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ON THE DUAL DARBOUX ROTATION AXIS OF THE DUAL SPACE CURVE

Abstract. In this paper, we obtain the dual correspondence of the "work named" Zum Drehvorgang Der Darboux Achse Einer Raumkurve given by Woldemar Barthel [1].

1. Introduction

In the Euclidean 3-dimensional space R^3, lines combined with one of their two directions can be represented by unit dual vectors over the ring of dual numbers. The important properties of real vector analysis are valid for the dual vectors.The oriented lines R^3 are in one to-one correspondence with the points of dual unit sphere D^3 [8].

A dual point on D^3 corresponds to a line in R^3; two different points of D^3 represents two skew lines in R^3. A differentiable curve on D^3 represents a ruled surface in R^3.

If φ ve φ^* are real numbers and $\varepsilon^2 = 0$ the combination

$$\hat{\varphi} = \varphi + \varepsilon\varphi^*$$

is called a dual number. The symbol ε designates the dual unit with the property $\varepsilon^2 = 0$. In analogy with the complex numbers W. K. Clifford defined the dual numbers and showed that they form an algebra, not a field. The pure dual numbers are εa^*.

According to the definition pure dual numbers εa^* are zero divisors. No number εa^* has an inverse in the algebra. But the others laws of the algebra of dual numbers $(a+ib,\, i^2 = -1)$. Later, E. Study introduced the dual angle subtended by two nonparallel lines in R^3 and defined it as $\hat{\varphi} = \varphi + \varepsilon\varphi^*$ in which φ and φ^* are, respectively, the projected angle and the shortest distance between the two lines [5].

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2. Preliminaries

The set

$$D = \{\hat{x} = x + \varepsilon x^* | x, x^* \in R\}$$

of dual numbers is a commutative ring with respect to the operations

i) $(x + \varepsilon x^*) + (y + \varepsilon y^*) = (x + y) + \varepsilon(x^* + y^*)$

ii) $(x + \varepsilon x^*) \cdot (y + \varepsilon y^*) = xy + \varepsilon(xy^* + yx^*)$.

The division $\frac{\hat{x}}{\hat{y}}$ is possible and unambiguous if $y \neq 0$ and it easily see that

$$\frac{\hat{x}}{\hat{y}} = \frac{x + \varepsilon x^*}{y + \varepsilon y^*} = \frac{x}{y} + \varepsilon \frac{x^* y - x y^*}{y^2}.$$

The set

$$\begin{aligned} D^3 = D \times D \times D = \{\hat{x} | \hat{x} &= (x_1 + \varepsilon x_1^*, x_2 + \varepsilon x_2^*, x_3 + \varepsilon x_3^*) \\ &= (x_1, x_2, x_3) + \varepsilon(x_1^*, x_2^*, x_3^*) \\ &= x + \varepsilon x^*, x \in R^3, x^* \in R^3\} \end{aligned}$$

is a module over the ring D. It is clear that any dual vector $\hat{x}$ in D^3, consist of any two real vector x and x^* in R^3 which are expressed in the natural right handed orthonormal frame in the 3-dimensional Euclidean space R^3. We call the elements of D^3 the dual vectors. If $x \neq 0$ the norm $\|\hat{x}\|$ of $\hat{x}$ is defined by

$$\|\hat{x}\| = (\hat{x} \cdot \hat{x})^{\frac{1}{2}} = \|x\| + \varepsilon \frac{x x^*}{\|x\|}.$$

Let

$$\begin{aligned} &\hat{x} : I \to D^3 \\ &\quad s \to \hat{x}(s) = x(s) + \varepsilon x^*(s) \end{aligned}$$

be a C^4 curve with the arc-length parameter s of the indicatrix. Then

$$\frac{d\hat{x}}{d\hat{s}} = \frac{d\hat{x}}{ds} \frac{ds}{d\hat{s}} = \hat{t}$$

is called the unit tangent vector of $\hat{x}(s)$. Since $\hat{t}$ has constant length 1, its differentiation with respect to $\hat{s}$, which is given by

$$\frac{d\hat{t}}{d\hat{s}} = \frac{d\hat{t}}{ds} \frac{ds}{d\hat{s}} = \frac{d^2\hat{x}}{d\hat{s}^2} = \hat{\kappa}\hat{n}, \qquad \hat{\kappa} = \kappa + \varepsilon \kappa^*$$

measures the way the curve is turning in D^3.

The norm of the vector $\frac{d\hat{t}}{d\hat{s}}$ is called curvature function of $\hat{x}(s)$. We impose the restriction that the function $\hat{\kappa} : I \to D$ is never pure-dual. Then the unit vector

$$\hat{n} = \frac{1}{\hat{\kappa}}\frac{d\hat{t}}{d\hat{s}}$$

is called the principal normal of $\hat{x}(s)$. The vector $\hat{b} = \hat{t} \times \hat{n}$ is then called the binormal of $\hat{x}(s)$. We call the vectors $\hat{t}$, $\hat{n}$, $\hat{b}$ Frenet trihedron of $\hat{x}(s)$ at the point $\hat{x}(s)$. Writing

$$\begin{aligned} \frac{d\hat{t}}{d\hat{s}} &= a_{11}\hat{t} + a_{12}\hat{n} + a_{13}\hat{b}, \\ \frac{d\hat{n}}{d\hat{s}} &= a_{21}\hat{t} + a_{22}\hat{n} + a_{23}\hat{b}, \\ \frac{d\hat{b}}{d\hat{s}} &= a_{31}\hat{t} + a_{32}\hat{n} + a_{33}\hat{b} \end{aligned}$$

and using the properties of inner product and differentiations of the inner products $\hat{t}$, $\hat{n}$, and $\hat{b}$, we may express Frenet formulas of the Frenet trihedron in the matrix form:

$$\begin{bmatrix} \hat{t}' \\ \hat{n}' \\ \hat{b}' \end{bmatrix} = \begin{bmatrix} 0 & \hat{\kappa} & 0 \\ -\hat{\kappa} & 0 & \hat{\tau} \\ 0 & -\hat{\tau} & 0 \end{bmatrix} \begin{bmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{bmatrix}.$$

We call the function $\hat{\tau} = \tau + \varepsilon\tau^* : I \to D$ such that $\frac{d\hat{b}}{d\hat{s}} = -\hat{\tau}\hat{n}$ the torsion of $\hat{x}(s)$ [7].

3. On the dual Darboux rotation axis of the dual space curve

Let $\{\hat{t}, \hat{n}, \hat{b}\}$ be the dual Frenet trihedron of the differentiable dual space curve in the dual space D^3. Then the Frenet equations are

$$\begin{aligned} \hat{t}' &= \kappa n + \varepsilon(\kappa^* n + \kappa n^*) \\ \hat{n}' &= -\kappa t + \tau b + \varepsilon(-\kappa^* t - \kappa t^* + \tau^* b + \tau b^*) \\ \hat{b}' &= -\tau n - \varepsilon(\tau^* n + \tau n^*), \end{aligned}$$

where $\hat{\kappa} = \kappa + \varepsilon\kappa^*$ is nowhere pure dual curvature and $\hat{\tau} = \tau + \varepsilon\tau^*$ is nowhere pure dual torsion.

These equations form a dual rotation motion with dual Darboux vector,

$$\hat{\partial} = \partial + \varepsilon\partial^* = \tau t + \kappa b + \varepsilon(\tau^* t + \tau t^* + \kappa b^* + \kappa^* b).$$

Also momentum dual rotation vector is expressed as follows:

$$\begin{aligned} \hat{t}' &= \hat{\partial} \times \hat{t} \\ \hat{n}' &= \hat{\partial} \times \hat{n} \\ \hat{b}' &= \hat{\partial} \times \hat{b}. \end{aligned}$$

Dual Darboux rotation of dual Frenet frame can be separated into two rotation motions: $\hat{t}$ tangent vectors rotates with a $\hat{\kappa}$ angular speed round $\hat{b}$ binormal vector, that is

$$\hat{t}' = (\hat{\kappa}\hat{b}) \times \hat{t}$$

and $\hat{b}'$ binormal vector rotates with a $\hat{\tau}$ angular speed round $\hat{t}$ tangent vector, that is

$$\hat{b}' = (\hat{\tau}\hat{t}) \times \hat{b}.$$

The separation of the rotation motion of the momentum dual Darboux axis into two rotation motions can be indicated as such:

$\frac{\hat{\partial}}{|\hat{\partial}|}$ vector rotates with a $\hat{w} = \frac{\hat{\kappa}\hat{\tau}' - \hat{\kappa}'\hat{\tau}}{\hat{\kappa}^2 + \hat{\tau}^2}$ angular speed round $\hat{n}$ principal normal, also

$$\left(\frac{\hat{\partial}}{|\hat{\partial}|}\right)' = (\hat{w}\hat{n}) \times \frac{\hat{\partial}}{|\hat{\partial}|},$$

and $\hat{n}$ principal normal vector rotates with a $|\hat{\partial}|$ angular speed round $\frac{\hat{\partial}}{|\hat{\partial}|}$ dual Darboux axis, also

$$\hat{n}' = \hat{\partial} \times \hat{n}.$$

From now on we shall do a further study of momentum dual Darboux axis. We obtain the unit vector $\hat{e}$ from dual Darboux vector,

$$\hat{e} = \frac{\hat{\partial}}{|\hat{\partial}|} = \frac{\tau t + \kappa b + \varepsilon(\tau^* t + \tau t^* + \kappa b^* + \kappa^* b)}{\sqrt{\tau^2 + \kappa^2} + \varepsilon\left(\frac{\tau\tau^* + \kappa\kappa^*}{\sqrt{\tau^2 + \kappa^2}}\right)}.$$

From $\hat{\partial}' = \hat{\tau}'\hat{t} + \hat{\kappa}'\hat{b}$, differentiation of $\hat{e}$

$$\hat{e}' = \left(\frac{\hat{\partial}}{|\hat{\partial}|}\right)' = \frac{|\hat{\partial}|\hat{\partial}' - \hat{\partial}|\hat{\partial}|'}{|\hat{\partial}|^2} = \frac{\hat{\kappa}\hat{\tau}' - \hat{\kappa}'\hat{\tau}}{\hat{\tau}^2 + \hat{\kappa}^2} \cdot \frac{\hat{\kappa}\hat{t} - \hat{\tau}\hat{b}}{\sqrt{\tau^2 + \kappa^2} + \varepsilon\left(\frac{\tau\tau^* + \kappa\kappa^*}{\sqrt{\tau^2 + \kappa^2}}\right)}$$

is found. From this,

$$\hat{e}' = -\hat{w}(\hat{e} \times \hat{n})$$

is written. According to the 2. Frenet formula,

$$\hat{n}' = |\hat{\partial}|\hat{e} \times \hat{n}$$

and

$$(\hat{e} \times \hat{n})' = \hat{e}' \times \hat{n} + \hat{e} \times \hat{n}' = \hat{w}\hat{e} - |\hat{\partial}|\hat{n}$$

are obtained. These three equations are in the form of the Frenet formulas that is

$$\hat{n}' = |\hat{\partial}|\hat{e} \times \hat{n}$$

$$(\hat{e} \times \hat{n})' = -|\hat{\partial}|\hat{n} + \hat{w}\hat{e}$$
$$\hat{e}' = -\hat{w}\hat{e} \times \hat{n},$$

where the first coefficient $|\hat{\partial}|$ is nowhere pure dual and second coefficient

$$\hat{w} = \frac{\hat{\kappa}\hat{\tau}' - \hat{\kappa}'\hat{\tau}}{\hat{\tau}^2 + \hat{\kappa}^2} = \frac{(\frac{\hat{\tau}}{\kappa})'}{1 + (\frac{\hat{\tau}}{\hat{\kappa}})^2}$$

is related only to $\frac{\hat{\tau}}{\hat{\kappa}}$ harmonic curvature. Thus, the vectors $\hat{n}$, $\hat{e} \times \hat{n}$, $\hat{e}$ define a dual rotation motion together the dual rotation vector,

$$\hat{\partial}_1 = \hat{w}\hat{n} + |\hat{\partial}|\hat{e} = \hat{w}\hat{n} + \hat{\partial}.$$

Also momentum dual rotation vector is expressed as follows:

$$\hat{n}' = \hat{\partial}_1 \times \hat{n}$$
$$(\hat{e} \times \hat{n})' = \hat{\partial}_1 \times (\hat{e} \times \hat{n})$$
$$(\hat{e})' = \hat{\partial}_1 \times \hat{e}.$$

This dual rotation motion of dual Darboux axis can be separated into two dual rotation motions again. Here $\hat{\partial}_1$ dual rotation vector is the addition of the dual rotation vectors of the dual rotation motions.

When continued in the similar way, the dual rotation motion of dual Darboux axis is done in a consecutive manner. In this way the series of dual Darboux vectors are obtained.

That is

$$\hat{\partial}_0 = \hat{\partial}, \hat{\partial}_1, \dots.$$

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