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LATTICES RESPECTING CONVEX DECOMPOSITIONS, II

Abstract. Let (L_1, L_2) be a convex decomposition of a lattice L. We prove that L is a lattice satisfying the atomic covering property provided L_1 and L_2 possess the same property. Moreover we show that L satisfies the general disjointness property (GD) whenever L_1 satisfies GD and L_2 is a modular lattice or whenever L_1 is a modular lattice with 0 and L_2 satisfies GD.

1. Introduction

The general scheme of our investigation in this paper is very similar to that of [2]. Here $L = \overrightarrow{cd}(L_1, L_2)$ denotes the fact that (L_1, L_2) is a convex decomposition of a lattice L.

We now fix some notations and conventions we use throughout the rest of the paper. We refer the reader to our paper [2] and to [1] for the basic theory of convex decompositions and for the background material.

Let $(L_1, \vee_1, \wedge_1)$ and $(L_2, \vee_2, \wedge_2)$ be sublattices of a lattice $(L, \vee, \wedge)$. The couple (L_1, L_2) is said to be a *convex decomposition* of L if L_1 and L_2 are proper sublattices of L, $L_1 \cap L_2 \neq \emptyset$, $L_1 \cup L_2 = L$, the order ideal $(L_1 \cap L_2]$ generated by $L_1 \cap L_2$ is equal to L_1 and the order filter $[L_1 \cap L_2)$ generated by $L_1 \cap L_2$ is equal to L_2.

In Figure 1, a convex decomposition of L_{13} is represented.

CONVENTION 1.1. Let us write for simplicity $b \in \bullet$ (or $\bullet \ni b$) if $b \in$ $\in L_1 \cap L_2$.

It is not difficult to show that *$L_1 \cap L_2$ is a convex subset in $L =$ $= \overrightarrow{cd}(L_1, L_2)$*, i.e., whenever $\bullet \ni a \leq b \leq c \in \bullet$, then $b \in \bullet$.

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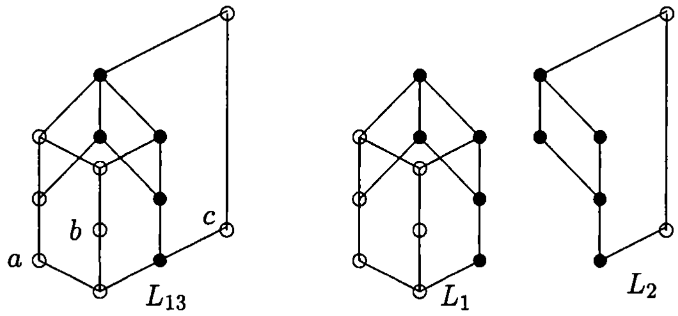

Figure 1

The following results valid for any convex decomposition (L_1, L_2) of L are a useful tool:

(1*) For every $a, b \in L_i$ $(i \in \{1,2\})$ $a \wedge b = a \wedge_i b$ and $a \vee b = a \vee_i b$.
(2*) Given $a \in L_1$ and $b \in L_2$, $a \vee b = (a \vee_1 b_+) \vee_2 b$ and $a \wedge b = a \wedge_1 (a^* \wedge_2 b)$ where b_+ and a^* are any elements of L such that $a \leq a^* \in \bullet$ and $\bullet \ni b_+ \leq b$.
(3*) If $a \in L_1$ and $b \in \bullet$, then $a \vee b \in \bullet$; if $c \in L_2$ and $d \in \bullet$, then $c \wedge d \in \bullet$.

2. AC–lattices

LEMMA 2.1. *Let* $L = \overrightarrow{cd}(L_1, L_2)$ *and* $a, b \in L_i$ *where* $i \in \{1,2\}$. *Then* b *covers* a *in* L_i *if and only if* b *covers* a *in* L, *i.e.,*

$$a \prec_i b \ \Leftrightarrow\ a \prec b.$$

Proof. Let $a, b \in L_1$ be such that $a \prec_1 b$. Now suppose that b fails to cover a in L. Then there exists $c \in L_2 \setminus L_1$ such that $a < c < b$. By the definition of a convex decomposition, there exist b_0 and c_0 such that $b \leq b_0 \in \bullet$ and $\bullet \ni c_0 \leq c$. Since $L_1 \cap L_2$ is convex in L, $c \in \bullet$, which contradicts the choice of c.

The remainder of the proof is straightforward, and will be omitted. ∎

A lattice L with 0 is said to satisfy the *atomic covering property* [4] if it satisfies the implication

$$(AC) \qquad (0 \prec p \ \&\ p \wedge a = 0) \ \Rightarrow\ a \prec a \vee p$$

for any elements a and p of L. We will call such a lattice an *AC–lattice*. See also [5], [6] and [3].

THEOREM 2.2. *Let* $L = \overrightarrow{cd}(L_1, L_2)$ *where* L_1 *and* L_2 *are AC–lattices. Then* L *is an AC–lattice.*

Proof. Assume that $a, p \in L$, $0 \prec p$, $p \wedge a = 0$ and that ω denotes the zero element in L_2.

We distinguish four cases.

Case I: $p \in L_1$ *and* $a \in L_1$. Since L_1 is an AC–lattice, $a \prec_1 a \vee_1 p$ and, by Lemma 2.1 and (1*), $a \prec a \vee p$.

Case II: $p \in L_2$ *and* $a \in L_2$. Then $L_2 \ni p \wedge a = 0$, a contradiction.

Case III: $p \in L_2$ *and* $a \in L_1$. Here $0 < \omega \leq p$ and $0 \prec p$. Hence $p = \omega \in L_1$ and we have the case I.

Case IV: $p \in L_1$ *and* $a \in L_2$. Then $p \wedge \omega \leq p \wedge a = 0$ and so $p \wedge_1 \omega = 0$. Since L_1 is an AC–lattice and $0 \prec_1 p$, we have $\omega \prec_1 p \vee_1 \omega$. Therefore, by Lemma 2.1, $\omega \prec_2 p \vee \omega$. At the same time $\omega \leq a \wedge_2 (p \vee \omega) \leq \leq p \vee \omega$.

Suppose $p \vee \omega = a \wedge_2 (p \vee \omega)$. Then $p \leq p \vee \omega \leq a$ and, consequently, $p \leq p \wedge a = 0$, a contradiction.

Hence $\omega = a \wedge_2 (p \vee \omega)$. Since L_2 is an AC–lattice, $\bullet \ni \omega \leq a$ and (2*) imply that $a \prec_2 (p \vee \omega) \vee_2 a = p \vee a$. Then in view of Lemma 2.1 we have $a \prec a \vee p$. ∎

3. GD–lattices

A lattice L with 0 satisfying the implication

$$(GD) \qquad (a \wedge b = 0 \& (a \vee b) \wedge c = 0) \Rightarrow a \wedge (b \vee c) = 0$$

for every $a, b, c \in L$ is said to satisfy the *general disjointness property* [4]. We call such a lattice briefly a *GD–lattice*. See also [6].

The following result appears in [4] as Proposition 4.2 and we include it here for completeness.

PROPOSITION 3.1. *Any modular lattice with 0 is a GD–lattice.*

We can now provide the following useful lemma:

LEMMA 3.2. *Let* $L = \overrightarrow{cd}(L_1, L_2)$, *let* L_1 *be a GD–lattice and let*

$$a \in L_2 \ \& \ b \in L_1 \ \& \ c \in L_1$$

be such that $a \wedge b = 0$ *and* $(a \vee b) \wedge c = 0$. *Then* $a \wedge (b \vee c) = 0$.

Proof. By (1*) and (2*),

$$a \wedge (b \vee c) = (b \vee_1 c) \wedge_1 [(b \vee_1 c)^* \wedge_2 a]$$

where $b \vee_1 c \leq (b \vee_1 c)^* \in \bullet$. Let $A := a \wedge_2 (b \vee_1 c)^*$, $B := b$ and $C := c$ so that $a \wedge (b \vee c) = A \wedge_1 (B \vee_1 C)$. Now, by (2*) and by assumption,

$$A \wedge_1 B = b \wedge_1 [(b \vee_1 c)^* \wedge_2 a] = b \wedge a = 0.$$

Moreover, by (1*), (3*) and by assumption,

$$(A \vee_1 B) \wedge_1 C = \{[(b \vee_1 c)^* \wedge_2 a] \vee_1 b\} \wedge_1 c = \\ = \{[(b \vee_1 c)^* \wedge a] \vee b\} \wedge c \leq (a \vee b) \wedge c = 0.$$

Since L_1 is a GD–lattice, $0 = A \wedge_1 (B \vee_1 C) = a \wedge (b \vee c)$. ∎

Let x, y and z be elements of a lattice L with 0. The triplet (x, y, z) is said to be a *GD–triplet,* if one of the following conditions

(i) $x \wedge y = 0 \& (x \vee y) \wedge z = 0$;
(ii) $x \wedge z = 0 \& (x \vee z) \wedge y = 0$;
(iii) $y \wedge z = 0 \& (y \vee z) \wedge x = 0$

is satisfied.

REMARK 3.3. If (x, y, z) is a GD–triplet in a GD–lattice, then it is immediate that the three conditions (i), (ii) and (iii) are fulfilled.

Let us now look again at Figure 1. It is worth pointing out that the thirteen–element lattice $L_{13} = \overrightarrow{cd}(L_1, L_2)$ illustrated in the figure is not a GD–lattice. A close inspection shows that the two sublattices L_1, L_2 of L_{13} are GD–lattices. This counter–example implies that we shall need stronger assumptions on L_1 or on L_2 as in [2, Thm 3.1 and Thm 3.2].

Our next two results deal with those convex decompositions (L_1, L_2) of L for which one of the lattices L_1 and L_2 is modular and the other is a GD–lattice.

THEOREM 3.4. *Let* $L = \overrightarrow{cd}(L_1, L_2)$ *where* L_1 *is a GD–lattice and* L_2 *is a modular lattice. Then* L *is a GD–lattice.*

Proof. Let $a, b, c \in L$ be such that

$$a \wedge b = 0 \tag{3.1}$$

and

$$(a \vee b) \wedge c = 0. \tag{3.2}$$

Then

$$a \wedge c = 0 \& b \wedge c = 0. \tag{3.3}$$

If a, b and c are elements of L_1, then the assertion is true by (1*). Since $0 \notin L_2$, it follows from (3.1) and (3.3) that no two elements from $\{a, b, c\}$ belong to L_2. Hence there are only three cases to consider.

Case I: $a \in L_2$, $b \in L_1$ *and* $c \in L_1$. Then the assertion follows from Lemma 3.2.

Case II: $a \in L_1$, $b \in L_1$ *and* $c \in L_2$. Let τ be any element of $L_1 \cap L_2$ such that $\tau \leq c$. Let $s := a \vee_1 b \vee_1 \tau$. Note that $a \leq s \in \bullet$. By (1*), $a \wedge_1 b = 0$. Moreover, in view of (3.2) and (2*) we have

$$(a \vee_1 b) \wedge_1 (s \wedge_2 c) = (a \vee b) \wedge c = 0.$$

Since L_1 is a GD–lattice,

$$a \wedge_1 [b \vee_1 (s \wedge_2 c)] = 0. \tag{3.4}$$

Let $e := a \wedge (b \vee c)$. By (2*),

$$e = a \wedge_1 [s \wedge_2 (b \vee c)] = a \wedge_1 \{s \wedge_2 [(b \vee_1 \tau) \vee_2 c]\}.$$

From $s \geq b \vee_1 \tau$, recalling that L_2 is modular, we have

$$e = a \wedge_1 [(b \vee_1 \tau) \vee_2 (s \wedge_2 c)].$$

By (3*), $s \wedge_2 c \in \bullet$. Now $s \geq \tau$ and $c \geq \tau$. Hence $s \wedge_2 c \geq \tau$ and taking (1*) into account,

$$e = a \wedge_1 [(b \vee_1 \tau) \vee_1 (s \wedge_2 c)] = a \wedge_1 [b \vee_1 (s \wedge_2 c)].$$

From (3.4) it follows that $a \wedge (b \vee c) = 0$.

Case III: $a \in L_1$, $b \in L_2$ *and* $c \in L_1$. Let $a' := b$, $b' := a$ *and* $c' := c$. By Lemma 3.2, $0 = a' \wedge (b' \vee c') = (a \vee c) \wedge b$. By assumption, $a \wedge c = 0$. We therefore have from Case II that $a \wedge (c \vee b) = 0$. ∎

THEOREM 3.5. *Let* $L = \overrightarrow{cd}(L_1, L_2)$ *where* L_1 *is a modular lattice with 0 and where* L_2 *is a GD–lattice. Then* L *is a GD–lattice.*

Proof. Let ω denote the zero element of L_2 and let $a, b, c \in L$ be such that (3.1) and (3.2) are true. However, by Proposition 3.1, L_1 is a GD–lattice. Consequently it follows by Lemma 3.2 that the assertion is true whenever $a \in L_2$, $b \in L_1$ and $c \in L_1$.

Similarly as in the proof of Theorem 3.4 *it suffices to consider the case where* $a \in L_1$, $b \in L_1$ *and* $c \in L_2$.

Let $e := a \wedge (b \vee c)$. Then, by (2*) and (3*),

$$e = a \wedge_1 \{s \wedge_2 [(b \vee_1 \omega) \vee_2 c]\}$$

where $s := a \vee_1 b \vee_1 \omega \in \bullet$. Let $\alpha := a \vee_1 \omega$ and let $\beta := b \vee_1 \omega$ so that $\alpha, \beta \in \bullet$ and $\alpha \vee_1 \beta = s$. Let $d := s \wedge_2 [(b \vee_1 \omega) \vee_2 c]$. Therefore, $d = s \wedge (\beta \vee c)$ and $e = a \wedge_1 d$. Now, $a \wedge_1 b = 0$ by (3.1), and $(a \vee_1 b) \wedge_1 \omega \leq (a \vee b) \wedge c = 0$ by (3.2). Hence (a, b, ω) is a GD–triplet in the GD–lattice L_1. Then by Remark 3.3 we see that $0 = b \wedge_1 (a \vee_1 \omega) = b \wedge_1 \alpha$. By modularity,

$$\alpha \wedge_1 \beta = (a \vee_1 \omega) \wedge_1 (b \vee_1 \omega) = [(a \vee_1 \omega) \wedge_1 b] \vee_1 \omega =$$
$$= (\alpha \wedge_1 b) \vee_1 \omega = \omega.$$

Thus, from (1*), we get

$$\alpha \wedge_1 \beta = \alpha \wedge_2 \beta = \omega. \tag{3.5}$$

Note that

$$0 \leq \omega \leq s \wedge c \leq s = \alpha \vee_1 \beta = a \vee_1 b \vee_1 \omega.$$

From (1*) it follows that

$$(a \vee_1 b) \wedge_1 (s \wedge_2 c) \leq (a \vee b) \wedge c = 0.$$

Let $A := \omega$, $B := a \vee_1 b$ and $C := s \wedge_2 c$. Then $A \leq C$, $B \vee_1 A = B \vee_1 C = a \vee_1 b \vee_1 \omega$ and $0 = B \wedge_1 C = B \wedge_1 A$. By the modularity of L_1, $A = C$ and we have $\omega = s \wedge c = (\alpha \vee_2 \beta) \wedge_2 c$. But then, by (3.5), (α, β, c) is a GD–triplet in the GD–lattice L_2. This yields $\alpha \wedge_2 (\beta \vee_2 c) = \omega$. Now $a \wedge d \leq d \leq \beta \vee c$ and $a \wedge d \leq \alpha$. Consequently, $e = a \wedge d \leq \alpha \wedge (\beta \vee c) = \omega \leq c$. Thus $e \leq a \wedge c = 0$. ∎

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