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ON THE SEQUENTIAL STRONG-WEAK CLOSEDNESS OF THE NEMYTSKIJ MULTIVALUED OPERATOR

Abstract. Let Ω be a measure space, and E, F be separable Banach spaces. Given a multifunction $f : \Omega \times E \to 2^F$, denote by $N_f(x)$ the set of all measurable selections of the multifunction $f(\cdot, x(\cdot)) : \Omega \to 2^F$, $s \mapsto f(s, x(s))$, for a function $x : \Omega \to E$. In this note we obtain a general theorem on the sequential strong-weak closedness for the Nemytskij multivalued superposition operator N_f acting into a Banach space of measurable F-valued functions in the *infinite-dimensional case* $\dim F = +\infty$, via discovering a new relation between the Q-upper limit and the M-upper limit of a sequence of subsets of F.

Introduction

Closedness-type theorems play important roles in many problems of the theories of differential / integral inclusions and optimal control. The first results of this kind were obtained in the work of C. Olech, A. Lasota, L. Cesari, C. Castaing, C. Castaing and M. Valadier, and others in 1960's decade (see references e.g. in [7, 9]).

The present note devotes the sequential strong-weak closedness problem for the Nemytskij multivalued superposition operator N_f generated by a multifunction $f : \Omega \times E \to 2^F$ (Ω is a measure space, and E, F are separable Banach spaces) and which acts into a Banach space Y of measurable F-valued functions. In the finite-dimensional case $\dim F < +\infty$ the general sequential strong-weak closedness result for N_f, at least in the case of the L_p-type space Y, can be immediately deduced from the above mentioned work. In the *infinite-dimensional case* $\dim F = +\infty$, various results on the sequential strong-weak closedness for N_f acting into the L_p-type space Y

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$(1 \leq p < \infty)$ were obtained via different proofs by C. Castaing, C. Castaing and M. Valadier, H. Attouch and A.Damlamian, J. P. Daures, N. S. Papageorgiou, and many others (see all different historical comments and numerous references in [1, 2, 4-7, 9-11, 14]).

In this note we present a general sequential strong-weak closedness theorem (see Theorem 2.1 in Section 2) which incorporates many of known results of this kind for N_f acting into the L_p-type space Y and which immediately extends these results to the case of the non-L_p-type space Y (such as Orlicz space, Banach lattice, Köthe-Bochner space [12], or Banach module [13]). Theorem 2.1/(2) together with its proof is crucially based on a new relation (see Lemma 2.1) between the Q-upper limit and the M-upper limit of a sequence of subsets of F in the case $\dim F = +\infty$. Note that Theorem 2.1/(2) allows immediately to refine recent existence theorems [2, 3] for nonlinear inclusions with nonpolynomial / exponential nonlinearities by droping such the additional assumption of [2, 3] that N_f maps an order bounded set into an order bounded set of Y.

The collection of all proofs of the results of Section 2 will be given in Section 3.

1. Some terminology and notations

First, we shall give some terminology and notations in set-valued analysis following, e.g., [7, 9, 11]. Given a multifunction $\Gamma : X \to 2^Y$ and $M \subset X$, define $\operatorname{Gr}\Gamma = \{(x,y) \in X \times Y : y \in \Gamma(x)\}$, $\operatorname{dom}\Gamma = \{x \in X : \Gamma(x) \neq \emptyset\}$, and $\Gamma(M) = \bigcup_{x\in M} \Gamma(x)$. Let F be a metric vector space. Denote by $\operatorname{cl}(M)$ (resp., $\operatorname{co}(M)$ and $\overline{\operatorname{co}}(M) = \operatorname{clco}(M)$) the closure (resp., the convex hull, the closed convex hull) of a set M in F. We denote [2] by $\mathrm{P}(F)$ (resp., $\mathrm{Cl}(F)$, $\mathrm{Bd}(F)$, $\mathrm{Cv}(F)$, etc.) the family of all nonempty (resp., and closed, and bounded, and convex, etc.) subsets of F. We denote by $\mathrm{Cl}(F_w)$ (resp., $\mathrm{Cp}(F_w)$, $\mathrm{CvCp}(F_w)$, etc.) the family of all nonempty w-weakly closed (resp., w-weakly compact, convex and w-weakly compact, etc.) subsets in F_w endowed with the w-weak topology $\sigma(F, F^*)$. Denote by $\mathcal{B}(F)$ the algebra of all Borel subsets of F. We recall the *Cesari's Q-upper limit*: $Q - \overline{\lim} A_n \stackrel{\text{def}}{=} \bigcup\{ \bigcap_{n=1}^{\infty} \operatorname{cl}\operatorname{co}\{ \bigcup_{k\geq n}^{\infty} x_k\} : x_k \in A_k\}$, and the *M-upper limit*: $w - \overline{\lim} A_n \stackrel{\text{def}}{=} \{u \in F_w : u = \lim u_{n_k} \text{ in } F_w,\ u_{n_k} \in A_{n_k}\}$. We recall also the *K-lower limit*: $\underline{\lim} A_n \stackrel{\text{def}}{=} \{u \in F : u = \lim u_n,\ u_n \in A_n\}$, and the *K-upper limit*: $\overline{\lim} A_n \stackrel{\text{def}}{=} \{u \in F : u = \lim u_{n_k},\ u_{n_k} \in A_{n_k}\}$. Now let E be a metric space. Then $f : E \to 2^F$ is called *Q-upper semicontinuous* if $Q - \overline{\lim}_{n\to\infty} f(u_n) \subset f(u)$ (a sequence $u_n \to u$ in E); *M-upper semi-continuous* (*M-u.s.c.*) if $w - \overline{\lim} f(u_n) \subset f(u)$ (a sequence $u_n \to u$ in E); *sequentially* *strong-weakly closed*, if the graph $\mathrm{Gr}\, f$ is sequentially closed in $E \times F_w$. It is known that (see references, e.g., in [11, 14]): If $f : E \to \mathrm{CvCl}\,(F_w)$ where F is a Banach space and E is a metric space, then "$f : E \to \mathrm{Cl}\,(F_w)$ is u.s.c." $\Rightarrow$ "f has sequentially closed graph in $E \times F_w$ " $\Leftrightarrow$ "f is M-u.s.c". Remember that given topological spaces X, Y and a multifunction $\Gamma : X \to 2^Y$, Γ is called *upper semicontinuous* (or *usc*) *at* $x_0 \in X$ if for any open set $V \subset Y$ such that $\Gamma(x_0) \subset V$, one can find an open neighbourhood $U \subset X$ of x_0 such that $\Gamma(x) \subset V$ for all $x \in U$. This multifunction Γ is called upper semicontinuous or usc, if it is usc at every $x \in X$.

Second, from this place, unless stated to the contrary, E and F, etc. denote separable Banach spaces; $(\Omega, \mathfrak{A}, \mu)$ denotes a fixed measure space with a complete σ-finite σ-additive measure μ on a σ-algebra $\mathfrak{A}$ of subsets of Ω; $S(\Omega, F)$ denotes the complete metric vector space of all (classes of equivalent) measurable functions $x : \Omega \to F$, equipped with the metric topology via the convergence in measure. Given a property P_s, we shall denote $P_s\,(mod0)$ if P_s is valid for almost all (a.a.) $s \in \Omega$. In Theorem 2.1 we need *a Banach space Y which is continuously embedded into $S(\Omega, F)$*. For example, it is known that every Köthe-Bochner space $Y = \mathbb{Y}[F]$ (and every Banach module Y in the sense [13]) always is continuously embedded into $S(\Omega, F)$. We recall some definitions. A Banach space $\mathbb{Y} \subset S(\Omega, \mathbb{R})$ with norm $\|.\|_{\mathbb{Y}}$ is called a Köthe space (also under the name, Banach lattice), if $x \in \mathbb{Y}$ and $y \in S(\Omega, \mathbb{R})$ and $|y(s)| \leq |x(s)|$ a.e. then $y \in \mathbb{Y}$ and $\|y\|_{\mathbb{Y}} \leq \|x\|_{\mathbb{Y}}$. Given a Köthe space $\mathbb{Y} \subset S(\Omega, \mathbb{R})$, define the Köthe-Bochner space $Y = \mathbb{Y}[F] \subset S(\Omega, F)$ as the Banach space of all measurable functions $x : \Omega \to F$ such that $\|x(\cdot)\|_F \in \mathbb{Y}$, with norm $\|x\|_Y \stackrel{\text{def}}{=} \|\|x(\cdot)\|_F\|_{\mathbb{Y}}$. Concrete examples of Köthe spaces are Lebesgue spaces L_p and many non-L_p-type spaces such as general Orlicz / Lorentz / Marcinkiewicz spaces and many others (see e.g. [12]).

Third, we denote by $\mathrm{Sel}\, g$ the set of all measurable selectors of a multifunction $g : \Omega \to 2^F$, i.e.

$$\mathrm{Sel}\, g = \{y \in S(\Omega, F) : y(s) \in g(s) \text{ a.e.}\}. \tag{1}$$

Further, a multifunction $f : \Omega \times E \to 2^F$ is called [2] *superpositionally* Sel-*measurable* on $G = S(\Omega, E)$ or shortly *sup*-Sel-*measurable*, if for every $x \in G$ the multifunction $\Gamma = f(\cdot, x(\cdot)) : \Omega \to 2^F$ is Sel-*measurable*, i.e. $\mathrm{Sel}\,\Gamma \neq \emptyset$.

2. The sequential strong-weak closedness

THEOREM 2.1. *Let F be a separable Banach space with* $\dim F = +\infty$, *Y be a Banach space embedded continuously into $S(\Omega, F)$. Let E be a complete* *separable metric space,* $f(\cdot,\cdot,\lambda) : \Omega \times E \to \mathrm{CvCl}(F_w) \cup \{\emptyset\}$ *for each* λ *of a metric "space of parameters"* Λ.

Suppose one of the following conditions (in fact, the condition 2 $\Rightarrow$ *the condition 1):*

(1) $f(\cdot,\cdot,\lambda)$ *is sup-Sel-measurable for each* λ, *and* $f(s,\cdot,\cdot)$ *is Q-upper semicontinuous for a.a.* $s \in \Omega$;

(2) $f(\cdot,u,\lambda)$ *is a Sel-measurable on* Ω *multifunction for each* $(u,\lambda) \in E \times \Lambda$, *for a.a.* $s \in \Omega$ $f(s,\cdot,\cdot)$ *on* $E \times \Lambda$ *is w-weakly pre-compact in* F_w *[in particular, bounded in F with F being reflexive] on each convergent sequence and it is Q-upper semicontinuous (*$\Leftrightarrow$ *M-u.s.c.* $\Leftrightarrow$ *sequentially strong-weakly closed).*

Then, if $N(x,\lambda) \stackrel{\text{def}}{=} \mathrm{Sel}\, f(\cdot, x(\cdot), \lambda) \subset Y$ *for every* (x,λ) *of some domain* $G \subset S(\Omega, E) \times \Lambda$, *the operator* $N : G \to \mathrm{CvCl}(Y, \sigma(Y, Y^*))$ *is sequentially strong-weakly closed. In particular, if X being another metric space is continuously embedded into* $S(\Omega,E)$ *and* $N(x,\lambda) \subset Y$ *for every* (x,λ) *of* $X \times \Lambda$, *then* $N : X \times \Lambda \to \mathrm{CvCl}(Y, \sigma(Y,Y^*))$ *is sequentially strong-weakly closed.*

Note that the proof of Theorem 2.1/(2) relies on Lemma 2.1 and Proposition 2.1. Note also that different particular cases in different forms of Theorem 2.1/(2) for M-u.s.c. $f(s,\cdot,\cdot)$ have been continuously and intensively used together with repeating their proofs (**different from our proof**) for the L_p-type space $Y = L_p[F]$ ($p \neq \infty$) by N.S. Papageorgiou and his co-workers since 1987 (see references in [5, 11, 14]). A particular case of Theorem 2.1/(2) is the case when Y is some L_p-type space and the generating function f satisfies the condition: $f(\cdot,u,\lambda)$ is a Sel-measurable on Ω multifunction for each $(u,\lambda) \in E \times \Lambda$, for a.a. $s \in \Omega$ $f(s,\cdot,\cdot)$ on $E \times \Lambda$ is w-weakly pre-compact in F_w [in particular, bounded in F with F being reflexive] on each convergent sequence and it is strong-weak upper semicontinuous from $E \times \Lambda$ into F_w (see different well-known forms of this case together with their proofs (**different from our proof**), and various references in [5, 7, 9, 11]; cf. also with [1, 4, 6]). The related closedness problems were treated, e.g., in [1, 4, 6, 7].

LEMMA 2.1. *Let F be a Banach space,* $\{A_n : n \in \mathbb{N}\} \subset \mathrm{Cv}(F)$, *and* $\cup_{n=1}^{\infty} A_n$ *be w-weakly pre-compact in* F_w *[in particular, if F is reflexive and* $\cup_{n=1}^{\infty} A_n$ *is bounded in F]. Then*

$$\overline{\mathrm{co}}\, w\text{-} \overline{\lim_{n\to\infty}}\, A_n = \mathrm{cl}(Q\text{-} \overline{\lim_{n\to\infty}}\, A_n\} \neq \emptyset. \tag{2}$$

Remark that Ch. Hess [10] (see also in [4]) shows that under the assumption of Lemma 2.1 the relation $\overline{\text{co}}\, w\text{-}\overline{\lim_{n\to\infty}} A_n = \bigcap_{n=1}^{\infty} \text{cl}\,\text{co}\{\bigcup_{k\geq n}^{\infty} A_k\}$ holds, but we need not use this relation here.

PROPOSITION 2.1. *Let F be a separable Banach space with* $\dim F = +\infty$, $f : \Omega \times E \to \text{CvCp}\,(F_w) \cup \{\emptyset\}$ *be a multifunction such that $f(\cdot, u)$ is* Sel-*measurable for all $u \in E$ and $f(s,\cdot)$ is Q-upper semicontinuous for a.a. $s \in \Omega$. Suppose that $f(s,\cdot)$ is w-weakly pre-compact (in particular, bounded with F being reflexive) on each convergent sequence [then, the above Q-upper semicontinuity $\Leftrightarrow$ M-upper semicontinuity $\Leftrightarrow$ sequential strong-weak closedness for $f(s,\cdot)$].*

Then f is sup-Sel-measurable on $S(\Omega, E)$.

3. Proofs of the results of Section 2

We recall the *support function for a set* A: $\sigma(u^*, A) = \sup\{(u^*, u) : u \in A\}$.

Proof of Lemma 2.1. First we remark that $Q\text{-}\overline{\lim_{n\to\infty}} A_n = \text{co}\{Q\text{-}\overline{\lim_{n\to\infty}} A_n\}$, since all A_n are convex. Putting $A = w\text{-}\overline{\lim_{n\to\infty}} A_n$, $B = Q\text{-}\overline{\lim_{n\to\infty}} A_n$, we must show $\overline{\text{co}}A = \text{cl}B$. Since $\{A_n : n \in \mathbb{N}\}$ is w-weakly pre-compact, via the Krein-Smulian Theorem [8], $B \subset \overline{\text{co}}\{A_n : n \in \mathbb{N}\} \in \text{CvCp}(F_w)$, so $\text{cl}B \in \text{CvCp}(F_w)$. On other hand, by the classical Banach-Saks-Mazur Theorem [8: Theorem II.5.2], $A \subset B$, and so we get

$$\emptyset \neq \overline{\text{co}}A \subset \text{cl}B \in \text{CvCp}\,(F_w), \tag{3}$$

$$\sigma(x^*, \overline{\text{co}}A) \leq \sigma(x^*, \text{cl}B) = \sigma(x^*, B) < +\infty \quad (x^* \in F^*). \tag{4}$$

We claim that

$$\sigma(x^*, B) \leq \underline{\lim_{n\to\infty}}\, (x^*, A_n) < +\infty \quad (x^* \in F^*), \tag{5}$$

$$\overline{\lim_{n\to\infty}}\, (x^*, A_n) \leq \sigma(x^*, \overline{\text{co}}A) \quad (x^* \in F^*). \tag{6}$$

To see (5), fix $x^* \in F^*$ and $y \in B$. Then by definition, $y \in B'$ where $B' = \overline{\lim_{n\to\infty}} \text{co}\{x_j : j \geq n\}$ for some $x_n \in A_n$. Remark that $B' = \overline{\lim_{n\to\infty}} \text{co}\{x_j : j \geq n\} = \underline{\lim_{n\to\infty}} \text{co}\{x_j : j \geq n\}$, since $\text{co}\{x_j : j \geq n\} \downarrow (n \uparrow)$. Then there exist $y_n \in \text{co}\{x_j : j \geq n\}$ such that $y_n \to y$ in norm of F. So we can find positive numbers $\alpha_{n(k)}$ and elements $x_{n(k)}\,(k = 1, \cdots, K_n)$ with $n \leq n(k) \in \mathbb{N}$ such that $y_n = \sum_{k=1}^{K_n} \alpha_{n(k)} x_{n(k)}$, $\sum_{k=1}^{K_n} \alpha_{n(k)} = 1$, $\alpha_{n(k)} > 0$. We get

$(x^*, y_n) = \sum_{k=1}^{K_n} \alpha_{n(k)}(x^*, x_{n(k)}) \leq \sup_{j\geq n} (x^*, x_j) \sum_{k=1}^{K_n} \alpha_{n(k)} = \sup_{j\geq n} (x^*, x_j)$, hence $(x^*, y) = \lim_{n\to\infty} (x^*, y_n) \leq \lim_{n\to\infty}\sup_{j\geq n} (x^*, x_j) = \overline{\lim_{n\to\infty}} (x^*, x_n) \leq \overline{\lim_{n\to\infty}} (x^*, A_n)$. So we get (5).

The inequality (6) is easily checked since $\{A_n : n \in \mathbb{N}\}$ is w-weakly precompact. From the inequalities (4), (5), (6) we get $\sigma(x^*, \overline{\text{co}}A) = \sigma(x^*, \text{cl}B) \in \mathbb{R}$ $(x^* \in F^*)$, and then by (3) and Hörmander's Theorem [7] we deduce $\overline{\text{co}}A = \text{cl}B$ (i.e. (2)). ∎

P r o o f o f P r o p o s i t i o n 2.1. Since $f(\cdot, u) : \Omega \to \text{Cv}(F) \cup \{\emptyset\}$ is Sel-measurable for all $u \in E$, f is sup-Sel-measurable on the set of all measurable step-functions. Further, fix a measurable function $x : \Omega \to E$ and let $\{x_n\}$ be a sequence of measurable step-functions such that $x_n(s) \to x(s)$ a.e. on Ω. For each x_n fix a measurable selection y_n of the Sel-measurable multifunction $f(\cdot, x_n(\cdot))$.

Consider the multifunction Δ, $\Delta(s) \stackrel{\text{def}}{=} Q- \overline{\lim_{n\to\infty}}\, y_n(s)$. Since $\text{cl}\,\text{co}\{y_k(s) : k \geq n\} \subset \text{cl}\,\text{co}\{f(s, x_k(s)) : k \geq n\}$ is w-weakly compact for a.a. $s \in \Omega$, we have, via the definition of Q-upper limit, $\Delta(s) = \bigcap_{n=1}^{\infty} \text{cl}\,\text{co}\{y_k(s) : k \geq n\} \neq \emptyset\,(mod 0)$. Via e.g. [10] for cl co and for $\overline{\lim_{n\to\infty}}$, the multifunction Δ is (mod0)-Gr-measurable, i.e. exists $D \in \mathfrak{A}$ with $\mu(\Omega \backslash D) = 0$ such that the graph of $\Delta : D \to 2^F$ belongs to the product algebra $\mathfrak{A} \times \mathcal{B}(F)$. So applying the known von Neumann-Aumann Selection Theorem [7], we get a measurable selection $y \in \text{Sel}\,\Delta$. Since $f(s, \cdot)$ is Q-u.s.c., we get $\Delta(s) \subset Q- \overline{\lim_{n\to\infty}} f(s, x_n(s)) \subset f(s, x(s))\,(mod 0)$. Hence, $y \in \text{Sel}\,f(\cdot, x(\cdot))$.

The equivalent relations inside of Proposition follow from $f(s, u) \in \text{CvCp}\,(F_w) \subset \text{CvBdCl}(F)(mod 0)$ and **Lemma 2.1** since we get then the equivalence of four following inclusions (a sequence $u_n \to u$ in E): $Q- \overline{\lim_{n\to\infty}} f(s, u_n) \subset f(s, u)$, $\text{cl}(Q- \overline{\lim_{n\to\infty}} f(s, u_n)) \subset f(s, u)$, $w- \overline{\lim_{n\to\infty}} f(s, u_n) \subset f(s, u)$, and $\overline{\text{co}}(w- \overline{\lim_{n\to\infty}} f(s, u_n)) \subset f(s, u)$. ∎

P r o o f o f T h e o r e m 2.1. First we suppose the condition 1 of Theorem. Then $N(x, \lambda) = \text{Sel}\,f(\cdot, x(\cdot), \lambda) \in \text{Cv}(Y)$ for every $(x, \lambda) \in G$. Now, fix (x_n, λ_n), $(x, \lambda) \in G$ and $y_n \in N(x_n, \lambda_n)$ such that $x_n \to x$ in $S(\Omega, E)$, $\lambda_n \to \lambda$ in Λ, $y_n \to y$ in $(Y, \sigma(Y, Y^*))$. Via Riesz's Theorem $x_{n_k} \to x$ a.e. for some subsequence n_k. Via Banach-Saks-Mazur's Theorem [8: Theorem II.5.2] there exist $z_k \in \text{co}\{y_{n_j} : j \geq k\}$ such that $z_k \to y$ in norm of Y, and so $z_k \to y$ in $S(\Omega, F)$, and therefore via Riesz's Theorem, by passing to a subsequence of k and denoting it again by k, $z_k(s) \to y(s)$ a.e. Hence,

$$y(s) \in \overline{\lim_{k\to\infty}}\, \overline{\text{co}}\{y_{n_j}(s) : j \geq k\} = Q- \overline{\lim_{k\to\infty}}\, y_{n_k}(s)\,(\text{mod}0).$$

Since $f(s,\cdot,\cdot)$ is Q-upper semicontinuous for a.a. $s \in \Omega$, we get then

$$y(s) \in Q-\overline{\lim_{k\to\infty}}\, y_{n_k}(s) \subset Q-\overline{\lim_{k\to\infty}}\, f(s, x_{n_k}(s), \lambda_{n_k}) \subset f(s, x(s), \lambda)\,(\mathrm{mod}0)$$

that proves $y \in N(x, \lambda)$, and so the sequential strong-weak closedness of N follows.

Second, via Proposition 2.1 and **via Lemma 2.1** (together with the analogous argument such as in the end of Proof of Proposition 2.1), from the condition 2 of Theorem the condition 1 of Theorem follows as well as we get all equivalent implications inside the condition 2 of Theorem. ∎

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