum of two fully invariant direct summands, each having D.S.S.P.; (1.3) completes the proof.

COROLLARY 3.2. *If H is a reduced torsion-free group, then the groups $H \oplus H$ and $Z \oplus H$ do not have D.S.S.P.*

The main result of this section is the following:

THEOREM 3.3. *The following statements are equivalent for any torsion-free group G:*

a) *G has D.S.S.P.*
b) *either*

(7) i) *G is divisible,*

or

(8) ii) *G is reduced indecomposable,*

or

(9) iii) $$G = \bigoplus_{i \in I} G_i,$$

where:

— *for every $i \in I$, G_i is a reduced indecomposable group,*
— *for every $i_1, i_2 \in I$, $i_1 \neq i_2$, $Hom(G_{i_1}, G_{i_2}) = Hom(G_{i_2}, G_{i_1}) = 0$.*

Proof. a) implies b) Let G be a torsion-free group with D.S.S.P. From (2.1)1) and (4.1) it follows that if H is a reduced group, then $Q \oplus H$ is not a direct summand in G. So G is either divisible or reduced. If G is either divisible or reduced indecomposable, then this gives the required result. If $G = \bigoplus_{i \in I} G_i$, where, for every $i \in I$, G_i is a reduced indecomposable group, then (3.1) completes the proof.

b) implies a) Let G be a group which satisfies one of the conditions from point b). If G is divisible, then (2.1)2) shows that it has D.S.S.P. If G is a reduced, indecomposable group, then it has, in a trivial way, D.S.S.P. If G is of the form (9), then, for every $i \in I$, G_i has D.S.S.P., and it is fully invariant in G and from (1.3) it follows that G has D.S.S.P.

COROLLARY 3.4. *The following statements are equivalent for any completely decomposable torsion-free group G:*

a) *G has D.S.S.P.*

b) *either*

(7) i) *G is divisible,*

or

(10) ii) *G is reduced (indecomposable), of rank one,*

or

(11) iii) $G = \bigoplus_{i \in I} G_i,$

where:

— *for every $i \in I$, G_i is a reduced (indecomposable) of rank one group,*

— *for every $i_1, i_2 \in I$, $i_1 \neq i_2$, $t(G_{i_1})$ and $t(G_{i_2})$ are incomparable types.*

Proof. For any H and K reduced, torsion-free, of rank one groups, $Hom(H,K) = Hom(K,H) = 0$ if and only if $t(H)$ and $t(K)$ are incomparable types. So, we can apply (3.3).

4. Splitting mixed groups

In this section we will study the (abelian) splitting mixed groups with D.S.S.P. We begin with:

LEMMA 4.1. *Let A be an abelian group and let $G = D \oplus A$, with D-divisible and torsion-free. The following statements are equivalent:*

a) *G has D.S.S.P.*

b) *A is of the form* (6).

Proof. a) implies b) We suppose that G has D.S.S.P. Then $Q \oplus A$ has this property. We consider a maximal independent system $\{x_i\}_{i \in I}$ of elements of infinite order and denote by $B = \bigoplus_{i \in I} \langle x_i \rangle$. Then any homomorphism of groups $f : A \to Q$ induces a homomorphism $g : B \to Q$ and vice-versa. But, for every $i \in I$, $\langle x_i \rangle \cong Z$; let $f_i : \langle x_i \rangle \to Z$ be an isomorphism. Then, for every $i \in I$, there is a homomorphism $h_i : Z \to Q$ such that $h_i f_i = g_i$, where $g_i : \langle x_i \rangle \to Q$ is a homomorphism of groups and $Img_i = Imh_i$. Since $Img = \sum_{i \in I} Img_i$, it follows that there is a homomorphism $g^* : B \to Q$ such that $Img^* \subset Q$. Then the homomorphism $f^* : A \to Q$, induced by g^*, is not epimorphism either, contradiction to (1.5)1). It follows that A does not have elements of infinite order; so A is a torsion group. Since A has D.S.S.P., it follows that it is of the form (6).

b) implies a) If A is a group of the form (6), then it has D.S.S.P. In this case, since the group G is a direct sum of two fully invariant subgroups (in G) and which have (each) D.S.S.P., (1.3) completes the proof.

PROPOSITION 4.2. *Let H be a reduced, indecomposable, torsion-free group and let $G = H \oplus Z(p^n)$, where p is a prime number and $n \in N^*$. Then the following two statements hold:*

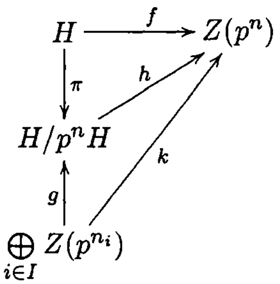
a) *If H is not p-divisible, then G has D.S.S.P. if and only if $n = 1$.*

b) *If H is p-divisible, then G has D.S.S.P. for every $n \geq 1$. Moreover, if $n \geq 2$, G has D.S.S.P. exactly if H is p-divisible.*

Proof. a) We suppose that H is not p-divisible. Of course if $n = 1$, then $G = A \oplus Z(p)$ has D.S.S.P.

Conversely, we suppose that $n \geq 2$ and $G = H \oplus Z(p^n)$ has D.S.S.P. We consider $f : H \to Z(p^n)$ any homomorphism of groups and let $\pi : H \to H/p^n H$ be the canonical projection of H along the quotient group H/p^nH. Since $p^nH = ker\pi \subseteq kerf$ there is a homomorphism $h : H/p^nH \to Z(p^n)$ such that $f = h\pi$. On the other side, $p^n(H/p^nH) = 0$, so H/p^nH is a bounded p-group, that is it is isomorphic to a direct sum of cyclic p-groups, of order $p^k \leq p^n$; let $H/p^nH \cong \bigoplus_{i \in I} Z(p^{n_i})$, where for every $i \in I$, $n_i \leq n$. Then there is a homomorphism $k : \bigoplus_{i \in I} Z(p^{n_i}) \to Z(p^n)$ such that $k = hg$, where $g : \bigoplus_{i \in I} Z(p^{n_i}) \to H/p^nH$ is an isomorphism, see the figure (12)

(12)



According to the hypothesis and to (1.5)2)ii), f is epimorphism. It follows that h is an epimorphism too. Then $Z(p^n) = Imf = Im(h\pi) = Imh = Im(hg) = Imk = \sum_{i \in I} Imk_i$, where, for every $i \in I$, $k_i : Z(p^{n_i}) \to Z(p^n)$ is a homomorphism of groups. Let $m \in N$, with the property that $1 \leq m \leq n-1$ and denote by $B = \langle p^m \rangle \subset Z(p^n)$. If for every $i \in I$, $k_i(1) = p^m$, then, for every $i \in I$, $Imk_i \subseteq B$. In this case $Z(p^n) = Imf = \sum_{i \in I} Imk_i \subseteq B \subset Z(p^n)$, what is impossible. It follows that $n = 1$.

b) If H is p-divisible, then G is a direct sum of two fully invariant direct summands, each having D.S.S.P. By (1.3) it follows that G has D.S.S.P.

Now, we suppose that $n \geq 2$ and G has D.S.S.P. According to what has been proved at point a), if H is not p-divisible then $n = 1$ - what is in contradiction to the hypothesis. Therefore, in this case, H is p-divisible.

COROLLARY 4.3. *Let H be a reduced, torsion-free indecomposable group, B a (reduced) p-group p^n-bounded and let $G = H \oplus B$, where p is a prime number and $n \in N^*$. Then the following two statements hold:*

a) *If H is not p-divisible, then G has D.S.S.P. if and only if B is elementary.*

b) *If H is p-divisible, then G has D.S.S.P. for any group B which is either indecomposable or elementary (so, B is of the form either* (3) *or* (4)). *Moreover, if $n \geq 2$, G has D.S.S.P. exactly if H is p-divisible and B is indecomposable.*

Proof. a) We suppose that H is not p-divisible and G has D.S.S.P. Then, according to (1.2), B has D.S.S.P. From (2.4) it follows that either $B = Z(p^n)$ or $B = (\oplus_{m_p} Z(p))$, where $n \in N^*$, $n \geq 2$, and m_p is any cardinal. Now from (4.2)a) it follows that B is elementary.

Conversely, we suppose that B is an elementary p-group and we consider T and S two direct summands of G. Since B is fully invariant in G, it follows that either $T = K \oplus U$ or $T = U$, and either $S = L \oplus V$ or $S = V$, where U and V are direct summands in B and K and L are isomorphic to H. It follows that $T + S$ is a direct summand in G and, thus, G has D.S.S.P.

b) If H is p-divisible, then G has D.S.S.P. if and only if B has this property. Now (2.4) completes the proof.

We suppose that $n \geq 2$. Then, according to the hypothesis, to (2.4) and to (4.2)b), G has D.S.S.P. if and only if $B = Z(p^n)$ and A is p-divisible.

The result from (4.3) can be generalized:

COROLLARY 4.4. *Let $H = \bigoplus_{i \in I} H_i$ be a reduced, torsion-free group, with the property that for every $i \in I$, H_i is indecomposable and let $B = \bigoplus_{p \in P'} B_p$ be a reduced, torsion group, decomposed according to [4,8.4], where P' is a set of prime numbers. Then the following statements are equivalent:*

1) *The group $G = H \oplus B$ has D.S.S.P.*

2) *The following two statements hold:*

a) *For every $p \in P'$, either*

i) *B_p is of the form* (3) *and for every $i \in I$, H_i is p-divisible, or*

ii) *B_p is of the form* (4).

b) *For every $i_1, i_2 \in I$, $i_1 \neq i_2$, $Hom(H_{i_1}, H_{i_2}) = Hom(H_{i_2}, H_{i_1}) = 0$.*

Proof. 1) implies 2) Let G be a group as in statement, with D.S.S.P. Then H, B and, for every $p \in P'$ and for every $i \in I$, $H_i \oplus B_p$ have (each) D.S.S.P. From (3.3) and (4.3) it follows that one of the conditions a) or b) holds.

2) implies 1) Suppose that, one of the conditions a) or b) holds. Then

$$G = H \oplus (\bigoplus_{p \in P_2} B_p) \oplus (\bigoplus_{p \in P_3} C_p), \tag{13}$$

where:

—P_2 and P_3 are disjoint subsets of the set P', with the property that $P_2 \cup P_3 = P'$,

—for every $p \in P_2$, B_p is an elementary p-group,

—for every $p \in P_3$, C_p is a reduced indecomposable (non-elementary) p-group.

Denote by $K = H \oplus (\bigoplus_{p \in P_2} B_p)$ and by $L = \bigoplus_{p \in P_3} B_p$. Any direct decomposition of K is of the form $K = H' \oplus (\bigoplus_{p \in P_2} B_p)$, where H' is isomorphic to H and any direct summand T of K is of the form $T = E \oplus F$, where E is isomorphic to a direct summand of H and F is a subgroup in $\bigoplus_{p \in P_2} B_p$. Since for any homomorphism $f : H \to \bigoplus_{p \in P_2} B_p$ $\operatorname{Im} f$ is a direct summand, it follows that the sum of any two direct summands of K is again a direct summand in K. On the other side, the summands K and L are fully invariant in G and L has D.S.S.P. (according to (2.5)). Now again (1.3) completes the proof.

LEMMA 4.5. *Let H be a reduced, indecomposable torsion-free group and let $G = H \oplus Z(p^\infty)$, where p is a prime number. Then the following statements are equivalent:*

1) *G has D.S.S.P.*

2) *H is p-divisible.*

Proof. 1) implies 2) We suppose that G has D.S.S.P., H is not p-divisible and $Z(p^\infty) = \langle c_1, c_2, \ldots, c_n, \ldots \rangle$, where $pc_1 = 0$, $pc_2 = c_1, \ldots$, $pc_n = c_{n-1}, \ldots$. Let $n \in N^*$ and let $f : H \to Z(p^n)$ be any homomorphism of groups. If $g : Z(p^n) \to Z(p^\infty)$ is a homomorphism with $g(1) = c_n$, then $g \cdot f : H \to Z(p^\infty)$ is a homomorphism, which is not epimorphism. So, according to (1.5),2), G does not have D.S.S.P.—contradiction to the hypothesis. Therefore if G has D.S.S.P., then H is p-divisible.

2) implies 1) It can be easily proved that any proper direct summand of G is either $Z(p^\infty)$ or isomorphic to H. So let $G = H \oplus Z(p^\infty) = K \oplus Z(p^\infty)$ be two direct decompositions of G. Then $H + K = H \oplus [Z(p^\infty) \cap (H+K)] = K \oplus [Z(p^\infty) \cap (H+K)]$ and $H/(H \cap K)$ is isomorphic to $Z(p^\infty) \cap (H+K)$, see [20]. Since $H/(H \cap K)$ is p-divisible, it follows that either $Z(p^\infty) \cap (H+K)$ is $Z(p^\infty)$ or 0. We conclude that G has D.S.S.P.

We consider the group

$$G = (\bigoplus_{p \in P_1} (\bigoplus_{n_p} Z(p^\infty))) \oplus (\bigoplus_{i \in I} H_i) \tag{14}$$

where:

—P_1 is a subset of prime numbers and n_p is any cardinal,

—I is any index set and for every $i \in I$, H_i is a reduced, indecomposable, torsion-free group.

Since the direct summand $D = \bigoplus_{p \in P_1} (\bigoplus_{n_p} Z(p^\infty))$ is fully invariant in this group G, using (3.1) and (4.5) it is straightforward to prove:

COROLLARY 4.6. *The following statements are equivalent for any group G of the form* (14)*:*

a) *G has D.S.S.P.*

b) *The following two statements hold:*

i) *for every $i \in I$ and every $p \in P_1$, H_i is p-divisible;*

ii) *for every $i_1, i_2 \in I$, $i_1 \neq i_2$, $Hom(H_{i_1}, H_{i_2}) = Hom(H_{i_2}, H_{i_1}) = 0$.*

Now we can present the structure of the splitting mixed groups with D.S.S.P.

THEOREM 4.7. *The following statements are equivalent for any splitting mixed group:*

1) *G has D.S.S.P.*

2) *G is of the form either*

(15) i) $G = D \oplus A$,

or

(16) ii) $G = H \oplus A$,

where:

—*D is torsion-free and divisible;*

—*A is a group of the form* (6)*;*

—*H is a reduced torsion-free group of the form either* (8) *or* (9), *for every non-elementary p-component of A any summand of H is p-divisible and for any two distinct indecomposable summands T and S of H*, $Hom(T,S)=0$.

Proof. 1) implies 2) Let $G=E\oplus F$ be a splitting group with D.S.S.P., where E is divisible and F is reduced. We suppose that $E=(\bigoplus_{m_0} Q)\oplus(\bigoplus_{p\in P_1} A_p)$, where P_1 is a subset of prime numbers and for every $p\in P_1$, A_p is a divisible p-group. If $m_0\neq 0$, then from (4.1) it follows that A is of the form (6) and G is of the form (15). If $m_0=0$ then (4.4) and (4.6) complete the proof.

2) implies 1) Suppose that G is a group which satisfies one of the conditions i) or ii) from the point 2). Then, according to (4.1), (4.4) and (4.6) it suffices to prove only that the group $G=H\oplus A$ has D.S.S.P. if the sets P_1,P_2 and P_3 are non-empty. So we consider such a group

$$G=(\bigoplus_{i\in I} H_i)\oplus(\bigoplus_{p\in P_1} A_p)\oplus(\bigoplus_{p\in P_2} B_p)\oplus(\bigoplus_{p\in P_3} C_p), \tag{17}$$

where:

—P_1,P_2 and P_3 are non-empty subsets of prime numbers, with the property that $P_1\cap P_2=P_1\cap P_3=P_2\cap P_3=\emptyset$,

—for every $p\in P_1$, A_p is a divisible p-group,

—for every $p\in P_2$, B_p is an elementary p-group,

—for every $p\in P_3$, C_p is a reduced indecomposable (non-elementary) p-group,

—for every $i\in I$, H_i is reduced indecomposable, torsion-free and p-divisible for every $p\in P_1\cup P_3$,

—for every $i_1,i_2\in I$, $i_1\neq i_2$, $Hom(H_{i_1},H_{i_2})=Hom(H_{i_2},H_{i_1})=0$.

Denote by $K=(\bigoplus_{i\in I} H_i)\oplus(\bigoplus_{p\in P_1} A_p)$ and by $L=(\bigoplus_{p\in P_2} B_p)\oplus(\bigoplus_{p\in P_3} C_p)$. It follows that K and L have (each) D.S.S.P., according to (4.6), respectively to (2.5). Since they are fully invariant in G and $G=K\oplus L$, again (1.3) completes the proof.

At the end of this work as a conclusion we must remark the following:

Remarks 4.8. 1) a) *There are abelian groups which have both D.S.I.P. and D.S.S.P.*

b) *There are abelian groups which do not have D.S.I.P., but which have D.S.S.P.*

c) *There are abelian groups which have D.S.I.P., but which do not have D.S.S.P.*

d) *There are abelian groups which have neither D.S.I.P. nor D.S.S.P.*

2) *For any torsion group T with D.S.S.P., there are two mixed groups G_1 and G_2 with the properties: G_1 and G_2 have (each) D.S.S.P., G_1 is not isomorphic to G_2, but $T(G_1) = T(G_2) = T$.*

Proof. 1) a) See (1.7).

b) According to [10,3.2] and (2.1)2), the group $G = Z(p^\infty) \oplus Z(p^\infty)$ satisfies the statement conditions.

c) See (1.8). (Otherwise, according to [10,3.2] and (2.2)2), the group $G = Z(p) \oplus Z(p^\infty)$ satisfies the statement conditions.

d) According to [10,3.2] and (2.2)2), the group $G = Z(p) \oplus Z(p^\infty) \oplus Z(p^\infty)$ satisfies the statement conditions.

2) See (4.7).

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