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# GEOMETRY OF $t$-PROJECTIVE STRUCTURES

**Abstract.** The main purpose of this paper is to define and study t-projective structure on even dimensional manifold $M$ as a certain reduction of the second order frame bundle over $M$. This t-structure reveals some similarities to the projective structures of Kobayashi-Nagano [KN1] but it is a completely different one. The structure group is the isotropy group of the tangent bundle of the projective space. With the t-structure we associate in a natural way the so called normal Cartan connection and we investigate its properties. We show that t-structures are closely related the almost tangent structures on $M$. Finally, we consider the natural cross sections and we derive the coefficients of the normal connection of a t-projective structure.

## 0. Introduction

Grassmannians of higher order appeared for the first time in a paper [Sz2] in the context of the Cartan method of moving frames. Recently, A. Szybiak has given in [Sz3] an explicit formula for the infinitesimal action of the second order jet group in dimension $n$ on the standard fiber of the bundle of second order grassmannians on an $n$-dimensional manifold.

In the present paper we introduce an another notion of a grassmannian of higher order in the case of a projective space which is in the natural way a homogeneous space. It is well-known that many interesting geometric structures can be obtained as structures locally modeled on homogeneous spaces. Interesting general approach to the Cartan geometries is developed in the book by R. W. Sharpe [Sh]. From the other hand T. Morimoto in his important paper [Mor] suggested the general universal procedure based on the theory of filtered manifolds and gave the general criterion to construct a Cartan connection associated with a geometric structure. However, we don't apply here his general method but we present a concrete example which is a good geometrical case in which we can construct a normal Cartan

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connection by direct calculations without relying on a rather big machinery. The detailed analysis of our particular case leads to deeper geometric results concerning the almost tangent structures. Other important geometric structures investigated with the use of Cartan connections can be found in [KN1], [Og], [Mi1], [Mi2], [Dh], [Moz2], [Ta], [Och], [Ya].

For the additional informations concerning the considered subject see [AG1], [AG2], [AG3], [Go], [Moz1], [Sz3], [MM], [Ru].

For the basic notions and notations in the jet theory see the book by I. Kolař, P. Michor, J. Slovák [KMS].

## 1. The projective space of first order

Let $\mathrm{T}^2_1(\mathbb{R}^{n+1}) = \mathrm{Reg}^2_0(\mathbb{R}^1, \mathbb{R}^{n+1})_0$, where $\mathrm{Reg}^2_0(\mathbb{R}^1, \mathbb{R}^{n+1})_0$ denotes the set of all regular jets of second order of mappings $\mathbb{R}^1 \longrightarrow \mathbb{R}^{n+1}$ of the source and target at 0. The manifold $\mathrm{T}^2_1(\mathbb{R}^{n+1})$ is called a Stiefel manifold of 1-frames of second order on $\mathbb{R}^{n+1}$ at 0. Note that the Stiefel manifold of 1-frames of first order is an ordinary Stiefel manifold (cf. [Sz2]).

Let $G = \{(a,b) : a, b \in \mathbb{R}, a \neq 0\}$ be a group with the following multiplication rule

$$(a,b) \cdot (c,d) = (ac, ad + bc).$$

The action of the group $G$ on $\mathrm{T}^2_1(\mathbb{R}^{n+1})$ is given by

$$(\tilde{x}, \tilde{y}) \cdot (a,b) = (a\tilde{x}, a\tilde{y} + b\tilde{x}),$$

where $(\tilde{x}, \tilde{y})$ are the coordinates of the jet $j^2_0 f \in \mathrm{T}^2_1(\mathbb{R}^{n+1})$. We then set $P^n_1 = \mathrm{T}^2_1(\mathbb{R}^{n+1})/G$ and call this space the projective space of first order.

On the other hand we consider a matrix group

$$\mathbb{G} = \{(A,B) : A \in \mathrm{GL}(n+1), B \in M(n+1)\},$$

where $M(n+1)$ denotes the set of all $(n+1) \times (n+1)$ real matrices and a multiplication is defined as follows

$$(A,B) \cdot (C,D) = (AC, BC + AD).$$

The group $\mathbb{G}$ is isomorphic to the subgroup of the group $\mathrm{GL}(2n+2)$ of matrices of the form

$$\begin{bmatrix} A & B \\ 0 & A \end{bmatrix},$$

where $\det A \neq 0$. The group $\mathbb{G}$ acts on $\mathrm{T}^2_1(\mathbb{R}^{n+1})$ by the formula

$$(A,B) \cdot (\tilde{x}, \tilde{y}) = (A\tilde{x}, A\tilde{y} + B\tilde{x}).$$

It is easy to see that this action factorises to the action on $P^n_1$.

Let's introduce the inhomogeneous coordinates on $P^n_1$. Let $(\tilde{x}, \tilde{y}) \in \mathrm{T}^2_1(\mathbb{R}^{n+1})$ and $(\tilde{x}, \tilde{y}) = (\tilde{x}^\alpha, \tilde{y}^\alpha)$, where $\alpha, \beta = 0, \dots n$. If $\tilde{x}^0 \neq 0$ then we set $a = \frac{1}{\tilde{x}^0}, b = -\frac{\tilde{y}^0}{(\tilde{x}^0)^2}$ and we get

$$(\tilde{x}, \tilde{y}) = (\tilde{x}^\alpha, \tilde{y}^\alpha) = \left(1, \frac{\tilde{x}^i}{\tilde{x}^0}, 0, \frac{\tilde{x}^0\tilde{y}^i - \tilde{y}^0\tilde{x}^i}{(\tilde{x}^0)^2}\right),$$

where $i = 1, \dots, n$. We call $x^i = \frac{\tilde{x}^i}{\tilde{x}^0}$, $y^i = \frac{\tilde{x}^0\tilde{y}^i - \tilde{y}^0\tilde{x}^i}{(\tilde{x}^0)^2}$ the inhomogeneous coordinate system of $P^n_1$. Note that the elements $(A, B)$ and $(aA, bA + aB)$, $a \neq 0$ in $\mathbb{G}$ induces the same transformation of $P^n_1$. Identifying such elements we get a group $\mathfrak{P}^1_n$. Now, we can define the inhomogeneous coordinates in $\mathfrak{P}^n_1$. If $A^0_0 \neq 0$ then

$$(A, B) = \left(A^\alpha_\beta, B^\alpha_\beta\right) = \left(\begin{bmatrix} 1 & \frac{A^0_i}{A^0_0} \\ \frac{A^i_0}{A^0_0} & \frac{A^i_j}{A^0_0} \end{bmatrix}, \begin{bmatrix} 0 & \frac{A^0_0 B^0_j - B^0_0 A^0_j}{(A^0_0)^2} \\ \frac{A^0_0 B^i_0 - B^0_0 A^i_0}{(A^0_0)^2} & \frac{A^0_0 B^i_j - B^0_0 A^i_j}{(A^0_0)^2} \end{bmatrix}\right).$$

We set

$$a_i = \frac{A^0_i}{A^0_0}, a^i = \frac{A^i_0}{A^0_0}, a^i_j = \frac{A^i_j}{A^0_0},$$
$$b_i = \frac{A^0_0 B^0_i - B^0_0 A^0_i}{(A^0_0)^2}, b^i = \frac{A^0_0 B^i_0 - B^0_0 A^i_0}{(A^0_0)^2}, b^i_j = \frac{A^0_0 B^i_j - B^0_0 A^i_j}{(A^0_0)^2}.$$

We call the above the inhomogeneous coordinate system in the neighborhood of the identity of $\mathfrak{P}^n_1$ defined by $A^0_0 \neq 0$. The induced action of $\mathfrak{P}^n_1$ on $P^n_1$ in terms of the introduced inhomogeneous coordinate system is given by the following formulas

$$\left(\left(\begin{bmatrix} 1 & a_i \\ a^i & a^i_j \end{bmatrix}, \begin{bmatrix} 0 & b_i \\ b^i & b^i_j \end{bmatrix}\right), (x^i, y^i)\right) \mapsto \left(\frac{a^i + a^i_j x^j}{1 + a_j x^j},\right.$$
$$\left.\frac{b^i + b^i_j x^j + a^i_j y^j - a_j a^i y^j + (a_j b^i_k - a^i_k b_j) x^j x^k + (a_k a^i_j - a_j a^i_k) x^k y^j}{(1 + a_j x^j)^2}\right).$$

## 2. The Maurer-Cartan equations

We are going to derive the structure equations of $\mathbb{G}$. The identity of $\mathbb{G}$ is given by $(I, 0)$ and

$$(A, B)^{-1} = \left(A^{-1}, -A^{-1}BA^{-1}\right).$$

If we set

$$(\bar{\omega}, \bar{\eta}) = (A, B)^{-1}(dA, dB)$$

then

$$\bar{\omega} = A^{-1}dA, \quad \bar{\eta} = A^{-1}dB - A^{-1}BA^{-1}dA.$$

From the above formula we get

LEMMA 2.1. (the structure equations of $\mathbb{G}$). *If* $(\bar{\omega},\bar{\eta}) = \left(\overline{\omega}^{\alpha}_{\beta}, \bar{\eta}^{\alpha}_{\beta}\right)$, $\alpha,\beta = 0,\dots,n$ *then*

$$d\bar{\omega}^{\alpha}_{\beta} = -\bar{\omega}^{\alpha}_{\mu} \wedge \bar{\omega}^{\mu}_{\beta} \tag{2.1}$$

$$d\bar{\eta}^{\alpha}_{\beta} = -\bar{\omega}^{\alpha}_{\mu} \wedge \bar{\eta}^{\mu}_{\beta} - \bar{\eta}^{\alpha}_{\mu} \wedge \bar{\omega}^{\mu}_{\beta}. \tag{2.2}$$

We shall find the structure equations of $\mathfrak{P}^n_1$. Let $\omega^i, \omega^i_j, \omega_j, \eta^i, \eta^i_j, \eta_j$ be the left invariant 1-forms on $\mathfrak{P}^n_1$ such that

$$\omega^i = da^i, \quad \omega^i_j = da^i_j, \quad \omega_j = da_j,$$
$$\eta^i = db^i, \quad \eta^i_j = db^i_j, \quad \eta_j = db_j$$

at the identity. From the definition of the inhomogeneous coordinates in $\mathfrak{P}^n_1$ we have at the identity

$$\omega^i = da^i = dA^i_0,\ \omega^i_j = da^i_j = dA^i_j - \delta^i_j dA^0_0,\ \omega_j = da_j = dA^0_j,$$
$$\eta^i = db^i = dB^i_0,\ \eta^i_j = db^i_j = dB^i_j - \delta^i_j dB^0_0,\ \eta_j = db_j = db^0_j.$$

Hence we have

THEOREM 2.1. *The Maurer-Cartan equations of* $\mathfrak{P}^n_1$ are

$$\begin{aligned}
d\omega^i &= -\omega^i_t \wedge \omega^t,\\
d\omega^i_j &= -\omega^i \wedge \omega_j - \omega^i_t \wedge \omega^t_j + \delta^i_j \omega_t \wedge \omega^t,\\
d\omega_j &= -\omega_t \wedge \omega^t_j,\\
d\eta^i &= \omega^t \wedge \eta^i_t + \eta^t \wedge \omega^i_t,\\
d\eta^i_j &= -\omega^i \wedge \eta_j - \eta^i \wedge \omega_j + \delta^i_j \eta_t \wedge \omega^t - \omega^i_t \wedge \eta^t_j - \eta^i_t \wedge \omega^t_j,\\
d\eta_j &= -\eta_t \wedge \omega^t_j - \omega_t \wedge \eta^t_j.
\end{aligned} \tag{2.3}$$

## 3. Cartan t-projective connections

Let $M$ be a manifold, dim $M = 2n$. We consider a principal fiber bundle $P$ over $M$ with a structure group $H$ - the isotropy group of $(0,\dots,0)$ with the a Cartan connection $\omega$ with values in the Lie algebra $\mathfrak{p}^n_1$ of $\mathfrak{P}^n_1$. With respect to the natural basis in the Lie algebra $\mathfrak{p}^n_1$ the form $\omega$ is given by the set of 1-forms

$$\omega = \left(\omega^i, \omega^i_j, \omega_j, \eta^i, \eta^i_j, \eta_j\right).$$

Such connection will be called a Cartan t-projective connection. The structure equations of the Cartan t-projective connection $\omega$ are given by

$$
(3.1)\quad
\begin{aligned}
d\omega^i &= -\omega^i_t \wedge \omega^t + \Omega^i,\\
d\omega^i_j &= -\omega^i \wedge \omega_j - \omega^i_t \wedge \omega^t_j + \delta^i_j \omega_t \wedge \omega^t + \Omega^i_j,\\
d\omega_j &= -\omega_t \wedge \omega^t_j + \Omega_j,\\
d\eta^i &= \omega^t \wedge \eta^i_t + \eta^t \wedge \omega^i_t + H^i,\\
d\eta^i_j &= -\omega^i \wedge \eta_j - \eta^i \wedge \omega_j + \delta^i_j \eta_t \wedge \omega^t - \omega^i_t \wedge \eta^t_j - \eta^i_t \wedge \omega^t_j + H^i_j,\\
d\eta_j &= -\eta_t \wedge \omega^t_j - \omega_t \wedge \eta^t_j + H_j.
\end{aligned}
$$

For the sake of simplicity we shall take these equations as a definition of the 2-forms $\Omega^i, \Omega^i_j, \Omega_j, H^i, H^i_j, H_j$. Each element $\tilde{A}$ of the Lie algebra $h$ of $H$ induces on $P$ a vector field $A^*$. With each element $(\xi, \gamma) = (\xi^1, \dots, \xi^n, \gamma^1, \dots, \gamma^n) \in \mathbb{R}^{2n}$ we can associate a unique vector field $D(\xi, \gamma)$ on $P$ with the following properties:

$$
\begin{aligned}
&\omega^i(D(\xi,\gamma)) = \xi^i, \quad \omega^i_j(D(\xi,\gamma)) = 0, \quad \omega_j(D(\xi,\gamma)) = 0,\\
&\eta^i(D(\xi,\gamma)) = \gamma^i, \quad \eta^i_j(D(\xi,\gamma)) = 0, \quad \eta_j(D(\xi,\gamma)) = 0.
\end{aligned}
$$

By a lengthy but straightforward calculations we obtain

LEMMA 3.1.

$$
[A^*, D(\xi,\gamma)] = D(A\xi, A\gamma + B\xi) - (\xi a + a\xi I, 0, \gamma a + \xi b + b\xi I + a\gamma I, 0).
$$

With this preparation we are now in the position to state

THEOREM 3.2.

$$
(3.2)\quad
\begin{aligned}
\Omega^i &= \frac{1}{2}K^i_{kl}\omega^k \wedge \omega^l + \frac{1}{2}L^i_{kl}\omega^k \wedge \eta^l + \frac{1}{2}M^i_{kl}\eta^k \wedge \eta^l\\
\Omega^i_j &= \frac{1}{2}K^i_{jkl}\omega^k \wedge \omega^l + \frac{1}{2}L^i_{jkl}\omega^k \wedge \eta^l + \frac{1}{2}M^i_{jkl}\eta^k \wedge \eta^l\\
\Omega_j &= \frac{1}{2}K_{jkl}\omega^k \wedge \omega^l + \frac{1}{2}L_{jkl}\omega^k \wedge \eta^l + \frac{1}{2}M_{jkl}\eta^k \wedge \eta^l\\
H^i &= \frac{1}{2}N^i_{kl}\omega^k \wedge \omega^l + \frac{1}{2}G^i_{kl}\omega^k \wedge \eta^l + \frac{1}{2}B^i_{kl}\eta^k \wedge \eta^l\\
H^i_j &= \frac{1}{2}N^i_{jkl}\omega^k \wedge \omega^l + \frac{1}{2}G^i_{jkl}\omega^k \wedge \eta^l + \frac{1}{2}B^i_{jkl}\eta^k \wedge \eta^l\\
H_j &= \frac{1}{2}N_{jkl}\omega^k \wedge \omega^l + \frac{1}{2}G_{jkl}\omega^k \wedge \eta^l + \frac{1}{2}B_{jkl}\eta^k \wedge \eta^l.
\end{aligned}
$$

We shall consider the situation whether we can find a Cartan connection in $P$ when the forms $\omega^i, \omega^i_j, \eta^i, \eta^i_j$ are given a priori.

THEOREM 3.3. *Suppose that on $P$ are given the forms $\omega^i, \omega^i_j, \eta^i, \eta^i_j$ with values in the Lie algebra $\mathfrak{p}^n_1$ of $\mathfrak{P}^n_1$ satisfying the following conditions:*

(1) *for each* $\tilde{A} \in h$, $\tilde{A} = (A^i_j, a_j, B^i_j, b_j)$

$$\omega^i(A^*) = 0, \omega^i_j(A^*) = A^i_j, \eta(A^*) = 0, \eta^i_j(A^*) = B^i_j$$

(2) $R^*_a(\omega^i, \omega^i_j, \eta^i, \eta^i_j) = ad_{a^{-1}}(\omega^i, \omega^i_j, \eta^i, \eta^i_j)$ *for* $a \in H$

(3) *if* $\omega^i(X) = \eta^i(X) = 0, i = 1, \dots, n$ *then* $X$ *is tangent to a fibre*

(4) $d\omega^i = -\omega^i_j \wedge \omega^j, \quad d\eta^i = -\eta^i_j \wedge \omega^j - \omega^i_j \wedge \eta^j$.

*Then there exists a unique Cartan connection* $(\omega^i, \omega^i_j, \omega_j, \eta^i, \eta^i_j, \eta_j)$ *such that*

$$\sum_i K^i_{ikl} = 0, \sum_i K^i_{jil} = 0, \sum_i L^i_{ikl} = 0, \sum_i L^i_{jil} = 0$$

$$\sum_i N^i_{ikl} = 0, \sum_i N^i_{jil} = 0, \sum_i G^i_{ikl} = 0, \sum_i G^i_{kil} = 0.$$

Proof. The following Cartan–Laptev lemma plays an essential role in our proof.

LEMMA 3.2. (cf. [Sz1]). *Suppose that we have a system of p-forms* $\psi_\nu$, $\nu = 1, \dots n$ *and a system of n linearly independent 1-forms* $\sigma^\nu$. *If the identity* $\psi_\nu \wedge \sigma^\nu = 0$ *holds then there exists a system of forms* $\Xi_{\mu\gamma}$ *such that we have*

$$\psi_\nu = \Xi_{\mu\gamma} \wedge \sigma^\nu,$$

$\Xi_{\mu\gamma}$ *being of degree* $p - 1$ *and being symmetric.*

Differentiating the formulas (4) and using the Cartan-Laptev lemma we obtain that the forms $\Omega^i_j$ and $H^i_j$ do not involve the terms of the form $\eta^i \wedge \eta^k$. We shall study the relationship between two Cartan connections $\omega = (\omega^i, \omega^i_j, \omega_j, \eta^i, \eta^i_j, \eta_j)$, $\overline{\omega} = (\omega^i, \omega^i_j, \bar{\omega}_j, \eta^i, \eta^i_j, \bar{\eta}_j)$. We can write

$$\omega_j - \bar{\omega}_j = A_{jk}\omega^k - B_{jk}\eta^k$$
$$\eta_j - \bar{\eta}_j = C_{jk}\omega^k + D_{jk}\eta^k.$$

Reasoning similarly as in [KN1] and [Og] we get the theorem. ∎

Applying the exterior differentiation to the structure equations we obtain

THEOREM 3.4. *Let* $P$ *be a principal fibre bundle over* $M$ *with the group* $H$ *as a structure group. If* $(\omega^i, \omega^i_j, \omega_j, \eta^i, \eta^i_j, \eta_j)$ *is a Cartan connection satisfying the conditions of Theorem 3.2 then*

(1) $\Omega^i_t \wedge \omega^t = 0$

(2) $\omega^i \wedge \Omega_i = 0$

(3) $\omega^t \wedge H^i_t + \eta^t \wedge \Omega^i_t = 0$

(4) $\eta^i \wedge \Omega_i + \omega^i \wedge H_i = 0$.

*If* $\Omega^i_j = 0$, $H^i_j = 0$ *and* $\dim M \geq 6$ *then* $\Omega_j = H_j = 0$.

## 4. t-projective structures

We consider now the isotropy group $H$ of the point $(0,\dots,0)$. Then $a^i = b^i = 0$ and an element

$$\left(\begin{bmatrix} 1 & a_i \\ 0 & a^i_j \end{bmatrix}, \begin{bmatrix} 0 & b_i \\ 0 & b^i_j \end{bmatrix}\right) \in H \tag{4.1}$$

acts on $P^1_n$ as a fractional transformation

$$(x^i, y^i) \mapsto \left(\frac{a^i_j x^j}{1 + a_j x^j}, \frac{b^i_j x^j + a^i_j y^j + (a_j b^i_k - a^i_j b_k) x^j x^k + (a_k a^i_k - a_j a^i_k) x^k y^j}{(1 + a_j x^j)^2}\right). \tag{4.2}$$

Let $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$,

$$f(x,y) = (f^i(x,y), f^{\bar{i}}(x,y)),$$

$\bar{i} = i + n$, be a smooth mapping such that $f(0) = 0$. We put

$$f^i_j = \frac{\partial f^i}{\partial x^j}(0), \quad f^{\bar{i}}_j = \frac{\partial f^{\bar{i}}}{\partial x^j}(0), \quad f^i_{\bar{j}} = \frac{\partial f^i}{\partial y^j}(0), \quad \dots \quad \textit{etc.}$$

Then the second order jet of $f$ has the following coordinates

$$\left(f^i_j, f^i_{\bar{j}}, f^{\bar{i}}_j, f^{\bar{i}}_{\bar{j}}, f^i_{jk}, f^i_{\bar{j}k}, f^i_{\bar{j}\bar{k}}, f^{\bar{i}}_{jk}, f^{\bar{i}}_{\bar{j}k}, f^{\bar{i}}_{\bar{j}\bar{k}}\right).$$

We associate to each element (4.1) of $H$ the second order jet of the fractional transformation (4.2). We get a mapping

$$\begin{aligned}(a_i, a^i_j, b_i, b^i_j) \mapsto (a^i_j, 0, b^i_j, a^i_j, &-a^i_j a_k - a^i_k a_j, \\ &0, 0, -b^i_j a_k - b^i_k a_j - a^i_j b_k - a^i_k b_j, -a_j a^i_k - a_k a^i_j, 0).\end{aligned}$$

Let $P^2(M)$ denotes the bundle of the second order frames over $M$, $\dim M = 2n$ with the structure group $G^2_{2n}$. We then have

THEOREM 4.1. *For each element $a \in H$ let $f$ be the fractional transformation (4.2). Then a mapping $a \mapsto j^2_0 f$ is an isomorphism of $H$ onto its image in $G^2_{2n}$ which shall be also denoted by $H$.*

DEFINITION 4.1. A subbundle $P$ of $P^2(M)$ with the structure group $H$ will be called a t-projective structure on $M$.

Consider the canonical form $T = (T^\alpha, T^\alpha_\beta)$, $\alpha, \beta = 1, 2, \dots, 2n$ on $L^2(M)$. Let

$$T^\alpha = (\vartheta^i, \vartheta^{\bar{i}}), \quad T^\alpha_\beta = \begin{bmatrix} \vartheta^i_j & \vartheta^i_{\bar{j}} \\ \vartheta^{\bar{i}}_j & \vartheta^{\bar{i}}_{\bar{j}} \end{bmatrix} \quad i, j = 1, \dots, n, \quad \bar{i} = i + n, \quad \bar{j} = j + n.$$

If we restrict the canonical form $T$ to the subbundle with the structure group $H$ we obtain

$$T^\alpha_\beta = \begin{bmatrix} \vartheta^i_j & 0 \\ \vartheta^{\bar{i}}_j & \vartheta^i_j \end{bmatrix}.$$

We shall prove

THEOREM 4.2. *For each t-projective structure on $M$ there is a unique t-projective connection $\omega = (\omega^i, \omega^i_j, \omega_j, \eta^i, \eta^i_j, \eta_j)$ such that*

$$\omega^i = \vartheta^i, \quad \eta^i = \vartheta^{\bar{i}}, \quad \omega^i_j = \vartheta^i_j, \quad \eta^i_j = \vartheta^{\bar{i}}_j \tag{4.3}$$

*and*

$$d\omega^i = -\omega^i_j \wedge \omega^j, \quad d\eta^i = -\eta^i_j \wedge \omega^j - \omega^i_j \wedge \eta^j \tag{4.4}$$

$$\sum_i \Omega^i_i = 0, \quad \sum_i H^i_i = 0, \quad \sum_i K^i_{jil} = 0, \quad \sum_i N^i_{jil} = 0. \tag{4.5}$$

Proof. Since

$$\vartheta^i_k = (x^i_t)^{-1} dx^t_k - (x^i_j)^{-1} x^j_{pk} dx^p$$

then $d\vartheta^i_k$ cannot involve the terms of the form $\omega^k \wedge \eta^l$ and $\eta^k \wedge \eta^l$. This means that $L^i_{jkl} = M^i_{jkl} = 0$. Using the local expression of $\vartheta^{\bar{i}}_k$ we show that $G^i_{jkl} = B^i_{jkl} = 0$. Let $j^2_0 f$ corresponds to $a \in H$ under the isomorphism of Theorem 4.1. It easy to see that $ad\ (a^{-1})$ and $ad\ (j^2_0 f^{-1})$ coincide. Then from Theorem 3.2 we get the theorem. ∎

DEFINITION 4.2. The unique t-projective connection for $P$ given by the above theorem we call the normal t-projective connection.

## 5. Linear connections and t-projective connections

Let us consider the isotropy group

$$H = \left( \begin{bmatrix} A & 0 \\ B & A \end{bmatrix}, \begin{bmatrix} C & 0 \\ D & C \end{bmatrix} \right) \subset L^2_{2n}$$

(via the isomorphism from theorem 4.1) and let

$$G_1 = \left\{ \begin{bmatrix} A & 0 \\ B & A \end{bmatrix} : A \in \mathrm{GL}(n), B \in \mathrm{M}(n) \right\}$$
$$= \left\{ \left( \begin{bmatrix} A & 0 \\ B & A \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) : A \in \mathrm{GL}(n), B \in \mathrm{M}(n) \right\} \subset H.$$

Let $P$ be a t-projective structure on $M$. The natural projection of $P^2(M)$ onto $P^1(M)$ restricted to $P$ gives a subbundle $P_1 \subset P^1(M)$. In this way we obtain a $G_1$-structure on $M$ which is known as an almost tangent structure [CG]. Then there exists a $(1,1)$-tensor $F$ such that $F^2 = 0$. From now on we suppose that this structure is integrable ([CG]). It follows that there exists an atlas with respect to which the components of $F$ are constant functions. A linear connection $\nabla$ in $L^1(M)$ is adapted to $G_1$-structure iff $\nabla F = 0$. Let $\varphi^\alpha_\beta$ be the transition function of $P_1$. Then

$$F^\beta_\gamma \varphi^\alpha_\beta = F^\alpha_\beta \varphi^\beta_\gamma, \quad \alpha, \beta, \gamma = 1, \dots, 2n. \tag{5.1}$$

Differentiating (5.1) we obtain

$$F^\beta_\gamma \varphi^\alpha_{\beta\delta} = F^\alpha_\beta \varphi^\beta_{\gamma\delta}.$$

The functions $\{\varphi^\alpha_\beta, \varphi^\alpha_{\beta\gamma}\}$ fulfill the cocycle condition. Then there exists a bundle $P^2_F(M)$ determined by these functions.

THEOREM 5.1.

(1) *The cross sections* $M \to P^2_F(M)/G_1$ *are in one-to-one correspondence with the connections adapted to almost tangent structure* $F$.

(2) *The cross sections* $M \to P^2_F(M)/H$ *are in one-to-one correspondence with the t-projective structures of* $M$.

Proof. Let $(x^\alpha, u^\alpha_\beta, u^\alpha_{\beta\gamma})$ be the coordinates of $P^2_F(M)$. On $P^2_F(M)/G_1$ we introduce the coordinates $(z^\alpha, z^\alpha_{\beta\gamma})$ given by

$$\begin{aligned} z^\alpha &= x^\alpha \\ z^\alpha_{\beta\gamma} &= u^\alpha_{\delta\mu} \left(u^\delta_\beta\right)^{-1} \left(u^\mu_\gamma\right)^{-1}. \end{aligned}$$

Then the cross section $\Gamma : M \to P^2_F(M)/G_1$ is given locally by a set of functions

$$z^\alpha_{\beta\gamma} = -\Gamma^\alpha_{\beta\gamma}, \quad \text{where} \quad \Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}.$$

It easy to see that $\Gamma^\alpha_{\beta\gamma}$ behave under the change of coordinate system as the coefficients of a linear connection. Obviously we have $\nabla F = 0$.

The second assertion is evident. ∎

Composing $\Gamma$ with a projection $P^2_F(M)/G_1 \to P^2_F(M)/H$ we get a t-projective structure (integrable).

DEFINITION 5.1. A linear connection $\Gamma$ adapted to an integrable almost tangent structure is said to belong to t-projective structure $P$ if $\Gamma$ induces $P$ in the manner described above. We say that two such connections are t-projectively related if they belong to the same t-projective structure.

From the action of $H$ on $P^2_F(M)$ it follows that connections $\Gamma$, $\tilde{\Gamma}$ are t-projectively related if

$$
(5.2)\qquad
\begin{aligned}
\tilde{\Gamma}^i_{jk} &= \Gamma^i_{jk} - \delta^i_j a_k - \delta^i_k a_j,\\
\tilde{\Gamma}^i_{\bar{j}k} &= 0 = \Gamma^i_{\bar{j}k},\\
\tilde{\Gamma}^i_{\bar{j}\bar{k}} &= 0 = \Gamma^i_{\bar{j}\bar{k}},\\
\tilde{\Gamma}^{\bar{i}}_{jk} &= \Gamma^{\bar{i}}_{jk} + \delta^i_j b_k + \delta^i_k b_j - \delta^i_j a_k - \delta^i_k a_j,\\
\tilde{\Gamma}^{\bar{i}}_{\bar{j}k} &= \Gamma^{\bar{i}}_{\bar{j}k} - \delta^i_j a_k - \delta^i_k a_j,\\
\tilde{\Gamma}^{\bar{i}}_{\bar{j}\bar{k}} &= 0 = \Gamma^{\bar{i}}_{\bar{j}\bar{k}}.
\end{aligned}
$$

Let $\Gamma : M \to P^2_F(M)/G_1$ be an adapted linear connection. Then we have a reduction of $L^2(M)$ to subbundle with a structure group $G_1$. This gives an isomorphism $\gamma : P_1 \to P^2_F(M)$.

THEOREM 5.2. *Let $\Gamma$ be an adapted linear connection and $\gamma : P_1 \to P^2_F(M)$ the above isomorphism. Let $(T^\alpha, T^\alpha_\beta)$ be the canonical form on $P^2_F(M)$. Then $\gamma^*(T^\alpha)$ is a canonical form on $P_1$ and $\gamma^*(T^\alpha_\beta)$ is an adapted connection to an almost tangent structure.*

The forms $\Omega^i_j$, $H^i_j$ given by (3.1) define a certain tensor on $M$ depending only on the t-projective structure $P$. This tensor is called the t-projective Weyl tensor.

## 6. Natural frames and coefficients of t-projective connections

Let $P$ be a t-projective structure on $M$ and $U$ be a coordinate neighborhood in $M$ with a local coordinate system $(x^1,\dots,x^n,y^1,\dots,y^n)$. let $\sigma : U \to P$ be a cross section and let $U \times H \approx P_{|U}$ be the isomorphism induced by $\sigma$. Let $(a^i_j, a_i, b^i_j, b_i)$ be the local coordinate system in $H$. Then we may take $(x^i, y^i, a^i_j, a_i, b^i_j, b_i)$ as a local coordinate system in $P_{|U}$. Let $(\omega^i, \omega^i_j, \omega_j, \eta^i, \eta^i_j, \eta_j)$ be the normal t-projective connection in $P$. Let us put

$$
(6.1)\qquad
\begin{aligned}
\psi^i &= \sigma^*(\omega^i) = \Pi^i_k dx^k,\\
\psi^i_l &= \sigma^*(\omega^i_l) = \Pi^i_{kl} dx^k,\\
\psi_j &= \sigma^*(\omega_j) = \Pi_{kj} dx^k + \bar{\Pi}_{kj} dy^k,\\
\phi^i &= \sigma^*(\eta^i) = \Xi^i_k dx^k + \bar{\Xi}^i_k dy^k,\\
\phi^i_j &= \sigma^*(\eta^i_j) = \Xi^i_{kj} dx^k,\\
\phi_j &= \sigma^*(\eta_j) = \Xi_{kj} dx^k + \bar{\Xi}_{kj} dy^k.
\end{aligned}
$$

Then after the tedious but simple calculations we have

$$
\begin{aligned}
\omega^i &= (a^i_k)^{-1} \psi^k,\\
\omega^i_j &= \alpha^i_j + \omega^i a_j + (a^i_k)^{-1} \psi^k_l a^l_j + \delta^i_j a_k \omega^k,
\end{aligned}
$$

$$\begin{aligned}
\omega_j =&\alpha_j - a_k\omega_j^k + a_j a_k \omega^k + \psi_k a_j^k,\\
\eta^i =&(a_p^i)^{-1}\phi^p - (a_s^i)^{-1} b_p^s \omega^p,\\
\eta_j^i =&(a_s^i)^{-1}\psi^s b_j + (a_s^i)^{-1}\psi_p^s b_j^p + (a_p^i)^{-1}\phi^p a_j - (a_s^i)^{-1} b_p^s (a_k^p)^{-1}\psi^k a_j +\\
&+ (a_p^i)^{-1}\phi_z^p b_j^z - (a_s^i)^{-1} b_p^s (a_k^p)^{-1}\psi_z^k b_j^z - a_s (a_p^s)^{-1}\phi^p \delta_j^i -\\
&- b_s (a_p^s)^{-1}\psi^p \delta_j^i + a_s (a_p^s)^{-1} b_r^p (a_k^r)^{-1}\psi^k \delta_j^i + a_t (a_s^t)^{-1}\psi^s (a_k^i)^{-1} b_j^k - \qquad (6.2)\\
&- a_t (a_s^t)^{-1}\psi^s (a_k^i)^{-1} b_p^k (a_z^p)^{-1} b_j^z + \beta_j^i,\\
\eta_j =&- a_t (a_s^t)^{-1}\psi^s b_j + \psi_s b_j^s - a_t (a_s^t)^{-1}\psi_p^s b_j^p - a_s (a_p^s)^{-1}\phi^p a_j -\\
&- b_s (a_p^s)^{-1}\psi^p a_j + a_s (a_p^s)^{-1} b_r^p (a_k^r)^{-1}\psi^k a_j + \phi_z b_j^z - a_s (a_p^s)^{-1}\phi_z^p b_j^z -\\
&- b_s (a_p^s)^{-1}\psi_z^p b_j^z + a_s (a_p^s)^{-1} b_r^p (a_k^r)^{-1}\psi_z^k b_j^z + a_t (a_s^t)^{-1}\psi^s b_j -\\
&- a_t (a_s^t)^{-1}\psi^s a_k (a_r^k)^{-1} b_j^r - a_t (a_s^t)^{-1}\psi^s b_r (a_z^r)^{-1} b_j^z +\\
&= a_t (a_s^t)^{-1}\psi^s a_r (a_p^r)^{-1} b_q^p (a_z^q)^{-1} b_j^z + \beta_j.
\end{aligned}$$

where $(\alpha_j^i, \alpha_j, \beta_j^i, \beta_j)$ is the canonical form on $H$.

THEOREM 6.1. *Let $P$ be a t-projective structure on $M$ and $(\omega^i, \omega_j^i, \omega_j, \eta^i, \eta_j^i, \eta_j)$ the normal t-projective connection. Let $U$ be a coordinate neighborhood in $M$ with local coordinate system $(x^i, y^i)$. Then there exists a unique local cross section $\sigma : U \to P$ such that*

$$\sigma^*(\omega^i) = dx^i, \quad \sigma^*(\eta^i) = dy^i,$$
$$\sum \sigma^*(\omega_i^i) = 0, \quad \sum \sigma^*(\eta_i^i) = 0.$$

*If we set for such a $\sigma$*

$$\sigma^*(\omega_j^i) = \Pi_{kj}^i dx^k, \quad \sigma^*(\omega_j) = \Pi_{kj} dx^k + \bar{\Pi}_{kj} dy^k,$$
$$\sigma^*(\eta_j^i) = \Xi_{kj}^i dx^k, \quad \sigma^*(\eta_j) = \Xi_{kj} dx^k + \bar{\Xi}_{kj} dy^k$$

*then*

$$\Pi_{kj}^i = \Pi_{jk}^i, \quad \Pi_{kt} = \Pi_{tk}, \quad \bar{\Pi}_{kt} = 0, \quad \Xi_{kt}^i = \Xi_{tk}^i, \quad \Pi_{tk} = \Xi_{kt}.$$

Proof. Let $\bar{\sigma}$ be a cross section such that $\bar{\sigma}^*(\omega^i) = dx^i$, $\bar{\sigma}^*(\eta^i) = dy^i$. In terms of the local coordinate system $(u^\alpha, u_\beta^\alpha, u_{\beta\gamma}^\alpha)$ in $P$ the cross section $\bar{\sigma}$ is given by

$$u^\alpha = (x^i, y^i),$$
$$u_\beta^\alpha = \delta_\beta^\alpha,$$
$$u_{\beta\gamma}^\alpha = -\Gamma_{\beta\gamma}^\alpha,$$

where $\Gamma_{\beta\gamma}^\alpha$ is a function of $x^i, y^i$. We take $\sigma$ as the cross section given by

$$\begin{aligned}
\bar{u}^{\alpha} &= (x^i, y^i),\\
\bar{u}^{\alpha}_{\beta} &= \delta^{\alpha}_{\beta},\\
\bar{u}^{\alpha}_{\beta\gamma} &= -\Pi^{\alpha}_{\beta\gamma},
\end{aligned}$$

where

$$\begin{aligned}
\Pi^{i}_{jk} &= \Gamma^{i}_{jk} - \frac{1}{n+1}\delta^{i}_{j}\Gamma^{z}_{kz} - \frac{1}{n+1}\delta^{i}_{k}\Gamma^{z}_{jz},\\
\Pi^{i}_{\bar{j}k} &= 0,\\
\Pi^{i}_{\bar{j}\bar{k}} &= 0,\\
\Pi^{\bar{i}}_{jk} &= \Gamma^{\bar{i}}_{jk} - \frac{1}{n+1}\delta^{i}_{j}\Gamma^{\bar{i}}_{k\bar{i}} - \frac{1}{n+1}\delta^{i}_{k}\Gamma^{\bar{i}}_{j\bar{i}},\\
\Pi^{\bar{i}}_{\bar{j}k} &= \Gamma^{i}_{jk} - \frac{1}{n+1}\delta^{i}_{j}\Gamma^{z}_{kz} - \frac{1}{n+1}\delta^{i}_{k}\Gamma^{z}_{jz},\\
\Pi^{\bar{i}}_{\bar{j}\bar{k}} &= 0.
\end{aligned} \tag{6.3}$$

It easy to see that this cross section is a unique one with the desired properties.

The remaining assertions follows easily from the facts that $\Omega^i = 0$, $H^i = 0$, $\sum \Omega^i_i = 0$, $\sum H^i_i = 0$. ■

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