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FINITE OPERATORS

Abstract. In this paper we give a class of finite operators of the form $A+K$, where $A \in \mathfrak{L}(H)$ and K is compact. These results are used to generalize the theorem of P.R.Halmos [2, Theorem 7] and the result given by J. P. Williams [7, Theorem 5] and we prove that $\mathfrak{W}_0(\delta_{A,B}) = \operatorname{co}\sigma(\delta_{A,B})$, where $\mathfrak{W}_0(\delta_{A,B})$, $\operatorname{co}\sigma(\delta_{A,B})$ denote respectively the numerical range of $\delta_{A,B}$ and the convex hull of $\sigma(\delta_{A,B})$ (the spectrum of $\delta_{A,B}$) for certain operators $A, B \in \mathfrak{L}(H)$. $\delta_{A,B}$ is the operator on $\mathfrak{L}(H)$ defined by $\delta_{A,B}(X) = AX - XB$ $(X \in \mathfrak{L}(H))$.

1. Introduction

Let $\mathfrak{L}(H)$ be the algebra of all bounded linear operators on an infinite dimensional complex and separable Hilbert space H.

An operator $A \in \mathfrak{L}(H)$ is called finite if $\|AX - XA - I\| \geq 1$ for each $X \in \mathfrak{L}(H)$. The class $\mathfrak{F}(H)$ of finite operators is uniformly closed. It contains every direct sum of a compact and normal operator [8]. For each integer $n \geq 1$, $\mathfrak{R}_n = \{T \in \mathfrak{L}(H) : T \text{ has an } n-\text{dimensional reducing subspace}\}$. It is known that $\overline{\mathfrak{R}_n} \subset \mathfrak{F}(H)$ for $n \geq 1$ where the bar indicates the norm closure of $\mathfrak{R}_n$ [8], each of the following conditions is a sufficient conditions for an operator A to belong to $\overline{\mathfrak{R}_1}$:

1) $\|A - \lambda I\| = r(A - \lambda I)$ [2, Theorem 8].

2) $A = T + K$, where T is hyponormal and K is compact [7, Theorem 2]. In [4] we show that the set of all finite operators is not invariant under similarity. In this paper we prove that every dominant operator is finite and every operator of the form (dominant + compact) is also finite. In addition we find a new class of finite operators which contains the class of operators $A \in \mathfrak{L}(H)$ such that $\|A - \lambda I\| = r(A - \lambda I)$ and we prove that every operator of this form + compact is finite. Finally we obtain a new proof of the containment in $\overline{\mathfrak{R}_1}$ of the algebra $\mathfrak{L}^\infty$ of all multiplication operators on

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$\mathfrak{L}^2$ and the algebra $\mathfrak{T}$ of all analytic Toeplitz operators. Consequently we obtain the following results:

$$\mathfrak{T} + \mathcal{K}(H) \subset \mathfrak{F}(H), \qquad \mathfrak{L}^\infty + \mathcal{K}(H) \subset \mathfrak{F}(H)$$

where $\mathcal{K}(H)$ is the ideal of all compact operators on H. For $A, B \in \mathfrak{L}(H)$, let $\delta_{A,B}$ denotes the operator on $\mathfrak{L}(H)$ defined by $\delta_{A,B}(X) = AX - XB$. If $A = B$, δ_A is called the inner derivation induced by $A \in \mathfrak{L}(H)$. J.H.Anderson and C.Foias [1] show that if A, B are normal operators, then $\mathfrak{W}_0(\delta_{A,B}) = \mathrm{co}\,\sigma(\delta_{A,B})$, where $\mathfrak{W}_0(\delta_{A,B})$, $\mathrm{co}\sigma(\delta_{A,B})$ denote respectively the numerical range of $\delta_{A,B}$ and the convex hull of $\sigma(\delta_{A,B})$ (the spectrum of $\delta_{A,B}$). Here we prove this result for a large class of operators $A, B \in \mathfrak{L}(H)$.

2. Finite operators

An operator $A \in \mathfrak{L}(H)$ is called *dominant* by J. G. Stampfli and B. L. Wadhwa [6] if, for all complex λ, $range(A - \lambda) \subseteq range(A - \lambda)^*$, or equivalently, if there is a real number $M_\lambda \geq 1$ such that $\|(A-\lambda)^* f\| \leq M_\lambda \|(A-\lambda)f\|$, for all $f \in H$.

If there exists a real number M such that $M_\lambda \leq M$ for all λ, the dominant operator A is said to be $M-hyponormal$. A $1-hyponormal$ is hyponormal.

For $A \in \mathfrak{L}(H)$ the set $\mathfrak{W}(A) = \{(Ax \mid x) : x \in H \text{ and } \|x\| = 1\}$ is called the numerical range of A. $\mathfrak{W}(A)$ is bounded, convex and $\sigma(A) \subset \overline{\mathfrak{W}(A)}$. Let $\sigma_{ar}(A)$ (resp. $\sigma_a(A)$) denotes the approximate reducing spectrum (resp. the approximate spectrum) of A defined by

$\sigma_{ar}(A) =$ the set $\lambda \in \mathbb{C}$ such that there exists a normed sequence (x_n); $n \in N^*$ for which $\lim_n (A-\lambda)x_n = 0$ and $\lim_n (A-\lambda)^* x_n = 0$.

$\sigma_a(A) =$ the set $\lambda \in \mathbb{C}$ such that there exists a normed sequence (x_n); $n \in N^*$ for which $\lim_n (A-\lambda)x_n = 0$.

LEMMA 1. [2] $\overline{\mathfrak{R}_1} = \{A \in \mathfrak{L}(H) : \sigma_{ar}(A) \neq \emptyset\}$.

LEMMA 2. *Let* $A \in \mathfrak{L}(H)$. *If* $\mathrm{Re}\, A \geq 0$, *then* $\{\lambda \in \sigma_a(A) : \mathrm{Re}\, A \geq 0\} \subset \sigma_{ar}(A)$.

Proof. Let $\lambda \in \sigma_a(A)$, then there exists a sequence (x_n) such that $(A-\lambda)x_n \longrightarrow 0$, then

$$B = \mathrm{Re}(A-\lambda) = \frac{1}{2}\left[(A-\lambda) + (A-\lambda)^*\right]$$

satisfies $(Bx_n \mid x_n) \longrightarrow 0$. Since $B \geq 0$, it results that $Bx_n \longrightarrow 0$, i.e,

$$\frac{1}{2}\left[(A-\lambda)x_n + (A-\lambda)^* x_n\right] \longrightarrow 0.$$

Since $(A-\lambda)x_n \longrightarrow 0$, then $(A-\lambda)^* x_n \longrightarrow 0$. ■

THEOREM 3. $\overline{\mathfrak{R}_1}$ *contains the following operators:*

1) $A \in \mathfrak{L}(H)$ *such that* $\partial\mathfrak{W}(A) \cap \sigma(A) \neq \emptyset$,
2) *dominant operators.*

Proof. 1) We have $\partial\mathfrak{W}(A) \cap \sigma(A) \subset \sigma_{ar}(A)$. Indeed, by the transformation

$$A \mapsto \alpha A + \beta$$

the hypothesis $\lambda \in \partial\mathfrak{W}(A) \cap \sigma(A) \subset \sigma_{ar}$ can be replaced by $0 \in \partial\mathfrak{W}(A) \cap \sigma(A) \subset \sigma_{ar}$ with $\operatorname{Re} A \geq 0$. Since $0 \in \partial\sigma(A) \subset \sigma_a(A)$, it results from Lemma 2 that $0 \in \sigma_{ar}(A)$. Since $\partial\mathfrak{W}(A) \cap \sigma(A) \neq \emptyset$, then $\sigma_{ar}(A) \neq \emptyset$.

2) If A is dominant we have $\sigma_a(A) \subset \sigma_{ar}(A)$, indeed, if $\lambda \in \sigma_a(A)$, then there exists a normed sequence $\{x_n\}$ such that $\lim_n (A - \lambda)x_n = 0$. Since A is dominant there exists a real number $M_\lambda \geq 1$ such that $\|(A - \lambda)^* \, x_n\| \leq M_\lambda \|(A - \lambda)x_n\|$, hence $\lim_n (A - \lambda)^* x_n = 0$, hence $\lambda \in \sigma_{ar}(A)$, that is, $\sigma_a(A) = \sigma_{ar}(A)$ and since $\partial\sigma(A) \subset \sigma_a(A)$, then $\sigma_{ar}(A) \neq \emptyset$. In the two cases we have $A \in \overline{\mathfrak{R}_1}$ by Lemma 1. ∎

LEMMA 4. *If there exists* $\lambda \in \mathbb{C}$ *such that* $\|A - \lambda I\| = r(A - \lambda I)$ *(the spectrum radius of* $(A - \lambda I)$*), then* $\lambda \in \partial\mathfrak{W}(A) \cap \sigma(A)$.

Proof. It suffices to consider the case where $\|A\| = r(A)$. If $\|A\| = r(A)$, then there exists $\lambda \in \sigma(A) : |\lambda| = \|A\|$; $\lambda \notin int\mathfrak{W}(A)$. Since $\sigma(A) \subset \overline{\mathfrak{W}(A)}$; $\partial\mathfrak{W}(A) = \overline{\mathfrak{W}(A)} \cap \overline{\complement\mathfrak{W}(A)}$ and $\mathfrak{W}(A) \subset \mathfrak{D}(0, \|A\|)$, we have $\complement\mathfrak{D}(0, \|A\|) \subset \complement\mathfrak{W}(A)$, where $\complement\mathfrak{W}(A)$ is the complementary of $\mathfrak{W}(A)$ and $\overline{\complement\mathfrak{D}(0, \|A\|)} = \{\lambda : \ |\lambda| \geq \|A\|\}$, so that $\|A\| \in \overline{\complement\mathfrak{D}(0, \|A\|)}$, then $\|A\| \in \overline{\complement\mathfrak{W}(A)}$ and we have $\|A\| \in \mathfrak{W}(A)$, that is, $\|A\| \in \partial\mathfrak{W}(A) \cap \sigma(A)$. ∎

REMARK 1. Theorem 3 generalizes the result given by P.R.Halmos [2], which asserts that if, $\|A - \lambda I\| = r(A - \lambda I)$, then $A \in \overline{\mathfrak{R}_1}$.

DEFINITION 1. An operator $A \in \mathfrak{L}(H)$ satisfies the condition $\mathfrak{C}$ if,

$$\|(A - \lambda I)^{-1}\| \leq [\operatorname{dist}(\lambda, \operatorname{co}\sigma(A))]^{-1}; \ \forall\lambda \notin \operatorname{co}\sigma(A),$$

where $\operatorname{dist}(\lambda, \operatorname{co}\sigma(A)\ )$ is the distance from λ to $\operatorname{co}\sigma(A)$.

THEOREM 5. $\overline{\mathfrak{R}_1}$ *contains the following operators*

1) $\|A\| = w(A)$, *where* $w(A)$ *is the numerical radius of* A.
2) $A \in \mathfrak{L}(H)$ *such that* A *satisfies* $\mathfrak{C}$.

Proof. 1) Let $\mathfrak{M}$ be the set $\{\lambda \in \overline{\mathfrak{W}(A)} : |\lambda| = \|A\|\}$. Since $\mathfrak{M} \subset \partial\mathfrak{W}(A)$ and $\partial\mathfrak{W}(A) \cap \sigma(A) \subset \sigma_{ar}(A)$. It suffices to prove that $\mathfrak{M} \subset \sigma_{ar}(A)$. Suppose that $\|A\| \in \mathfrak{M}$, let (x_n) be a normed sequence such that $\|A\| = \lim_n (Ax_n \mid x_n)$. Since,

$$|(Ax_n \mid x_n)| \leq \|Ax_n\| \leq \|A\|$$

we have

$$\lim_n \|Ax_n\| = \|A\|.$$

Then,

$$\begin{aligned}\lim_n \|(A - \|A\|)\, x_n\|^2 &= \lim_n ((A - \|A\|)x_n \mid (A - \|A\|)x_n) \\ &= \lim_n \Big(\|Ax_n\|^2 + \|A\|^2 - \|A\|\,(Ax_n \mid x_n) - \|A\|\,(x_n \mid Ax_n)\Big) \\ &= \|A\|^2 + \|A\|^2 - \|A\|^2 - \|A\|^2 = 0.\end{aligned}$$

So, $\|A\| \in \sigma_a(A) \subset \sigma(A)$, that is, $\|A\| \in \sigma_{ar}(A)$. Hence it suffices to apply Lemma 1.

2) Since the equality

$$\overline{\mathfrak{W}(A)} = \operatorname{co}\sigma(A)\ (*)$$

implies $\partial\mathfrak{W}(A) \cap \sigma(A) \neq \emptyset$, it suffices to prove that (*) is a consequence of $\mathfrak{C}$. In other words, since $\overline{\mathfrak{W}(A)} \supset \operatorname{co}\sigma(A)$ we prove that if, $\lambda \notin \operatorname{co}\sigma(A)$ then, $\lambda \notin \overline{\mathfrak{W}(A)}$. By applying the transformation

$$A \mapsto \alpha A + \beta$$

we can suppose that $[\lambda < 0, 0 \in \operatorname{co}\sigma(A) \ \subset \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}]$. For every $\forall c < 0$ the estimate

$$\operatorname{dist}(c, \operatorname{co}\sigma(A)) \geq |c|$$

implies

$$\left\|(A - c)^{-1}\right\| \leq |c|^{-1},$$

so

$$c^2 \|x\|^2 \leq ((A - c)\, x \mid (A - c)\, x).$$

This implies (after letting c tend to minus infinity) that

$$(Ax \mid x) + (x \mid Ax) \geq 0.$$

Hence,

$$\overline{W(A)} \subset \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\},$$

that is, $\lambda \notin \overline{W(A)}$. It results that $\partial\mathfrak{W}(A) \cap \sigma(A) \neq \emptyset$. Then, it results from Theorem 3 that $A \in \overline{\mathfrak{R}_1}$. ∎

Corollary 6. *$\mathfrak{J}(H)$ contains the following operators:*

1) *$A \in \mathfrak{L}(H)$ such that $\partial\mathfrak{W}(A) \cap \sigma(A) \neq \emptyset$*
2) *dominant operators*
3) *$A \in \mathfrak{L}(H)$ such that $\mathfrak{M} \neq \emptyset$*
4) *$A \in \mathfrak{L}(H)$ such that A satisfies $\mathfrak{C}$.*

Proof. In these cases we have $A \in \overline{\mathfrak{R}_1}$ and since $\overline{\mathfrak{R}_n} \subset \mathfrak{F}(H); n \geq 1$, then $A \in \mathfrak{F}(H)$. ∎

COROLLARY 7. *Every quasinilpotent operator A such that* $\operatorname{Re} A \geq 0$ *belongs to* $\overline{\mathfrak{R}_1}$.

Proof. It follows immediatly by Lemma 2 and Lemma 1. ∎

DEFINITION 2. Let $\mathfrak{U}$ be the set $\{A \in \mathfrak{L}(H) : \ \mathfrak{M} \ \neq \emptyset\}$.

REMARK 2. We can define an element satisfies $\mathfrak{C}$ (resp. belongs to $\mathfrak{U}$) in the C*− algebra $\mathfrak{A}$ by a satisfies $\mathfrak{C}$ (resp. $a \in \mathfrak{U}$) if,

$$\|(a - \lambda e)^{-1}\| \leq [\operatorname{dist}(\lambda, \operatorname{co}\sigma(a))]^{-1}; \forall\lambda \notin \operatorname{co}\sigma(a)$$

(resp. $a \in \mathfrak{U}$ such that $\{\lambda \in \overline{\mathfrak{W}(a)} : |\lambda| = \|a\|\} \neq \emptyset$. An element $a \in \mathfrak{A}$ is called dominant if, there is m_λ such that

$$(a - \lambda)^*(a - \lambda) - m_\lambda^{-2}(a - \lambda)(a - \lambda)^* \geq 0; \forall\lambda \in \mathbb{C}.$$

If $\|A\| = r(A)$, then there exists $\lambda_0 \in \sigma(A)$ such that $|\lambda_0| = \|A\|$ and we have $\lambda_0 \in \sigma(A) \subset \overline{\mathfrak{W}(A)}$, that is, $\mathfrak{M} \neq \emptyset$.

THEOREM 8. *Let a be an element of $\mathfrak{A}$, then $a \in \mathfrak{F}(\mathfrak{A})$ in each of the following cases:*

1) *a dominant*
2) *a satisfies $\mathfrak{C}$*
3) *$a \in \mathfrak{U}$.*

Proof. It is known [3, p.97] that there exists a *-isometric homomorphism φ and a Hilbert space H $(\varphi : \mathfrak{A} \longmapsto \mathfrak{L}(H))$. Then $\varphi(a)$ is dominant, $\varphi(a)$ satisfies $\mathfrak{C}$ and $\varphi(a) \in \mathfrak{U}$. Since φ is isometric it results from Corollary 6 that $a \in \mathfrak{F}(\mathfrak{A})$. ∎

COROLLARY 9. *$\mathfrak{F}(H)$ contains the following operators:*

1) *$T = A + K$, K compact and A dominant*
2) *$T = A + K$, K compact and A satisfies $\mathfrak{C}$*
3) *$T = A + K$, K compact and $A \in \mathfrak{U}$.*

Proof. Since the Calkin algebra $\mathfrak{B}$ is a C*− algebra then $[A] \in \mathfrak{B}$ is dominant, $[A]$ satisfies $\mathfrak{C}$ and $[A] \in \mathfrak{U}$. Hence it follows from Theorem 8 that $[A] \in \mathfrak{F}(\mathfrak{A})$ and we have

$$\|I - TX - XT\| \geq \|[I] - [A][X] - [X][A]\| \geq \|I\| = 1. \ \blacksquare$$

REMARK 3. Since $\mathfrak{U}$ contains the class of operators A such that $\|A\| = r(A)$, hence the above corollary generalizes the result given by J. P. Williams [7] which asserts that $\mathfrak{F}(H)$ contains every operator of the form $T + K$, where $\|T\| = r(T)$ and T is compact.

LEMMA 10. *Every operator which commutes with S, where S is the unilateral or the bilateral shift belongs to $\overline{\mathfrak{R}_1}$.*

Proof. If S is the bilateral shift then, the commutant of S is the algebra of all multiplication operators $\mathfrak{L}^\infty = \{M_\phi : \phi \in \mathfrak{L}^\infty(x, \mu)\}$ [5, p.37]. If S is the unilateral shift then, the commutant of S is the algebra $\mathfrak{T}$ of analytic Toeplitz operators [5, p.37]. In the two cases we have $\|A\| = r(A)$, it results from Lemma 4 and Theorem 3 that $A \in \overline{\mathfrak{R}_1}$ and we can deduce that $\overline{\mathfrak{R}_1}$ contains the algebra of all multiplication operators and the algebra of analytic Toeplitz operators. ∎

COROLLARY 11. *$\mathfrak{F}(H)$ contains the following operators:*

1) *Every quasinilpotent operator such that* $\operatorname{Re} A \geq 0$
2) *The algebra of all multiplication and Toeplitz operators.*

Proof. It follows immediately from Corollary 9 and Lemma 10. ∎

REMARK 4. According to Corollary 6 and Lemma 10 we can affirm that

$$\mathfrak{T} + \mathcal{K}(H) \subset \mathfrak{F}(H), \qquad \mathfrak{L}^\infty + \mathcal{K}\,(H) \subset \mathfrak{F}(H).$$

3. Numerical ranges

Let $\mathcal{A}$ denote a complex Banach algebra with identity e.

Let $\mathfrak{A}$ be a C^* - algebra with identity and let $P(\mathfrak{A})$ be the set

$$\left\{ f \in \mathfrak{A}' : f(I) = \|f\| = 1 \right\}.$$

For $A \in \mathfrak{A}$, the set

$$W_0(A) = \{f(A) : f \in P(\mathfrak{A})\}$$

is said to be the numerical range of A. Let $A \in \mathfrak{A}$ and L_A, R_A defined respectively by $X \mapsto AX$, $X \mapsto XA$. Adapted from J. H. Anderson and C. Foias [1] we have

$$W_0(A) = W_0(L_A) = W_0(R_A), \tag{3.1}$$

and if $\mathfrak{A} = \mathfrak{L}(H)$, then $W_0(A) = \overline{\mathfrak{W}(A)}$.

LEMMA 12. *Let $A \in \mathfrak{L}(H)$, then $\overline{W_0(A)} = \operatorname{co} \sigma(A)$, if and only if,*

$$\|(A - \lambda)^{-1}\| \leq [\operatorname{dist}(\lambda, \operatorname{co} \sigma(A)]^{-1}$$

for all $\lambda \notin \operatorname{co} \sigma(A)$.

$\Rightarrow$) Let $\lambda \notin \operatorname{co}\sigma(A)$. We can suppose that the pair (λ, A) satisfies

(3.2) $\quad [\lambda < 0, 0 \in \operatorname{co}\sigma(A)$ then, $\operatorname{co}\sigma(A) \subset \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}]$

by the transformation

$$A \longmapsto \alpha A + \beta.$$

Let $\overline{W(A)} = \operatorname{co}\sigma(A)$, we prove that for all $\lambda \notin \operatorname{co}\sigma(A)$ we have

$$\|(A-\lambda)^{-1}\| \leq [\operatorname{dist}(\lambda, \operatorname{co}\sigma(A))]^{-1}.$$

It suffices to consider the cases (λ, A) satisfies (3.2). Then for every $x \in H$ we have

$$\|(A-\lambda)x\|^2 = \|Ax\|^2 - \lambda[(Ax \mid x) + (x \mid Ax)] + \lambda^2 \|x\|^2.$$

Since $(A-\lambda)$ is invertible, then for every $x \in H$ we have

$$\|x\|^2 \geq \lambda^2 \|(A-\lambda)^{-1}x\|^2,$$

then

$$|\lambda|^{-1} \geq \|(A-\lambda)^{-1}\|.$$

Or

$$|\lambda| = \operatorname{dist}(\lambda, \operatorname{co}\sigma(A)).$$

$\Leftarrow$) See the proof of Theorem 5. ■

THEOREM 13. *Let $A, B \in \mathfrak{L}(H)$ such that $\|(A-\lambda)^{-1}\| \leq [\operatorname{dist}(\lambda, \operatorname{co}\sigma(A))]^{-1}$ for all $\lambda \notin \operatorname{co}\sigma(A)$ and $\|(B-\lambda)^{-1}\| \leq [\operatorname{dist}(\mu, \operatorname{co}\sigma(B))]^{-1}$ for all $\mu \notin \operatorname{co}\sigma(B)$. Then,*

$$W_0(\delta_{A,B}) = \operatorname{co}\sigma(\delta_{A,B}).$$

Proof. We have

$$W_0(\delta_{A,B}) = \{f(\delta_{A,B}) : f \in P(\mathfrak{L}(\mathfrak{L}(H)))\} = \{f(L_A - R_B) : f \in P(\mathfrak{L}(\mathfrak{L}(H)))\},$$

it results that

$$W_0(\delta_{A,B}) \subseteq \{f(L_A) : f \in P(\mathfrak{L}(\mathfrak{L}(H)))\} - \{g(R_B) : f \in P(\mathfrak{L}(\mathfrak{L}(H)))\} = W_0(L_A) - W_0(R_B).$$

Then, it follows by (3.1) that

$$W_0(L_A) = W_0(A), W_0(R_B) = W_0(B)$$

by applying Lemma 12 we obtain $W_0(\delta_{A,B}) \subseteq W_0(A) - W_0(B) = \operatorname{co}\sigma(A) - \operatorname{co}\sigma(B) = \operatorname{co}(\sigma(A) - \sigma(B)) = \operatorname{co}(\sigma(\delta_{A,B}))$.

Since $\sigma(\delta_{A,B}) \subset W_0(\delta_{A,B})$, then $\operatorname{co}(\sigma(\delta_{A,B})) \subset W_0(\delta_{A,B})$. Hence

$$W_0(\delta_{A,B}) = \operatorname{co}\sigma(\delta_{A,B}). \blacksquare$$

LEMMA 14. *Let $A \in \mathfrak{L}(H)$ be hyponormal then,*

$$\left\|(A-\lambda)^{-1}\right\| \leq [\operatorname{dist}(\lambda, \operatorname{co}\sigma(A))]^{-1} \textit{ for all } \lambda \notin \operatorname{co}\sigma(A).$$

This lemma is a very well-known result in an even stronger version, estimating the norm of the resolvant by the inverse of the distance from λ to the spectrum of a hyponormal operator contain it.

COROLLARY 15. *If A, B are hyponormal operators, then*

$$W_0(\delta_{A,B}) = \operatorname{co}\sigma(\delta_{A,B}).$$

Proof. It suffices to apply Lemma 14. ∎

REMARK 5. The inclusions relating to the classes of nonnormal operators listed above are follows: Normal⊂Quasinormal⊂Subnormal⊂Hyponormal; the above inclusions are all proper, if A is hyponormal operator then, $\|(A-\lambda)^{-1}\| \leq [\operatorname{dist}(\lambda, \operatorname{co}\sigma(A))]^{-1}$ for all $\lambda \notin \operatorname{co}\sigma(A)$, by the above lemma. This fact shows that our result generalizes ([2], Th. 5.7) to certain non normal cases.

Let $\mathcal{A}$ denote a complex Banach algebra with identity e. Let $A \in \mathcal{A}$. Then $0 \in W_0(A)$ if, and only if, $|\lambda| \leq \|A-\lambda\|$ (*) for all $\lambda \in \mathbb{C}$, (see [8]). Write $AX - XA$ instead of A in (*) and choose $\lambda = 1$, that is, A is finite if, and only if, $0 \in W_0(AX - XA)$. This fact shows that there is a relation between finite operators and the numerical range of a derivation. In [8], J. P. Williams shows that, if $a \in \mathcal{A}$, then the following statements are equivalent:

LEMMA 16 [8]. *For $a \in \mathcal{A}$, the following statements are equivalent:*

(1) $0 \in W_0(ax - xa)$, *for all $x \in \mathcal{A}$*
(2) $\|ax - xa - e\| \geq 1$, *for all $x \in \mathcal{A}$*
(3) *there exists a state f such that $f(ax) = f(xa)$, for all $x \in \mathcal{A}$.*

4. Generalized finite operators

The present paper initiates a study of a more general class of finite operators defined by

$$\mathcal{GF} = \{(A,B) \in \mathcal{L}(\mathcal{H}) \times \mathcal{L}(\mathcal{H}) : \|I - (AX - XB)\| \geq 1, \text{ for each } X \in \mathfrak{L}(\mathcal{H})\}.$$

We call such operators generalized finite operators.

THEOREM 17. *$\mathcal{GF}(\mathcal{A})$ is closed in $\mathcal{A}$.*

Proof. Let $(a_n)_n, (b_n)_n$ $n \in \mathbb{N}^*$ be two Cauchy's sequences in $\mathcal{GF}(\mathcal{A})$, converge respectively to a and to b. We have for all $x \in \mathcal{A}$

$$1 \leq \|e - (a_n x - x b_n)\| \leq \|e - (ax - xb)\| + \|a_n - a\|\,\|x\| + \|b_n - b\|\,\|x\|.$$

Hence for all $\varepsilon > 0$, by the choice of $n > N_\varepsilon$, we get

$$1 - 2\varepsilon \|x\| \leq \|e - (ax - xb)\|,$$

then

$$\|e - (ax - xb)\| \geq 1,$$

hence $a, b \in \mathcal{GF}(\mathcal{A})$. ∎

THEOREM 18. *For $a, b \in \mathcal{A}$ the following statements are equivalent:*

(i) $\|ax - xb - e\| \geq 1$, *for all* $x \in \mathcal{A}$

(ii) *there exists a state f such that $f(ax) = f(xb)$, for all $x \in \mathcal{A}$*

(iii) $0 \in W_0(ax - xb)$, *for all* $x \in \mathcal{A}$.

Proof. (i)⇒(ii): By hypothesis, $\overline{R(\delta_{a,b})} \cap \{e\} = \emptyset$, we define $f_0 \in G^*$, where $G = \overline{R(\delta_{a,b})} \otimes \{e\}$, by $f_0(e) = 1$ and $f_0(R(\delta_{a,b}) = 0$. Since for $y = e - (ax - xb)$, $f_0(y) = 1$ and $\|y\| \geq 1$, we have $f_0(y) \leq \|y\|$, hence $\|f_0\| = 1$.

According to the Hahn Banach theorem f_0 can be prolonged on an element f of $\mathcal{P} = \{f \in \mathcal{A}^* : f(e) = 1 = \|f\|\}$, satisfying (ii).

(ii)⇒(i): If f satisfies (ii), then $f(e - (ax - xb)) = f(e) = 1$ and

$$1 \leq \|f\| \, \|e - (ax - xb)\| = \|e - (ax - xb)\|,$$

hence (i) is fulfilled.

(iii)⇒(i): For an arbitrary $x \in \mathcal{A}$, let $f_x \in \mathcal{P}$ such that $f_x(ax - xb) = 0$, then

$$1 = f_x(e - (ax - xb)) \leq \|f_x\| \, \|e - (ax - xb)\| = \|e - (ax - xb)\|.$$

Hence (i) is fulfilled. ∎

Finally we recall the following open questions of L. A. Fialkow, D. Herrero and J. P. Williams:

1. Is $\overline{\mathfrak{R}_n} = \mathfrak{F}(H)$?

2. For all T is there an operator A similar to T such that $A \in \mathfrak{F}(H)$?

3. Is $\operatorname{dist}(I, R(\delta_A)) = r \in]0, 1[$ implies $\operatorname{dist}(I, R(\delta_{SAS^{-1}})) = r$ for all S invertible?

At the end of this paper I would like to thank the referee for his useful remarks.

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