DEMONSTRATIO MATHEMATICA
Vol. XXXV No 2 2002

Christoph Schmoeger

ON LOGARITHMS OF LINEAR OPERATORS ON HILBERT SPACES

1. Introduction and terminology

Throughout this paper $\mathcal{H}$ denotes a complex Hilbert space and $\mathcal{L}(\mathcal{H})$ the Banach algebra of all bounded linear operators on $\mathcal{H}$. For $A \in \mathcal{L}(\mathcal{H})$ the spectrum and the spectral radius of A are denoted by $\sigma(A)$ and $r(A)$, respectively. For the resolvent set of A we write $\rho(A)$.

DEFINITIONS. An operator $A \in \mathcal{L}(\mathcal{H})$ is said to be

(a) *normal* if $AA^* = A^*A$,

(b) *unitary* if $AA^* = I = A^*A$, where I denotes the identity operator $\mathcal{H}$,

(c) *symmetric* if $A^* = A$,

(d) *positive* if A is symmetric and $(Ax|x) \geq 0$ for all $x \in \mathcal{H}$, where $(\cdot|\cdot)$ denotes the inner product on $\mathcal{H}$.

For $A \in \mathcal{L}(\mathcal{H})$ we denote by e^A the operator

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

In [5], C. R. Putnam has proved the following result:

THEOREM A. *If $A \in \mathcal{L}(\mathcal{H})$ is positive, $T \in \mathcal{L}(\mathcal{H})$,*

$$e^T = A \quad \textit{and} \quad \|T\| < 2\log 2$$

then T is symmetric.

S. Kurepa has shown in [3, Theorem 3] that it is sufficient to assume that $\|T\| < 2\pi$:

1991 *Mathematics Subject Classification*: 47A10, 47B15.
Key words and phrases: logarithm, normal operators.

THEOREM B. *If $A \in \mathcal{L}(\mathcal{H})$ is positive, $T \in \mathcal{L}(\mathcal{H})$,*

$$e^T = A \quad \text{and} \quad \|T\| < 2\pi$$

then T is symmetric.

Since $e^{2\pi i I} = I$, the condition $\|T\| < 2\pi$ cannot be replaced by $\|T\| \leq 2\pi$ without changing the conclusion.

The following result is also due to S. Kurepa (see [3, Theorem 2]).

THEOREM C. *Suppose that $N \in \mathcal{L}(\mathcal{H})$ is normal, $0 \leq \alpha \leq \frac{1}{2}$, $T \in \mathcal{L}(\mathcal{H})$,*

$$\sigma(N) \subseteq \{re^{i\phi} : -\alpha\pi \leq \phi \leq \alpha\pi,\ r \geq 0\},$$

$$e^T = N \quad \text{and} \quad \|T\| < \left(1 - \frac{\alpha^2}{4}\right)\pi$$

then T is normal.

The aim of the present paper is to prove some generalizations and improvements of Theorem B and Theorem C. Furthermore we shall extend Corollary 1 in [3]. To this end we need some preparations which we will give in this section. In Section 2 we consider logarithms of normal operators. Section 3 deals with logarithms of symmetric operators. Logarithms of positive operators are considered in Section 4. In Section 5 we are concerned with logarithms of unitary operators.

DEFINITION. A set $\Omega \subseteq \mathbb{C}$ is called *$2\pi i$-congruence-free* if $\lambda_1, \lambda_2 \in \Omega$ and $\lambda_1 \equiv \lambda_2 \pmod{2\pi i}$ imply that $\lambda_1 = \lambda_2$.

1.1. PROPOSITION. *Let $A, B \in \mathcal{L}(\mathcal{H})$.*

(a) *If $\sigma(A)$ is $2\pi i$-congruence-free and $e^A = e^B$ then $AB = BA$.*

(b) *If $\sigma(A)$ and $\sigma(B)$ are $2\pi i$-congruence-free and $e^A e^B = e^B e^A$ then $AB = BA$.*

(c) *If $e^A e^B = e^{A+B} = e^B e^A$ and $\sigma(A + B)$ is $2\pi i$-congruence-free then $AB = BA$.*

(d) *A is normal if and only if*

$$e^A e^{A^*} = e^{A+A^*} = e^{A^*} e^A.$$

Proof. (a) is shown in [2].

Proofs of (b) can be found in [6] or [8].

(c) is proved in [7, Theorem 2].

(d) Since $A + A^*$ is symmetric, $\sigma(A + A^*) \subseteq \mathbb{R}$. Hence $\sigma(A + A^*)$ is $2\pi i$-congruence-free. Now use (c) to get the result. ∎

1.2. PROPOSITION. *Let $A \in \mathcal{L}(\mathcal{H})$.*

(a) *If A is symmetric, then e^A is positive.*

(b) *If A is normal, then $r(A) = \|A\|$.*

(c) *Let A be normal. Then:*

$$A \text{ is symmetric } \iff \sigma(A) \subseteq \mathbb{R}$$

and

$$A \text{ is positive } \iff \sigma(A) \subseteq [0, \infty).$$

Proof. (a) It is clear that e^A is symmetric. For each $x \in \mathcal{H}$ we have

$$(e^A x|x) = (e^{\frac{A}{2}} e^{\frac{A}{2}} x|x) = (e^{\frac{A}{2}} x|e^{\frac{A}{2}} x) = \|e^{\frac{A}{2}} x\|^2 \geq 0,$$

hence e^A is positive.

(b) is shown in [4, Lemma 4.3.11].

(c) follows from Proposition 4.4.7 in [4]. ∎

DEFINITIONS. Let $A \in \mathcal{L}(\mathcal{H})$.

(a) The *real part* $\mathrm{Re}(A)$ of A is defined by

$$\mathrm{Re}(A) = \frac{1}{2}(A + A^*).$$

(b) If A is positive there is a unique positive operator, denoted by $A^{\frac{1}{2}}$, satisfying $(A^{\frac{1}{2}})^2 = A$ (see [4, Proposition 3.2.11]). $A^{\frac{1}{2}}$ is called the *square root* of A.

(c) The *absolute value* $|A|$ of A is defined by

$$|A| = (A^* A)^{\frac{1}{2}}$$

(observe that $A^* A$ is positive).

(d) We denote the set of eigenvalues of A by $\sigma_p(A)$.

(e) The set $\sigma_\pi(A) = \{\lambda \in \sigma(A) : |\lambda| = r(A)\}$ is called the *peripheral spectrum* of A.

1.3. PROPOSITION. *Let $A, B \in \mathcal{L}(\mathcal{H})$ and $AB = BA$. Then $r(A + B) \leq r(A) + r(B)$ and $r(AB) \leq r(A)r(B)$.*

Proof. [1, Satz 13.11]. ∎

2. Logarithms of normal operators

Throughout this section N denotes a normal operator in $\mathcal{L}(\mathcal{H})$.

2.1. THEOREM. *If $T \in \mathcal{L}(\mathcal{H})$, $e^T = N$ and $\sigma(T)$ is $2\pi i$-congruence-free, then T is normal.*

Proof. From $e^{T^*} = (e^T)^* = N^*$ we get

$$e^{T^*} e^T = N^* N = N N^* = e^T e^{T^*}.$$

Now use Proposition 1.1(b) to derive $TT^* = T^*T$. ∎

Our next result generalizes Theorem C:

2.2. COROLLARY. *Suppose that* $T \in \mathcal{L}(\mathcal{H})$,

$$e^T = N \quad and \quad r(T) < \pi.$$

Then T *is normal.*

Proof. Since $r(T) < \pi$, $\sigma(T)$ is $2\pi i$-congruence-free. The normality of T follows from Theorem 2.1. ∎

EXAMPLE. Let $\mathcal{H} = \mathbb{C}^2$ and the operator T be given by the matrix

$$T = \begin{pmatrix} i\pi & 0 \\ z & -i\pi \end{pmatrix},$$

where $z \in \mathbb{C} \setminus \{0\}$ is arbitrary. Then $\sigma(T) = \{i\pi, -i\pi\}$, $r(T) = \pi$ and $e^T = -I$. But T is *not* normal. This shows that the condition $r(T) < \pi$ in Corollary 2.2 cannot be replaced by $r(T) \leq \pi$.

2.3. THEOREM. *Suppose that* $T \in \mathcal{L}(\mathcal{H})$ *and* $e^T = N$. *Then*

$$T \text{ is symmetric} \iff \sigma(T) \subseteq \mathbb{R}.$$

In this case N *is positive.*

Proof. ($\Longrightarrow$): If T is symmetric, $\sigma(T) \subseteq \mathbb{R}$. By Proposition 1.2(a), $N = e^T$ is positive.

($\Longleftarrow$): Since $\sigma(T) \subseteq \mathbb{R}$, $\sigma(T)$ is $2\pi i$-congruence-free, thus, by Theorem 2.1, T is normal. Now use Proposition 1.2(c) to see that T is symmetric. ∎

2.4. COROLLARY. *If* $T \in \mathcal{L}(\mathcal{H})$ *and* $e^T = N$ *then*

$$T \text{ is positive} \iff \sigma(T) \subseteq [0, \infty).$$

Proof. Theorem 2.3 and Proposition 1.2(c). ∎

2.5. THEOREM. *Let* $T \in \mathcal{L}(\mathcal{H})$ *and* $e^T = N$. *The following assertions are equivalent*:

(a) T *is normal*;
(b) $e^{T+T^*} = N^*N$;
(c) $e^{\mathrm{Re}(T)} = |N|$.

Proof. If T is normal,

$$e^{T+T^*} = e^{T^*}e^T = N^*N,$$

thus (a) implies (b).

Suppose that (b) holds. It follows from Proposition 1.2(a) that e^{T+T^*} and $e^{\mathrm{Re}(T)}$ are positive. Hence $e^{\mathrm{Re}(T)}$ is the square root of e^{T+T^*}. Therefore

$$e^{\mathrm{Re}(T)} = (N^*N)^{\frac{1}{2}} = |N|,$$

hence (c) is valid.

Now assume that (c) holds. Then

$$e^{T+T^*} = |N|^2 = N^*N = NN^*,$$

hence

$$e^{T+T^*} = e^{T^*}e^T = e^Te^{T^*}.$$

It follows from Proposition 1.1(d) that T is normal. ∎

3. Logarithms of symmetric operators

Throughout this section A denotes a symmetric operator in $\mathcal{L}(\mathcal{H})$.

As an immediate consequence of our results in Section 2 we have:

3.1. THEOREM. *Let $T \in \mathcal{L}(\mathcal{H})$ and $e^T = A$.*

(a) *If $\sigma(T)$ is $2\pi i$-congruence-free, then T is normal.*
(b) *T is normal $\iff e^{T+T^*} = A^2$.*
(c) *T is symmetric $\iff \sigma(T) \subseteq \mathbb{R}$.*
(d) *T is positive $\iff \sigma(T) \subseteq [0,\infty)$.*

DEFINITIONS. For $j \in \mathbb{Z}$ put

$$\Omega_j = \{\alpha + j\pi i : \alpha \in \mathbb{R}\},$$
$$\Omega_+ = \bigcup_{j\in\mathbb{Z}\setminus\{0\}} \Omega_{2j}, \qquad \Omega_- = \bigcup_{j\in\mathbb{Z}} \Omega_{2j+1},$$
$$\Omega = \Omega_+ \cup \Omega_-.$$

The following lemma is easily verified.

3.2. LEMMA.

(a) *$\Omega_0 = \mathbb{R}$ and $\Omega \subseteq \mathbb{C}\setminus\mathbb{R}$.*
(b) *For $\lambda \in \mathbb{C}$ we have*
   (i) *$e^\lambda \in \mathbb{R}$ and $e^\lambda > 0 \iff \lambda \in \Omega_0 \cup \Omega_+$, and*
   (ii) *$e^\lambda \in \mathbb{R}$ and $e^\lambda < 0 \iff \lambda \in \Omega_-$.*
(c) *If $K = \{\lambda \in \mathbb{C} : |\lambda| \le 2\pi\}$ then*

$$K \cap \Omega_+ = \{2\pi i,\ -2\pi i\}$$

and

$$K \cap \Omega_- = \{\alpha \pm i\pi : \alpha \in \mathbb{R},\ |\alpha| \le \sqrt{3}\pi\}.$$

3.3. THEOREM. *Let $T \in \mathcal{L}(\mathcal{H})$ and $e^T = A$. Then*

$$T \text{ is symmetric } \iff \sigma(T) \cap \Omega = \emptyset.$$

In this case A is positive.

Proof. ($\Longrightarrow$): If T is symmetric, $\sigma(T) \subseteq \mathbb{R}$, thus $\sigma(T) \cap \Omega = \emptyset$.

($\Longleftarrow$): Since e^T is invertible in $\mathcal{L}(\mathcal{H})$, $0 \in \rho(A)$, therefore $\sigma(A) \subseteq \mathbb{R}\setminus\{0\}$. Take $\lambda \in \sigma(T)$, then, by the spectral mapping theorem ([1, Satz 99.2]), $e^\lambda \in \sigma(A)$. Part (b) of Lemma 3.2 yields, since $\sigma(T) \cap \Omega = \emptyset$, $\lambda \in \Omega_0 = \mathbb{R}$. Therefore we have that $\sigma(T) \subseteq \mathbb{R}$. Use Theorem 3.1(c) to derive the symmetry of T. ∎

3.4. COROLLARY. *Let $T \in \mathcal{L}(\mathcal{H})$, $e^T = A$ and $r(T) \leq 2\pi$. The following assertions are equivalent*:

(a) *T is symmetric.*
(b) *$-2\pi i$, $2\pi i \notin \sigma(T)$ and $\sigma(T) \cap \{\alpha \pm i\pi : \alpha \in \mathbb{R},\ |\alpha| \leq \sqrt{3}\pi\} = \emptyset$.*

Proof. Lemma 3.2(c) and Theorem 3.3. ∎

4. Logarithms of positive operators

Throughout this section let $A \in \mathcal{L}(\mathcal{H})$ be positive and $T \in \mathcal{L}(\mathcal{H})$.

4.1. THEOREM. *If $e^T = A$ then*

$$T \text{ is normal } \Longleftrightarrow e^{\mathrm{Re}(T)} = A.$$

Proof. Since A is positive, Theorem 2.5 gives

$$T \text{ is normal } \Longleftrightarrow e^{\mathrm{Re}(T)} = |A| = (A^2)^{\frac{1}{2}} = A. \ ∎$$

4.2. THEOREM. *If $e^T = A$ then*

$$T \text{ is symmetric } \Longleftrightarrow \sigma(T) \cap \Omega_+ = \emptyset.$$

Proof. If $\lambda \in \sigma(T)$ then $e^\lambda \in \sigma(A)$. Since $0 \in \rho(A)$, we get $e^\lambda > 0$, by Proposition 1.2(c). Lemma 3.2(b) gives then that $\sigma(T) \cap \Omega_- = \emptyset$. Now use Theorem 3.3 to complete the proof. ∎

4.3. COROLLARY. *If $e^T = A$ and $r(T) \leq 2\pi$ then*

$$T \text{ is symmetric } \Longleftrightarrow 2\pi i \notin \sigma(T) \text{ and } -2\pi i \notin \sigma(T).$$

Proof. Lemma 3.2(c) and Theorem 4.2. ∎

REMARK. As an immediate consequence of Corollary 4.3 we get Theorem B (Section 1).

4.4. THEOREM. *Suppose that $B \in \mathcal{L}(\mathcal{H})$ is symmetric and that $e^T = e^B$. then the following assertions are equivalent*:

(a) $T = B$.
(b) $\sigma(T) \subseteq \mathbb{R}$.
(c) $\sigma(T) \cap \Omega_+ = \emptyset$.

Proof. The implications (a)$\Longrightarrow$(b)$\Longrightarrow$(c) are clear. Now suppose that (c) holds. Since e^B is positive (Proposition 1.2(a)), Theorem 4.2 shows that T

is symmetric. Hence $T - B$ is symmetric. From $e^T = e^B$ and Proposition 1.1(a) we derive $TB = BT$ and so $e^{T-B} = I$.

Now take $\lambda \in \sigma(T - B)$. Then $\lambda \in \mathbb{R}$ and $e^\lambda = 1$, thus $\lambda = 0$. This gives $\sigma(T - B) = \{0\}$, hence $r(T - B) = 0$. Since $T - B$ is symmetric, Proposition 1.2(b) shows that $\|T - B\| = 0$, hence (a) is valid. ■

4.5. COROLLARY. *Suppose that $B \in \mathcal{L}(\mathcal{H})$ is symmetric,*

$$e^T = e^B \text{ and } r(T) \le 2\pi.$$

(a) $T = B \iff 2\pi i \notin \sigma(T)$ *and* $-2\pi i \notin \sigma(T)$.

(b) *If* $r(T) < 2\pi$ *then* $T = B$.

P r o o f. (a) Since $r(T) \le 2\pi$, we get from Lemma 3.2(c) that

$$\sigma(T) \cap \Omega_+ \subseteq \{2\pi i, -2\pi i\}.$$

Consequently, by Theorem 4.4, $T = B$ if and only if $2\pi i, -2\pi i \notin \sigma(T)$.

(b) follows from (a). ■

DEFINITION. $T \in \mathcal{L}(\mathcal{H})$ is called *isoloid* if every isolated point of $\sigma(T)$ belongs to $\sigma_p(T)$.

From [1, Satz 112.2] we get:

4.6. LEMMA. *If T is normal, then T is isoloid.*

4.7. THEOREM. *Let $e^T = A$ and $r(T) \le 2\pi$.*

(a) *If T is invertible in $\mathcal{L}(\mathcal{H})$, $2\pi i \notin \sigma(T)$ or $-2\pi i \notin \sigma(T)$ then T is normal.*

(b) *Let $2\pi i \notin \sigma_p(T)$ and $-2\pi i \notin \sigma_p(T)$. Then*

$$T \text{ is isoloid} \iff T \text{ is normal} \iff T \text{ is symmetric.}$$

P r o o f. (a) We assume that $-2\pi i \notin \sigma(T)$ (the proof for the case $2\pi i \notin \sigma(T)$ is similar). Take $\lambda \in \sigma(T)$. Then $e^\lambda > 0$, thus $\lambda = \alpha + 2k\pi i$ for some $\alpha \in \mathbb{R}$ and $k \in \mathbb{Z}$. Then

$$|\lambda|^2 = \alpha^2 + 4k^2\pi^2 \le r(T)^2 \le 4\pi^2,$$

hence $k \in \{0, 1, -1\}$. Suppose $k = -1$. Thus $\lambda = \alpha - 2\pi i$, which gives $|\lambda|^2 = \alpha^2 + 4\pi^2 \le 4\pi^2$, hence $\alpha = 0$ and so $\lambda = -2\pi i$, a contradiction. Therefore $k = 0$ or $k = 1$. If $k = 1$ then $\lambda = 2\pi i$. This shows that

$$\sigma(T) \subseteq ([-2\pi, 2\pi] \cup \{2\pi i\}) \setminus \{0\}.$$

But the last set is $2\pi i$-congruence-free, hence $\sigma(T)$ has this property. Theorem 2.1 shows that T is normal, as desired.

(b) Because of Lemma 4.6, the implications

$$T \text{ symmetric} \implies T \text{ normal} \implies T \text{ isoloid}$$

are clear.

Now suppose that T is isoloid. Assume that $2\pi i \in \sigma(T)$. As in the proof of (a), $2\pi i$ is an isolated point of $\sigma(T)$, hence $2\pi i \in \sigma_p(T)$, a contradiction. Thus $2\pi i \notin \sigma(T)$. The same argument gives $-2\pi i \notin \sigma(T)$. Use Corollary 4.3 to get the symmetry of T. ∎

In what follows we investigate logarithms of the operator $e^{i\theta}A$, where $\theta \in (0, 2\pi)$. The following result is due to S. Kurepa ([3, Corollary 1]). We will give a slightly different proof.

4.8. THEOREM. *Let* $e^T = e^{i\theta}A$ *and* $\theta \in (0, 2\pi)$.

(a) *If* $\theta \in (0, \pi]$ *then* $r(T) \geq \theta$.

(b) *If* $\theta \in [\pi, 2\pi)$ *then* $r(T) \geq 2\pi - \theta$.

Proof. (a) Assume to the contrary that $r(T) < \theta$. Then $r(T - i\theta I) \leq r(T) + \theta < 2\theta \leq 2\pi$. From $e^{T - i\theta I} = A$ and Corollary 4.3 we see that $T - i\theta I$ is symmetric, thus T is normal and $T - T^* = 2i\theta I$. Furthermore, by Proposition 1.2(b),

$$2\theta = r(T - T^*) = \|T - T^*\| \leq \|T\| + \|T^*\| = 2\|T\| = 2r(T) < 2\theta,$$

a contradiction.

(b) Put $\vartheta = 2\pi - \theta$, then $\vartheta \in (0, \pi]$ and

$$e^{T^*} = (e^T)^* = e^{-i\theta}A = e^{i(2\pi - \theta)}A = e^{i\vartheta}A.$$

Now use (a) to derive

$$r(T) = r(T^*) \geq \vartheta = 2\pi - \theta. \ ∎$$

Our final result in this section reads as follows:

4.9. THEOREM. *Let* $e^T = e^{i\theta}A$ *and* $\theta \in (0, 2\pi)$.

(a) *If* $\theta \in (0, \pi)$ *and* $r(T) = \theta$ *then* T *is normal and*

$$\sigma_\pi(T) = \{i\theta\}.$$

(b) *If* $\theta = \pi$ *and* $r(T) = \theta$ *then*

$$\sigma_\pi(T) = \{i\pi, -i\pi\}.$$

(c) *If* $\theta \in (\pi, 2\pi)$ *and* $r(T) = 2\pi - \theta$ *then* T *is normal and*

$$\sigma_\pi(T) = \{i(\theta - 2\pi)\}.$$

EXAMPLE. Let $\mathcal{H} = \mathbb{C}^2$ and

$$T = \begin{pmatrix} i\pi & 0 \\ z & -i\pi \end{pmatrix},$$

where $z \in \mathbb{C} \setminus \{0\}$ (see Example 2.6). Then

$$r(T) = \pi, \quad e^T = -I = e^{i\pi}I$$

and

$$\sigma(T) = \sigma_\pi(T) = \{i\pi, -i\pi\}.$$

But T is not normal. This example shows that in part (b) of Theorem 4.9 in general T is not normal and $\sigma_\pi(T)$ is not a singleton.

Proof of Theorem 4.9. (a) Let $\lambda \in \sigma_\pi(T)$. Then $e^\lambda = e^{i\theta}\alpha$ for some $\alpha > 0$. Thus there is $k \in \mathbb{Z}$ such that $\lambda = \log\alpha + i(\theta + 2k\pi)$. From

$$|\lambda|^2 = (\log\alpha)^2 + (\theta + 2k\pi)^2 = r(T)^2 = \theta^2,$$

we derive $|\theta + 2k\pi| \le \theta$. It follows that $k\pi \ge -\theta$ and $k \le 0$. Since $\theta < \pi$, $-1 < k \le 0$, thus $k = 0$. Hence $\lambda = \log\alpha + i\theta$. Again by $|\lambda| = \theta$, we get $\lambda = i\theta$ and so $\sigma_\pi(T) = \{i\theta\}$. From $r(T) = \theta < \pi$ we see that $\sigma(T)$ is $2\pi i$-congruence-free. Since $A = e^{T - i\theta I}$, $e^{T^* + i\theta I} = A^* = A = e^{T - i\theta}$. Hence

$$e^T = e^{T^* + 2i\theta I}.$$

Proposition 1.1(a) shows then that $T(T^* + 2i\theta I) = (T^* + 2i\theta I)T$, thus T is normal.

(c) Put $\vartheta = 2\pi - \theta$. As in the proof of Theorem 4.8(b) we have $\vartheta \in (0, \pi)$ and $e^{T^*} = e^{i\vartheta}A$. Since $r(T^*) = r(T) = 2\pi - \theta = \vartheta$, (a) gives the normality of T and $\sigma_\pi(T) = \{\overline{\lambda} : \lambda \in \sigma_\pi(T^*)\} = \{-i\vartheta\} = \{i(\theta - 2\pi)\}$.

(b) Take $\lambda \in \sigma_\pi(T)$. Then $\lambda = \log\alpha + i(\pi + 2k\pi)$ for some $\alpha > 0$ and $k \in \mathbb{Z}$. Use

$$|\lambda|^2 = (\log\alpha)^2 + (2k+1)^2\pi^2 = \pi^2$$

to derive $k \in \{0, -1\}$. Thus $\lambda = i\pi$ or $\lambda = -i\pi$. ∎

5. Logarithms of unitary operators

Throughout this section let U denote an unitary operator in $\mathcal{L}(\mathcal{H})$ and let $T, S \in \mathcal{L}(\mathcal{H})$.

5.1. LEMMA. $\sigma(U) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

Proof. [1, Satz 118.1]. ∎

5.2. THEOREM. *If $e^S = U$, then*

$$S \text{ is normal} \iff S = -S^*.$$

Proof. ($\Longleftarrow$): Clear.

($\Longrightarrow$): By hypothesis and Theorem 2.5,

$$e^{S+S^*} = U^*U = I.$$

Take $\lambda \in \sigma(S + S^*)$. Then $\lambda \in \mathbb{R}$ and $e^\lambda = 1$, thus $\lambda = 0$. Therefore $\sigma(S + S^*) = \{0\}$ and $r(S + S^*) = 0$. Proposition 1.2(b) gives $S = -S^*$. ∎

5.3. COROLLARY. *Let $e^{iT} = U$. Then the following assertions are equivalent*:

(a) *T is normal.*

(b) *T is symmetric.*

Proof. Put $S = iT$ and use Theorem 5.2 to derive

T is normal $\Longleftrightarrow S$ is normal $\Longleftrightarrow S = -S^* \Longleftrightarrow iT = iT^* \Longleftrightarrow T = T^*$. ∎

5.4. COROLLARY. *Suppose that $e^{iT} = U$ and $r(T) \le \pi$. If $\pi \notin \sigma(T)$ or $-\pi \notin \sigma(T)$ then T is symmetric.*

Proof. Take $\lambda \in \sigma(iT)$. Then $e^\lambda \in \sigma(U)$, thus $|e^\lambda| = 1$, by Lemma 5.1. It follows that $\lambda = i\beta$ for some $\beta \in \mathbb{R}$. Since $|\beta| = |\lambda| \le r(T) \le \pi$,

$$\sigma(iT) \subseteq \{i\beta : \beta \in [-\pi, \pi]\}.$$

By hypothesis, $i\pi \notin \sigma(iT)$ or $-i\pi \notin \sigma(iT)$, hence $\sigma(iT)$ is $2\pi i$-congruence-free. From

$$e^{-iT^*} = (e^{iT})^* = U^* = U^{-1} = e^{-iT},$$

we get $e^{iT} = e^{iT^*}$. From Proposition 1.1(a) we then derive that T is normal. Corollary 5.3 shows therefore that T is symmetric. ∎

REMARK. Example 2.6 shows that we cannot drop the condition "$\pi \notin \sigma(T)$ or $-\pi \notin \sigma(T)$" in Corollary 5.4 without changing the conclusion.

5.5. COROLLARY. *Suppose that e^{iT} is unitary and that $r(T) < \pi$. Then T is symmetric.*

References

[1] H. Heuser, *Funktionalanalysis*, Teubner (1991).

[2] E. Hille *On roots and logarithms of elements of a complex Banach algebra*, Math. Ann. 136 (1958), 46–57.

[3] S. Kurepa, *A note on logarithms of normal operators*, Proc. Amer. Math. Soc. 13 (1962), 307–311.

[4] G. K. Petersen, *Analysis Now*, Springer (1988).

[5] C. R. Putnam, *On square roots and logarithms of self-adjoint operators*, Proc. Glasgow Math. Assoc. 4 (1958), 1–2.

[6] Ch. Schmoeger, *Remarks on commuting exponentials in Banach algebras*, Proc. Amer. Math. Soc. 127 (1999), 1337–1338.

[7] Ch. Schmoeger, *Remarks on commuting exponentials in Banach algebras II*, Proc. Amer. Math. Soc. 128 (2000), 3405–3409.

[8] E. M. E. Wermuth, *A remark on commuting operator exponentials*, Proc. Amer. Math. Soc. 125 (1997), 1685–1688.

MATHEMATISCHES INSTITUT I
UNIVERSITÄT KARLSRUHE
D-76128 KARLSRUHE, GERMANY
E-mail: christoph.schmoeger@math.uni-karlsruhe.de

Received June 11, 2001; revised version November 6, 2001.