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ON CR-SUBMANIFOLD OF NEARLY AND CLOSELY PARA COSYMPLECTIC MANIFOLDS

1. Introduction

CR-submanifold of an almost contact metric manifold have been studied by ([7], [8]). Para cosymplectic manifold have been studied by [2]. The purpose of the present paper is to study the CR-submanifold of nearly and closely para cosymplectic manifolds.

Section 2 is preliminary in which we recall some definitions. Section 3 is devoted to basic results. In section 4 integrability, autoparallelness and nearly autoparallelness of the distribution $D^1 \oplus \{U\}$ on submanifold are studied. Integrability of distributions D^0, $D^0 \oplus \{U\}$ are also studied in this section. In section 5 totally umbilical and totally geodesic submanifolds are studied. Section 6 is devoted to the study of para contact umbilical and totally para contact geodesic submanifolds.

2. Preliminaries

Let $\overline{V}$ be an almost para contact metric manifold [3] with structure tensors $(F,\ U,\ u,\ g)$,where F is a $(1,1)$ tensor field, a vector field U, a 1-form u associated with U and a metric tensor g satisfying the following relations:

$$F^2 = I - u \otimes U,\ u(U) = 1,\ F(U) = 0,\ uoF = 0, \tag{1}$$

$$g(FX, FY) = g(X,Y) - u(X)u(Y), \tag{2}$$

$$g(X, FY) = g(FX, Y),\ g(X,U) = u(X),\ \forall X,\ Y \in T\overline{V}. \tag{3}$$

An almost para contact metric manifold is called a para cosymplectic manifold [2] if

$$(\overline{\nabla}_X F)Y = 0. \tag{4}$$

An almost para contact metric manifold is called nearly para cosymplectic if F is a killing, that is,

$$(\overline{\nabla}_X F)Y + (\overline{\nabla}_Y F)X = 0, \tag{5}$$

where $\overline{\nabla}$ is the operator of covariant differentiation with respect to g. On nearly para cosymplectic manifold, U is a killing vector field. That is,

$$g(\overline{\nabla}_X U, Y) + g(\overline{\nabla}_Y U, X) = 0, \forall, X, Y \in T\overline{V}. \tag{6}$$

An almost para contact metric manifold is called closely para cosymplectic if F is a killing and u is a closed. On closely para cosymplectic manifold we have

$$\overline{\nabla}_U F = 0, \overline{\nabla} U = 0, \overline{\nabla} u = 0. \tag{7}$$

Let V be a submanifold of a Riemannian manifold $\overline{V}$ with a Riemannian metric g. Then Gauss and Wiengarten formulae are given respectively by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), X, Y \in TV, \tag{8}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, N \in T^{\perp} V, \tag{9}$$

where $\overline{\nabla}$, ∇ and $\nabla^{\perp}$ are Riemannian, induced Riemannian and induced normal connections in $\overline{V}$, V and the normal bundle $T^{\perp}V$ of V respectively and h is the second fundamental form related to A by

$$g(h(X, Y), N) = g(A_N X, Y). \tag{10}$$

F is a $(1, 1)$ tensor field on V, for $X \in TV$ and $N \in T^{\perp}V$ we have [5]

$$\begin{aligned} (11)\quad & (\overline{\nabla}_X F)Y \\ &= ((\nabla_X P)Y - A_{QY}Y - th(X, Y)) + ((\nabla_X Q)Y + h(X, PY) - fh(X, Y)), \end{aligned}$$

$$\begin{aligned} (12)\quad & (\overline{\nabla}_X F)N \\ &= ((\nabla_X t)N - A_{FN}Y - PA_N X) + ((\nabla_X f)N + h(X, tN) + h(X, tN)), \end{aligned}$$

where,

$$FX \equiv PX + QX, PX \in TV, QX \in T^{\perp}V, \tag{13}$$

$$FN \equiv tN + fN, tN \in TV, fN \in T^{\perp}V, \tag{14}$$

where PX and tN are tangential parts, while QX and fN are normal parts of FX and FN respectively

$$(\nabla_X P)Y \equiv \nabla_X PY - P\nabla_X Y, \tag{15}$$

$$(\nabla_X Q)Y \equiv \nabla^{\perp}_X QY - Q\nabla_X Y, \tag{16}$$

$$(\nabla_X t)N \equiv \nabla_X tN - t\nabla^{\perp}_X N, \tag{17}$$

$$(\nabla_X f)N \equiv \nabla^{\perp}_X fN - f\nabla^{\perp}_X N. \tag{18}$$

The submanifolds V is said to be totally geodesic in $\overline{V}$ if $h = 0$ and totally umbilical in $\overline{V}$ if

$$h(X,Y) = g(X,Y)K.$$

For a distribution D on V, V is said to be D-totally geodesic if $h(X,Y) = 0$, $\forall X, Y \in D$. V is said to be D-totally umbilical if we have $h(X,Y) = h(X,Y)K$, where K is a normal vector field $\forall X, Y \in D$. V is said to be (D,E)-mixed totally geodesic if $h(X,Y) = 0$, $\forall X \in D$ and $YinE$.

Let D and E be two distributions defined on a manifold V. D is said to be E-parallel if we have $\nabla_X Y \in D$, $\forall X \in E$ and $Y \in D$. If D is D-parallel then it is called autoparallel. D is said to be X-parallel if we have $\nabla_X Y \in D$, $\forall X \in TV$ and $Y \in D$. D is said to be parallel if $\forall X \in TV$ and $Y \in D$, $\nabla_X Y \in D$.

If a distribution D on V is autoparallel then it is integrable, and by Gauss formula D is totally geodesic in V. If D is parallel then the orthogonal complementary distribution $D^{\perp}$ is also parallel.

A submanifold V of an almost para contact metric manifold $\overline{V}$ with $U \in TV$ is called a CR-submanifold of $\overline{V}$ if for each $x \in V$, $T_x V = D_x^1 \oplus D_x^0 \oplus \{U\}_x$, where,

$$D_x^1 = \mathrm{Ker}(Q|_{\{U\}^{\perp}})_x = \{X_x \in \{U\}_{x^{\perp}} : \|X_x\| = \|PX_x\|\} = T_x V \cap F(T_x V),$$
$$D_x^0 = \mathrm{Ker}(Q|_{\{U\}^{\perp}})_x = \{X_x \in \{U\}_{x^{\perp}} : \|X_x\| = \|QX_x\|\} = T_x V \cap F(T_x^{\perp} V).$$

The condition $T_x V = D_x^1 \oplus D_x^0 \oplus \{U\}_x$ implies that $P^3 - P = 0$ [6] on V and hence $\mathrm{Dim}(D_x^1) = \mathrm{Rank}(P_x)$ is independent of $x \in V$ and so is that D_x^0.

Now we have $TV = D^1 \oplus D^0 \oplus \{U\}$. These distributions are also differentiable. We have $T^{\perp} V = \overline{D}^1 \oplus \overline{D}^0$, where $\overline{D}^1 = \mathrm{Ker}(t) = T^{\perp} V \cap F(T^{\perp} V)$, $\overline{D}^0 = \mathrm{Ker}(f) = T^{\perp} V \cap F(TV)$, $QD^0 = \overline{D}^0$ and $t\overline{D}^0 = D^0$.

A CR-submanifold of an almost para contact manifold reduces to invariant submanifold [1, 8] (resp. anti-invariant submanifold [1, 8]) if $D^0 = \{0\}$ (resp. $D^1\{0\}$).

3. Some results

Let V be a submanifold of a nearly para cosymplectic manifold, tangent to U. By virtue of equation (5) and equation (11) we have

$$\begin{aligned}(19)\quad &((\nabla_X P)Y + (\nabla_Y P)X - A_{QY}X - A_{QX}Y - 2th(X,Y)) \\ &+((\nabla_X Q)Y + (\nabla_Y Q)X + h(X,PY) + h(PX,Y) - 2fh(X,Y)) = 0.\end{aligned}$$

PROPOSITION 3.1. *Let V be a submanifold of a nearly para cosymplectic manifold. If $U \in TV$ then $\forall X, Y \in TV$ we have*

$$(20)\qquad (\nabla_X P)Y + (\nabla_Y P)X - A_{QY}X - A_{QX}Y - 2th(X,Y) = 0,$$

(21) $(\nabla_X Q)Y + (\nabla_Y Q)X + h(X, PY) + h(PX, Y) - 2fh(X, Y) = 0.$

Proof. Equating tengential and normal parts of equation (19), we get the desired results

PROPOSITION 3.2. *Let V be a submanifold of a nearly para cosymplectic manifold. If $U \in TV$ then $\forall X, Y \in TV$ we have*

(22) $\overline{\nabla}_X FY + \overline{\nabla}_Y FX - F[X, Y]$
$= 2((\nabla_X P)Y - A_{QY}Y - th(X, Y)) + 2((\nabla_X Q)Y + h(X, PY) - fh(X, Y)).$

The proof is obvious and hence omitted.

THEOREM 3.3. *Let V be a submanifold of a nearly para cosymplectic manifold. If $U \in TV$ then $\forall X, Y \in TV$ we get*

(23) $P[X, Y] = -\nabla_X PY - \nabla_Y PX + A_{QY}X + A_{QX}Y + 2P\nabla_X Y + 2th(X, Y),$

(24) $Q[X, Y] = -\nabla^{\perp}_X QY - \nabla^{\perp}_Y QX - h(X, PY) - h(PX, Y)$
$+2Q\nabla_X Y + 2fh(X, Y).$

Proof. By virtue of equation (11) and (22) we get

$$(\nabla_X PY - P\nabla_X Y - \nabla_Y PX + P\nabla_Y X - A_{QY}X + A_{QX}Y - 2\nabla_X PY + 2P\nabla_X Y + 2A_{QY}X + 2th(X, Y)) + (\nabla^{\perp}_X QY - Q\nabla_X Y - \nabla^{\perp}_Y QX - Q\nabla_Y X + h(X, PY) - h(PX, Y) - 2\nabla^{\perp}_X QY + 2Q\nabla_X Y - 2h(X, PY) + 2fh(X, Y)) = 0.$$

Now equating tangential and normal parts of the above equation we get equation (23) and (24).

PROPOSITION 3.4. *Let V be a submanifold of a nearly para cosymplectic manifold. Then (P, U, u, g) is a nearly para cosymplectic structure on the distribution $D^1 \oplus \{U\}$ if $th(X, Y) = 0 \forall X, Y \in D^1 \oplus \{U\}$.*

Proof. Using $D^1 \oplus \{U\} = \mathrm{Ker}(Q)$ and $P^2 + tQ = I - u \otimes U$ we obtain $P^2 = I - u \otimes U$ on $D^1 \oplus \{U\}$. We also get $PU = 0$, $u(U) = 1$, $uoP = 0$. Using $D^1 \oplus \{U\} = \mathrm{Ker}(Q)$ and $th(X, Y) = 0$ in equation (20) we have $(\nabla_X P)Y + (\nabla_Y P)X = 0, \forall X, Y \in D^1 \oplus \{U\}$.

This proves our assertion.

THEOREM 3.5. *Let V be a CR-submanifold of a nearly para cosymplectic manifold. We have*

(a) *if $D^0 \oplus \{U\}$ is autoparallel then*

$$A_{QY}X + A_{QX}Y + 2th(X, Y) = 0, \forall X, Y \in D^0 \oplus \{U\},$$

(b) *if $D^1 \oplus \{U\}$ is autoparallel then*

$$h(X, PY) + h(PX, Y) = 2fh(X, Y), \forall X, Y \in D^1 \oplus \{U\}.$$

Proof. Using equation (20) and autoparallelness of $D^0 \oplus \{U\}$ we get (a) and using equation (21) and autoparallelness of $D^1 \oplus \{U\}$ we get (b). This completes the proof.

THEOREM 3.6. *Let V be a submanifold of a nearly para cosymplectic manifold with $U \in TV$. If V is invariant then V is nearly para cosymplectic manifold. Moreover*

$$h(X, PY) + h(PX, Y) - 2fh(X, Y) = 0, \forall X, Y \in TV. \tag{25}$$

Proof. From $D^1 \oplus \{U\} = \mathrm{Ker}(Q)$ and equation (21) we get equation (25).

4. Intigrability conditions

LEMMA 4.1. *Let V be a CR-submanifold of a nearly para cosymplectic manifold, $\forall X, Y \in D^1 \oplus \{U\}$ we get*

$$Q[X, Y] = -h(X, PY) - h(PX, Y) + 2Q\nabla_X Y + 2fh(X, Y) \tag{26}$$

or equivalently

$$-h(X, PX) + Q\nabla_X X + fh(X, X) = 0. \tag{27}$$

Proof. Using $D^1 \oplus \{U\} = \mathrm{Ker}(Q)$ and equation (24) we get equation (26) and using $X = Y$ in equation (26) we get the required result.

THEOREM 4.2. *The distribution $D^1 \oplus \{U\}$ on a CR-submanifold of a nearly para cosymplectic manifold is integrable if and only if*

$$h(X, PY) + h(PX, Y) = 2(Q\nabla_X Y + fh(X, Y)). \tag{28}$$

Proof. From $D^1 \oplus \{U\} = \mathrm{Ker}(Q)$ and using equation (26) we get the result.

DEFINITION 4.3. Let V be a Remannian manifold with a Riemannian connection ∇. A distribution D on V is said to be nearly autoparallel if $\forall X, Y \in D$ we have $(\nabla_X Y + \nabla_Y X) \in D$ or equivalently $\nabla_X X \in D$.

We have
Parallel ⇒ Autoparallel ⇒ Nearly autoparallel,
Parallel ⇒ Integrable,
Autoparallel ⇒ Integable, and
Nearly autoparallel + Integable ⇒ Autoparallel.

THEOREM 4.4. *Let V be a CR-submanifold of a nearly para cosymplectic manifold. Then the following statements*:

(I) *the distribution $D^1 \oplus \{U\}$ is autoparallel,*

(II) $h(X, PY) + h(PX, Y) = 2fh(X, Y), X, Y \in D^1 \oplus \{U\}$,

(III) $h(X, PX) = fh(X, X), X \in D^1 \oplus \{U\}$,

(IV) *the distribution $D^1 \oplus \{U\}$ is nearly autoparallel, are related by* (I)⇒(II)⇔(III)⇒(IV). *In particular if $D^1 \oplus \{U\}$ is integrable then the above four statements are equivalent.*

Proof. (I)$\Rightarrow$(II) follows from Theorem (3.5)(b). Putting $X = Y$ in (II) we get (II)$\Leftrightarrow$(III). From (27) we get (III)$\Rightarrow$(IV). This completes the proof of the theorem.

THEOREM 4.5. *Let V be a CR-submanifold of a nearly para cosymplectic manifold, such that V is $D^1 \oplus \{U\}$—totally umbilical, then*

(I) *the distribution $D^1 \oplus \{U\}$ is nearly autoparallel.*

Consequently, the following two statements becomes equivalent:

(II) *the distribution $D^1 \oplus \{U\}$ is integable,*

(III) *the distribution $D^1 \oplus \{U\}$ is autoparallel.*

Proof. From proposition (5.2), V is $D^1 \oplus \{U\}$—totally geodesic that is $h = 0$. Thus from (27) we get $\nabla_X X = 0$ the statement (I) holds. Hence from definition (4.3) we get (II)$\Leftrightarrow$(III).

COROLLARY 4.6. *In a totally umbilical CR-submanifold of a nearly para cosymplectic manifold, $D^1 \oplus \{U\}$ is autoparallel.*

Proof. Using Theorem (4.5) we get the result.

LEMMA 4.7. *Let V be a CR-submanifold of a nearly para cosymplectic manifold. Then*

$$3(A_{QX}Y - A_{QY}X) = P[X, Y], X, Y \in D^0 \oplus \{U\}. \tag{29}$$

Proof. Let $X, Y \in D^0 \oplus \{U\}$ and $Z \in TV$. we have from equation (8) and (9)

$$\begin{aligned} A_{FX}Z + \nabla_Z^{\perp}FX = \overline{\nabla}_Z FX &= (\overline{\nabla}_Z F)X + F\overline{\nabla}_Z X \\ &= -(\overline{\nabla}_X F)Z + F\nabla_Z X + Fh(Z, X) \end{aligned}$$

so that,

$$Fh(Z, X) = -A_{FX}Z + \nabla_Z^{\perp}FX - F\nabla_Z X + (\overline{\nabla}_X F)Z,$$

and hence

$$\begin{aligned} g(Fh(Z, X), Y) &= -g(A_{FX}Z, Y) + g((\overline{\nabla}_X F)Z, Y) \\ &= g(A_{FX}Y, Z) + g((\overline{\nabla}_X F)Y, Z). \end{aligned}$$

Now we have

$$g(Fh(Z, X), Y) = g(h(Z, X), FY) = g(A_{FY}X, Z).$$

Thus from the above two equations, we have

$$g(A_{FY}X, Z) = g(A_{FX}Y, Z) + g((\overline{\nabla}_X F)Y, Z). \tag{30}$$

Now for $X, Y \in D^0 \oplus \{U\}$, we have

$$\overline{\nabla}_X FY - \overline{\nabla}_Y FX = A_{FX}Y - A_{FY}X + \nabla_X^{\perp}FY - \nabla_Y^{\perp}FX$$

and $\overline{\nabla}_X FY - \overline{\nabla}_Y FX = (\overline{\nabla}_X F)Y - (\overline{\nabla}_Y F)X + F[X,Y]$ from above two equations, we have

$$(\overline{\nabla}_X F)Y - (\overline{\nabla}_Y F)X = A_{FX}Y - A_{FY}X + \nabla^{\perp}_X FY - \nabla^{\perp}_Y FX - F[X,Y].$$

Using equation (5) and above equation, we get

$$(\overline{\nabla}_X F)Y = \frac{1}{2}(A_{FX}Y - A_{FY}X + \nabla^{\perp}_X FY - \nabla^{\perp}_Y FX - F[X,Y]).$$

From this and equation (30) we get the required result.

THEOREM 4.8. *Let V be a CR-submanifold of a nearly para cosymplectic manifold. Then the distribution $D^0 \oplus \{U\}$ is integrable if and only if*

$$A_{QX}Y - A_{QY}X = 0, \quad \forall X, Y \in D^0 \oplus \{U\}.$$

Proof. From $D^0 \oplus \{U\} = \mathrm{Ker}(P)$ and equation (29), we get the result.

THEOREM 4.9. *Let V be a CR-submanifold of a nearly para cosymplectic manifold. Then the distribution D^0 is integrable if and only if*

$$A_{QX}Y - A_{QY}X = 0, \forall X, Y \in D^0.$$

Proof. By definition of D^0 and equation (29), we get the result.

THEOREM 4.10. *Let V be a CR-submanifold of a para cosymplectic manifold. Then the distributions D^0 and $D^0 \oplus \{U\}$ are integrable.*

Proof. The result follows from Theorem (4.8) and Theorem (4.9).

5. Totally umbilical and totally geodesic submanifolds

LEMMA 5.1. *Let V be a submanifold of a closely para cosymplectic manifold, tangent to U. Then the integral curve of U in V is geodesic in V, and U is an asymptotic direction.*

Proof. Since in a closely para cosymplectic manifold we have $\overline{\nabla} U = 0$. Now in view of equation (8), we get $h(U,U) = 0$. This completes the proof.

PROPOSITION 5.2. *Let D be a distribution on a submanifold V of a closely para cosymplectic manifold such that $U \in TV$. If V is D-totally umbilical then V is D-totally geodesic.*

Proof. For D-totally umbilical we have

$$h(X,Y) = g(X,Y)K, \forall X, Y \in D.$$

A direction U at a point of V is an asymptotic direction if normal vector field $K = 0$, which implies that $h(X,Y) = 0$, which shows that V is D-totally geodesic.

PROPOSITION 5.3. *Every totally umbilical submanifold of a closely para cosymplectic manifold, tangent to U, is totally geodesic.*

Proof. The proof follows from proposition (5.2).

6. Totally para contact umbilical and totally para contact geodesic submanifolds

Let V be a submanifold of an almost para contact metric manifolds, tangent to U. In this case $TV = \{U\} \oplus \{U\}^{\perp}$, where $\{U\}$ is the distribution spanded by $\{U\}$ and $\{U\}^{\perp}$ is the complementary orthogonal distribution of $\{U\}$ in V.

DEFINITION 6.1. A submanifold V of an almost para contact metric manifold, tangent to U, is called

(1) totally para contact umbilical if it is $\{U\}^{\perp}$-totally umbilical, and

(2) totally para contact geodesic if it is $\{U\}^{\perp}$-totally geodesic.

The condition of totally para contact umbilical and totally para contact geodesic are respectively

$$h(F^2X, F^2Y) = g(F^2X, F^2Y)K, X, Y \in TV, \tag{31}$$

$$h(F^2X, F^2Y) = 0, X, Y \in TV, \tag{32}$$

where K is normal vector field. Using (1) in (31) and (32), the following respectively.

$$h(X, Y) = g(FX, FY)K + u(X)h(Y, U) + u(Y)h(X, U), \tag{33}$$

$$h(X, Y) = u(X)h(Y, U) + u(Y)h(X, U). \tag{34}$$

THEOREM 6.2. *If V is a totally para contact umbilical CR-submanifold of a closely para cosymplectic manifold, then V is (D^1, D^0)-mixed totally geodesic.*

Proof. Now we have $h(X, Y) = g(X, Y)K$, for $X, Y \in \{U\}^{\perp}$ $h(U, U) = g(U, U)K$ $0 = g(U, U)K$, by using Gauss equation $\Rightarrow K = 0$. Therefore V is (D^1, D^0)-mixed totally geodesic. This completes our assertions.

THEOREM 6.3. *Let V be a totally para contact umbilical submanifold of a closely para cosymplectic manifold, then either $D^0 = \{0\}$ or $\mathrm{Dim}(D^0) = 1$ or the normal vector field K is orthogonal to FD^0.*

Proof. If $\mathrm{Dim}(D^0) > 1$, for each $H \in D^0 \exists X \in D^0$ such that $g(X, H) = 0$ and $\|X\| = 0$, then

$$\begin{aligned} g(K, FH) &= g(h(X, X), FH) = g(A_{FH}X, X) \\ &= g(A_{FX}H, X) = g(h(X, H), FX) = 0. \end{aligned}$$

This gives desired result.

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