DEMONSTRATIO MATHEMATICA
Vol. XXXV No 2 2002

M. Ganster, S. Jafari, G. B. Navalagi

# ON SEMI-g-REGULAR AND SEMI-g-NORMAL SPACES

**Abstract.** The aim of this paper is to introduce and study two new classes of spaces, called semi-g-regular and semi-g-normal spaces. Semi-g-regularity and semi-g-normality are separation properties obtained by utilizing semi-generalized closed sets. Recall that a subset $A$ of a topological space $(X, \tau)$ is called semi-generalized closed, briefly sg-closed, if the semi-closure of $A \subseteq X$ is a subset of $U \subseteq X$ whenever $A$ is a subset of $U$ and $U$ is semi-open in $(X, \tau)$.

## 1. Introduction and preliminaries

In 1970, Levine [15] introduced a new and significant notion in General Topology, namely the notion of a generalized closed set. A subset $A$ of a topological space $(X, \tau)$ is called *generalized closed*, briefly g-closed, if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$. This notion has been studied extensively in recent years by many topologists. The investigation of generalized closed sets has led to several new and interesting concepts, e.g. new covering properties and new separation axioms weaker than $T_1$. Some of these separation axioms have been found to be useful in computer science and digital topology. As an example, the well-known digital line is a $T_{3/4}$ space but fails to be a $T_1$ space (see e.g. [7]). In 1987, Ganguly et al. [11] generalized the usual notions of regularity and normality by replacing "closed set" with "g-closed set" in the definitions, thus obtaining the notions of g-regularity and g-normality.

The aim of our paper is to introduce and investigate the notions of semi-g-regularity and semi-g-normality. We shall do so by utilizing the concept of a semi-generalized closed set.

1991 *Mathematics Subject Classification*: Primary: 54A05; Secondary: 54D10.
*Key words and phrases*: semi-open, semi-generalized closed, semi-g-regular, semi-g-normal.

A subset $A$ of a topological space $(X, \tau)$ is called *semi-open* [14] if $A \subseteq cl(int(A))$, where $cl(A)$ and $int(A)$ denote the closure and the interior of $A$. The complement of a semi-open set is called a *semi-closed* set. The *semi-closure* [4] of $A$, denoted by $scl(A)$, is the intersection of all semi-closed sets containing $A$. The *semi-interior* of $A$, denoted by $sint(A)$, is the largest semi-open set contained in $A$. It is well known that $scl(A) = A \cup int(cl(A))$ and $sint(A) = A \cap cl(int(A))$ for any subset $A \subseteq X$. Recall that a subset $A$ is said to be *preopen* if $A \subseteq int(cl(A))$.

DEFINITION 1. A topological space $(X, \tau)$ is said to be

(i) *semi-regular* [9] if for each semi-closed $A \subseteq X$ and each point $x \notin A$ there exist disjoint semi-open sets $U, V \subseteq X$ such that $x \in U$ and $A \subseteq V$,

(ii) *semi-normal* if for each pair $A, B \subseteq X$ of disjoint semi-closed sets there exist disjoint semi-open sets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$.

DEFINITION 2. A subset $A$ of a topological space $(X, \tau)$ is called *semi-generalized closed* [2], briefly sg-closed, if $scl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open.

The complement of a semi-generalized closed set is called *semi-generalized open.*

It is obvious that every semi-closed set is sg-closed. Spaces in which every sg-closed set is semi-closed have been called *semi-*$T_{1/2}$ [2]. One can easily show that a space $(X, \tau)$ is semi-$T_{1/2}$ if and only if every singleton is either semi-open or semi-closed. Recall also that a space $(X, \tau)$ is said to be *semi-*$T_2$ if distinct points can be separated by disjoint semi-open sets.

In [13], Jankovic and Reilly pointed out that every singleton $\{x\}$ of a space $(X, \tau)$ is either nowhere dense or preopen. This yields a decomposition $X = X_1 \cup X_2$, where $X_1 = \{x \in X : \{x\}$ is nowhere dense$\}$ and $X_2 = \{x \in X : \{x\}$ is preopen$\}$. The usefulness of this decomposition, which we call the *Jankovic-Reilly decomposition*, is illustrated by the following result.

LEMMA 1.1 [8]. *A subset $A$ of a space $(X, \tau)$ is sg-closed if and only if $X_1 \cap scl(A) \subseteq A$.*

DEFINITION 3. A function $f : (X, \tau) \to (Y, \sigma)$ is called

(i) *irresolute* [5] if $f^{-1}(B)$ is semi-open in $X$ for every semi-open set $B \subseteq Y$,

(ii) *pre-semi-closed* [12] if $f(F)$ is semi-closed in $Y$ for every semi-closed set $F \subseteq X$.

The following result has been proved by Garg and Sivaraj [12].

LEMMA 1.2. *A function $f : (X, \tau) \to (Y, \sigma)$ is pre-semi-closed if and only if $scl(f(A)) \subseteq f(scl(A))$ for every subset $A \subseteq X$.*

## 2. Semi-g-regular spaces

DEFINITION 4. A topological space $(X, \tau)$ is said to be *semi-g-regular* if for each sg-closed set $A$ and each point $x \notin A$, there exist disjoint semi-open sets $U, V \subseteq X$ such that $A \subseteq U$ and $x \in V$.

LEMMA 2.1. *A space $(X, \tau)$ is semi-g-regular if and only if $(X, \tau)$ is semi-regular and semi-$T_{1/2}$.*

Proof. Suppose that $(X, \tau)$ is semi-g-regular. Then clearly $(X, \tau)$ is semi-regular. Now let $A \subseteq X$ be sg-closed. For each $x \notin A$ there exists a semi-open set $V_x$ containing $x$ such that $V_x \cap A = \emptyset$. If $V = \bigcup\{V_x : x \notin A\}$, then $V$ is semi-open and $V = X \setminus A$, hence $A$ is semi-closed.

The converse is obvious. ∎

We will now show that there exist semi-regular spaces which are not semi-g-regular. Recall that a space $(X, \tau)$ is said to be *locally indiscrete* if every open subset is closed. It is well known that $(X, \tau)$ is locally indiscrete if and only if every subset of $X$ is preopen. If $(X, \tau)$ is locally indiscrete, then every semi-closed set is closed and thus clopen, and, by Lemma 1.1, every subset of $X$ is sg-closed.

EXAMPLE 2.2. Let $Y$ and $Z$ be disjoint infinite sets, and let $X = Y \cup Z$. Then $\tau = \{\emptyset, Y, Z, X\}$ is a topology on $X$. Clearly $(X, \tau)$ is locally indiscrete and thus every semi-closed set is clopen, hence $(X, \tau)$ is semi-regular. If $\emptyset \neq A \subseteq Y$ and $x \in Y \setminus A$ then $A$ is sg-closed, but $A$ and $x$ cannot be separated by disjoint semi-open sets. Thus $(X, \tau)$ fails to be semi-g-regular.

Our next result characterizes semi-regular spaces. We shall call subset $A \subseteq X$ *semi-clopen* in $(X, \tau)$ if $A$ is both semi-open and semi-closed. Observe that for any semi-open set $V$, $scl(V)$ is always semi-clopen.

THEOREM 2.3. *For a topological space $(X, \tau)$ the following are equivalent:*

1) *$(X, \tau)$ is semi-g-regular.*
2) *Every sg-open set $U$ is a union of semi-clopen sets.*
3) *Every sg-closed set $A$ is an intersection of semi-clopen sets.*

Proof. (1) $\Rightarrow$ (2) : Let $U$ be sg-open and let $x \in U$. If $A = X \setminus U$, then $A$ is sg-closed. By assumption there exist disjoint semi-open subsets $W_1$ and $W_2$ of $X$ such that $x \in W_1$ and $A \subseteq W_2$. If $V = scl(W_1)$, then $V$ is semi-clopen and $V \cap A \subseteq V \cap W_2 = \emptyset$. It folllows that $x \in V \subseteq U$. Thus $U$ is a union of semi-clopen sets.

(2) $\Leftrightarrow$ (3) : This is obvious.

(3) $\Rightarrow$ (1) : Let $A$ be sg-closed and let $x \notin A$ . By assumption, there exists a semi-clopen set $V$ such that $A \subseteq V$ and $x \notin V$. If $U = X \setminus V$, then $U$ is a semi-open set containing $x$ and $U \cap V = \emptyset$. Thus $(X, \tau)$ is semi-g-regular. ∎

Sg-open sets give rise to various separation properties of which we offer the following.

DEFINITION 5. A topological space $(X, \tau)$ is called a

(i) *sg–$T_0$* space if for each pair of distinct points there exists an sg-open set containing one point but not the other.

(ii) *(s,sg)–$R_0$* space if $scl(\{x\}) \subseteq U$ whenever $U$ is sg-open and $x \in U$.

Observe that the space in Example 2.2 is neither sg–$T_0$ nor (s,sg)–$R_0$. Also note that every semi-$T_2$ space is semi-$T_{1/2}$, and every semi-$T_{1/2}$ space is sg-$T_0$.

THEOREM 2.4. *Every semi-g-regular space $(X, \tau)$ is both semi-$T_2$ and (s,sg)–$R_0$.*

Proof. Let $(X, \tau)$ be semi-g-regular, and let $x, y \in X$ such that $x \neq y$. By Lemma 2.1, $\{x\}$ is either semi-open or semi-closed. If $\{x\}$ is semi-open, hence sg-open, then $\{x\}$ is semi-clopen by Theorem 2.3. Thus $\{x\}$ and $X \setminus \{x\}$ are separating semi-open sets. If $\{x\}$ is semi-closed, then $X \setminus \{x\}$ is semi-open and so, by Theorem 2.3, the union of semi-clopen sets. Hence there is a semi-clopen set $V \subseteq X \setminus \{x\}$ containing $y$. This proves that $(X, \tau)$ is semi-$T_2$.

By Theorem 2.3 it follows immediately that $(X, \tau)$ also has to be (s,sg)–$R_0$. ∎

## 3. Semi-g-normal spaces

DEFINITION 6. A topological space $(X, \tau)$ is said to be *semi-g-normal* if for every pair of disjoint sg-closed sets $A$ and $B$ of $(X, \tau)$, there exist disjoint semi-open sets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$.

REMARK 3.1. Obviously every semi-g-normal space is semi-normal. Example 2.2 shows that the converse is false.

Recall that a space $(X, \tau)$ is said to be *semi-symmetric* [3] if for any two points $x, y \in X$, if $x \in scl(\{y\})$ then $y \in scl(\{x\})$. Caldas [3] has shown that a space $(X, \tau)$ is semi-symmetric if and only if $\{x\}$ is sg-closed for every $x \in X$.

REMARK 3.2. Observe that, if $(X, \tau)$ is semi-normal and $F \cap A = \emptyset$, where $F$ is semi-closed and $A$ is sg-closed, then $F \cap sclA = \emptyset$, hence there exist disjoint semi-open sets $U, V \subseteq X$ such that $F \subseteq U$ and $A \subseteq V$.

As a consequence we have the following result.

THEOREM 3.3. (i) *Every semi-normal, semi-symmetric topological space* $(X, \tau)$ *is semi-regular.*

(ii) *Every semi-g-normal, semi-symmetric topological space* $(X, \tau)$ *is semi-g-regular.*

Proof. (i) Suppose that $A \subseteq X$ is semi-closed and $x \notin A$. Then $\{x\}$ is sg-closed. By Remark 3.2, $A$ and $\{x\}$ can be separated by disjoint semi-open sets.

(ii) is obvious. ∎

THEOREM 3.4. *For a topological space* $(X, \tau)$ *the following are equivalent:*

(1) $(X, \tau)$ *is semi-g-normal.*

(2) *For every sg-closed set* $A$ *and every sg-open set* $U$ *containing* $A$, *there is a semi-clopen set* $V$ *such that* $A \subseteq V \subseteq U$.

Proof. (1) $\Rightarrow$ (2) : Let $A$ be sg-closed and $U$ be sg-open with $A \subseteq U$. Now we have $A \cap (X \setminus U) = \emptyset$, hence there exist disjoint semi-open sets $W_1$ and $W_2$ such that $A \subseteq W_1$ and $X \setminus U \subseteq W_2$. If $V = scl(W_1)$, then $V$ is semi-clopen satisfying $A \subseteq V \subseteq U$.

(2) $\Rightarrow$ (1) : This is obvious. ∎

We now investigate the behaviour of semi-g-normal spaces under certain mappings.

LEMMA 3.5. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be irresolute and pre-semi-closed. If* $B \subseteq Y$ *is sg-closed, then* $f^{-1}(B) \subseteq X$ *is sg-closed.*

Proof. Let $B \subseteq Y$ be sg-closed, and let $f^{-1}(B) \subseteq U$ where $U \subseteq X$ is semi-open. If $V = Y \setminus f(X \setminus U)$, then $V \subseteq Y$ is semi-open since $f$ is pre-semi-closed. Moreover, we have $f^{-1}(V) \subseteq U$ and $B \subseteq V$. Since $B$ is sg-closed, we conclude that $scl(B) \subseteq V$. Since $f$ is irresolute, we now have $scl(f^{-1}(B)) \subseteq f^{-1}(scl(B)) \subseteq f^{-1}(V) \subseteq U$. Hence $f^{-1}(B)$ is sg-closed. ∎

THEOREM 3.6. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be irresolute, pre-semi-closed and onto. If* $(X, \tau)$ *is semi-g-normal, then* $(Y, \sigma)$ *is semi-g-normal.*

Proof. Let $B_1, B_2 \subseteq Y$ be disjoint sg-closed sets. By Lemma 3.5, $f^{-1}(B_1)$ and $f^{-1}(B_2)$ are sg-closed, hence there exist disjoint semi-open sets $U_1, U_2 \subseteq X$ such that $f^{-1}(B_1) \subseteq U_1$ and $f^{-1}(B_2) \subseteq U_2$. As in the proof of Lemma 3.5 there exist semi-open sets $V_1, V_2 \subseteq Y$ such that $B_i \subset V_i$ and $f^{-1}(V_i) \subseteq U_i$ for $i = 1, 2$. Hence $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$ and, since $f$ is onto, we have $V_1 \cap V_2 = \emptyset$, i.e. $(Y, \sigma)$ is semi-g-normal. ∎

Recall that a function $f : (X, \tau) \to (Y, \sigma)$ is said to be *contra-semi-continuous* [6] if the inverse image of every closed set in $(Y, \sigma)$ is semi-open in $(X, \tau)$.

DEFINITION 7. (i) A space $(X, \tau)$ is called *weakly sg-normal* if disjoint sg-closed sets can be separated by disjoint closed sets.

(ii) A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *always sg-closed* if the image of each sg-closed set in $(X, \tau)$ is sg-closed in $(Y, \sigma)$.

THEOREM 3.7. *If $f : (X, \tau) \to (Y, \sigma)$ is an injective contra-semi-continuous always sg-closed function and $(Y, \sigma)$ is weakly sg-normal, then $(X, \tau)$ is semi-g-normal.*

Proof. Suppose that $A_1, A_2 \subseteq X$ are sg-closed and disjoint. Since $f$ is always sg-closed and injective, $f(A_1), f(A_2) \subseteq Y$ are sg-closed and disjoint. Since $(Y, \sigma)$ is weakly sg-normal, $f(A_1)$ and $f(A_2)$ can be separated by disjoint closed sets $B_1, B_2 \subseteq Y$. Since $f$ is contra-semi-continuous, $A_1$ and $A_2$ can be separated by the disjoint semi-open sets $f^{-1}(B_1)$ and $f^{-1}(B_2)$. Thus $(X, \tau)$ is semi-g-normal. ∎

## References


[1] S. P. Arya and M. P. Bhamini, *A generalization of normal spaces*, Mat. Vesnik 35 (1983), 1–10.

[2] P. Bhattacharyya and B. K. Lahiri, *Semi-generalized closed sets in topology*, Indian J. of Math. 29 (3) (1987), 375–382.

[3] M. Caldas, *Semi $T_{1/2}$-spaces*, Pro Math. 8 (1994), 115–121.

[4] S. G. Crossley and S. K. Hildebrand, *Semi-closure*, Texas J. Sci. 22 (1971), 99–112.

[5] S. G. Crossley and S. K. Hildebrand, *Semi-topological properties*, Fund. Math. 74 (1972), 233–254.

[6] J. Dontchev and T. Noiri, *Contra-semicontinuous functions*, Math. Pannonica 10 (1999), 159–168.

[7] J. Dontchev and M. Ganster, *On $\delta$-generalized closed sets and $T_{3/4}$ spaces*, Mem. Fac. Sci Kochi Univ. 17 (1996), 15–31.

[8] J. Dontchev and H. Maki, *On sg-closed sets and semi-$\lambda$-closed sets*, Q & A in General Topology 15 (1997), 259–266.

[9] C. Dorsett, *Semi-regular spaces*, Soochow J. Math. 8 (1982), 45–53.

[10] C. Dorsett, *Semi-normal spaces*, Kyungpook Math. J. 25 (1985), 173–180.

[11] A. Ganguly and R. S. Chandel, *Some results on general topology*, J. Indian Acad. Math. 9(2)(1987), 87–91.

[12] G. L. Garg and D. Shivaraj, *Presemiclosed mappings*, Periodica Math. Hungar. 19(2) (1988), 97–106.

[13] D. Jankovic and I. Reilly, *On semi-separation axioms*, Indian J. Pure Appl. Math. 16 (1985), 957–964.

[14] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly 70 (1963), 36–41.
[15] N. Levine, *Generalized closed sets in topology*, Rend. Circlo. Mat. Palermo 19 (2) (1970), 89–96.

Addresses of Authors:

M. Ganster
DEPARTMENT OF MATHEMATICS A
GRAZ UNIVERSITY OF TECHNOLOGY
Steyrergasse 30
A-8010 GRAZ, AUSTRIA
E-mail: ganster@weyl.math.tu-graz.ac.at

S. Jafari
DEPARTMENT OF MATHEMATICS AND PHYSICS
ROSKILDE UNIVERSITY
POSTBOX 260
4000 ROSKILDE, DENMARK
E-mail: jafari@post12.tele.dk

G. B. Navalagi
DEPARTMENT OF MATHEMATICS
G. H. COLLEGE
HAVERI-581110
KARNATAKA, INDIA
E-mail: gnavalagi@hotmail.com

*Received June 1st., 2001.*