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INTEGRAL OF FUNCTION WITH VALUES IN COMPLETE MODULAR SPACE

Abstract. The theorem on existence of an integral of a function with values in a modular space and some fundamental properties of this integral are given.

Let X be a real vector space. A functional $\rho : X \to \overline{R}_+$, where $\overline{R}_+ = [0, +\infty]$, is called a *convex pseudomodular* on X if $\rho(0) = 0$, $\rho(-u) = \rho(u)$ and $\rho(\alpha u + \beta v) \le \alpha\rho(u) + \beta\rho(v)$ for all $u, v \in X$ and $\alpha, \beta \ge 0$, $\alpha + \beta = 1$. If, additionally, $\rho(u) = 0$ only for $u = 0$, then ρ is called *a convex modular* on X. The vector space $X_\rho = \{u \in X : \rho(au) < \infty \textit{ for some } a > 0\}$ is called *the modular space* generated by ρ. Examples of modular spaces, e.g. Orlicz spaces, may be found in [3].

A sequence (u_k) of elements of X_ρ is called *modular convergent to* u, $u \in X_\rho$, if there exists a $\lambda > 0$ such that $\rho(\lambda(u_k - u)) \to 0$ as $k \to \infty$. A sequence (u_k) is called *a Cauchy sequence in* X_ρ, if $\rho(\lambda(u_k - u_l)) \to 0$ as $k, l \to \infty$, for some $\lambda > 0$.

The modular space X_ρ is called *complete* if every Cauchy sequence in X_ρ is convergent in X_ρ. In the following by X_ρ we shall mean a complete modular space.

We assume henceforth that Ω is a non-empty abstract set and let (Ω, Σ, μ) be a finite measure space with a complete and positive measure on Σ.

Let $\{B_1, B_2, \ldots, B_k\}$ be a finite collection of mutually disjoint, Σ - measurable subsets of Ω and let $\{c_1, c_2, \ldots, c_k\}$ be a corresponding collection of points of X_ρ. The mapping f on Ω into X_ρ defined by

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$$f(x) = \sum_{i=1}^{k} c_i \chi_{B_i}(x),$$

where χ_B is the characteristic function of B, is called *a simple function.*

An arbitrary function f defined almost everywhere on Ω into X_ρ, is said to be *a $\rho - \Sigma-$measurable function* (*briefly ρ - measurable*) if there exists a sequence (f_n) of simple functions such that

(1) $\rho(\lambda_1(f_n(x) - f(x))) \to 0$ as $n \to \infty$ for some $\lambda_1 > 0$

and for almost all $x \in \Omega$ and

(2) $\rho(\lambda_2 f_n(x)) \to \rho(\lambda_2 f(x))$ as $n \to \infty$ for some $\lambda_2 > 0$

and for almost all $x \in \Omega$.

Let us remark that if $\lambda_2 \geq \lambda_1$, then the constants λ_1 and λ_2 may be taken identical.

LEMMA 1. *Let ρ be a convex pseudomodular in a real vector space X. If $u, v \in X_\rho$ and $\rho(v) < \infty$, then for arbitrary α such that $0 < \alpha \leq 1$ the inequality*

$$\rho(u) - \rho(v) \leq \frac{1}{2}\alpha\rho\left(\frac{2}{\alpha}(u - v)\right) + \frac{1}{2}\alpha\rho(2v)$$

holds.

Proof. Let $0 < \alpha \leq 1$ and $\beta = 1 - \alpha$. By convexity of ρ we have

$$\rho(u) \leq \alpha\rho\left(\frac{u}{\alpha} - \frac{\beta}{\alpha}v\right) + \beta\rho(v).$$

Hence

$$\rho(u) - \rho(v) \leq \alpha\rho\left(\frac{1}{\alpha}(u - v) + v\right) \leq \frac{1}{2}\alpha\rho\left(\frac{2}{\alpha}(u - v)\right) + \frac{1}{2}\alpha\rho(2v).$$

The following statement shows that if the condition (1) holds for every $\lambda > 0$, then (2) follows from (1).

PROPOSITION. *If $\rho(cu) < \infty$ for some $c > 0$ and $\rho(a(u_n - u)) \to 0$ as $n \to \infty$ for every $a > 0$, then there exists a constant $a_1 > 0$ such that $\rho(a_1 u_n) \to \rho(a_1 u)$ as $n \to \infty$.*

Proof. Let $\epsilon > 0$ be arbitrary and let $\rho\left(\frac{1}{4}cu_n\right) \geq \rho\left(\frac{1}{4}cu\right)$. Let $\alpha \in (0, 1]$ be so small that $\frac{1}{4}\alpha\rho(cu) < \frac{1}{2}\epsilon$. For given α

$$\frac{1}{2}\alpha\rho\left(\frac{c}{2\alpha}(u_n - u)\right) < \frac{1}{2}\epsilon$$

holds for sufficiently large n. Thus, by Lemma 1, we have

$$0 \leq \rho\left(\frac{1}{4}cu_n\right) - \rho\left(\frac{1}{4}cu\right) \leq \frac{1}{2}\alpha\rho\left(\frac{c}{2\alpha}(u_n - u)\right) + \frac{1}{4}\alpha\rho(cu) < \epsilon$$

for sufficiently large n and $\alpha \in (0, 1]$.

Let now $\rho\left(\frac{1}{4}cu_n\right) < \rho\left(\frac{1}{4}cu\right)$. Then

$$0 < \rho\left(\frac{1}{4}cu\right) - \rho\left(\frac{1}{4}cu_n\right) \leq \frac{1}{2}\alpha\rho\left(\frac{c}{2\alpha}(u - u_n)\right) + \frac{1}{2}\alpha\rho(cu_n).$$

For the sequence $\left(\rho\left(\frac{1}{2}cu_n\right)\right)$ there exists a constant $M > 0$ such that

$$\rho\left(\frac{1}{2}cu_n\right) \leq \rho(c(u_n - u)) + \rho(cu) \leq M$$

for every n. Hence, we obtain $0 < \rho\left(\frac{1}{4}cu\right) - \rho\left(\frac{1}{4}cu_n\right) < \epsilon$ for sufficiently small $\alpha \in (0, 1]$ and sufficiently large n. Finally, we have $\left|\rho\left(\frac{1}{4}cu_n\right) - \rho\left(\frac{1}{4}cu\right)\right| < \epsilon$ for sufficiently large n. ∎

Let $f : \Omega \to X_\rho$ be a simple function. The $\rho -$ *Bochner integral of* f is defined by

$$\int_\Omega f(x)d\mu = \sum_{i=1}^{k} c_i\mu(B_i).$$

Immediately from above definition it follows

LEMMA 2. (i) *Let* $f : \Omega \to X_\rho$ *be a simple function. Then*

$$\rho\left(\int_\Omega f(x)d\mu\right) \leq \frac{1}{c}\int_\Omega \rho(cf(x))d\mu,$$

where $c = \mu(\Omega)$.

(ii) *Let* f, g *be two simple functions*, $\alpha, \beta \in R$. *Then* $\alpha f + \beta g$ *is also a simple function and*

$$\int_\Omega (\alpha f(x) + \beta g(x))d\mu = \alpha \int_\Omega f(x)d\mu + \beta \int_\Omega g(x)d\mu.$$

A function $f : \Omega \to X_\rho$ is said to be $\rho -$ *Bochner integrable* if there exists a sequence (f_n) of simple functions satisfying (1) and (2) such that

$$\lim_{n\to\infty} \int_\Omega \rho(\lambda_3(f_n(x) - f(x)))d\mu = 0 \qquad \textit{for some} \quad \lambda_3 > 0, \tag{3}$$

$$\lim_{n,m\to\infty} \int_\Omega \left|\rho(\lambda_2 f_n(x)) - \rho(\lambda_2 f_m(x))\right| d\mu = 0, \tag{4}$$

where the constant λ_2 is the same as in (2).

For any set $B \in \Sigma$ and for any ρ-measurable and integrable f we define the *ρ – Bochner integral of f over B* by

$$\int_B f(x)d\mu = \lim_{n\to\infty} \int_\Omega \chi_B(x) f_n(x) d\mu.$$

The above limit exists and its value is independent of the approximating sequence of simple functions (f_n).

Let us denote

$$s_n = \int_\Omega \chi_B(x) f_n(x) d\mu.$$

The existence of the limit follows from the inequality

$$\rho\left(\frac{\lambda_3}{2c}(s_n - s_m)\right) \le \frac{1}{2c}\int_B \rho(\lambda_3(f_n(x) - f(x)))d\mu + \frac{1}{2c}\int_B \rho(\lambda_3(f_m(x) - f(x)))d\mu$$

where $c = \mu(\Omega)$, and from the completeness of X_ρ. The constant λ_3 is chosen as in (3).

THEOREM 1. *Let ρ be a convex modular on X. A ρ-measurable function $f : \Omega \to X_\rho$ is ρ-Bochner integrable if and only if the function $\rho(cf(x))$ is μ-intergrable with the constant c from* (2).

Proof. Since f is ρ-measurable there exists a sequence (f_n) of simple functions satisfying (1) and (2). Define functions y_n, $n = 1, 2, \ldots$, on Ω by

$$y_n(x) = \begin{cases} f_n(x) & \text{if} \quad x \in A_n \\ 0 & \text{otherwise} \end{cases},$$

where $A_n = \{x \in \Omega : \rho(\lambda_2 f_n(x)) \le 2\rho(\lambda_2 f(x))\}$. Obviously every y_n is simple. We put $B = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A'_k$, where $A'_k = \Omega - A_k$. Then we have $\mu(B) = 0$, (see [2]). Hence $\rho(\lambda_1(y_n(x) - f(x))) \to 0$ as $n \to \infty$ almost everywhere on Ω and $\rho(\lambda_2 y_n(x)) \to \rho(\lambda_2 f(x))$ as $n \to \infty$ almost everywhere on Ω.

Moreover, if $\lambda_1 \le \lambda_2$ then

$$\rho\left(\frac{1}{2}\lambda_1(y_n(x) - f(x))\right) \le \rho(\lambda_2 y_n(x)) + \rho(\lambda_2 f(x)) \le 3\rho(\lambda_2 f(x))$$

for almost all $x \in \Omega$. The dominated convergence theorem yields

$$\lim_{n\to\infty} \int_\Omega \rho\left(\frac{1}{2}\lambda_1(y_n(x) - f(x))\right) d\mu = 0.$$

In the case $\lambda_1 > \lambda_2$ the constants in (1) and (2) may be taken identical puting the smaller of them. Then

$$\rho\left(\frac{1}{2}\lambda_2(y_n(x) - f(x))\right) \leq 3\rho(\lambda_2 f(x))$$

and

$$\lim_{n\to\infty} \int_\Omega \rho\left(\frac{1}{2}\lambda_2 y_n(x) - f(x))\right) d\mu = 0.$$

Arguing in a similar manner we may easily show that

$$\lim_{m,n\to\infty} \int_\Omega |\rho(\lambda_2 y_n(x)) - \rho(\lambda_2 y_m(x))|\, d\mu = 0.$$

Thus f is ρ-Bochner integrable.

Let the function f be ρ-Bochner integrable and let (f_n) be a sequence of simple functions with properties (1) - (4). Let us denote $z_n(x) = \rho(\lambda_2 f_n(x))$. The functions z_n are simple and the sequence (z_n) is convergent to $\rho(\lambda_2 f(x))$ for almost all $x \in \Omega$. Applying (4) we obtain for any fixed $\epsilon > 0$

$$\left|\int_\Omega z_n(x)d\mu - \int_\Omega z_m(x)d\mu\right| < \epsilon$$

for sufficiently large m, n. Hence $\rho(\lambda_2 f(x))$ is μ- integrable over Ω. ∎

Let us consider the modular space X_ρ with Luxemburg norm $\| \|_\rho$ generated by the convex modular ρ. We shall investigate the connection between Bochner integral of f and ρ-Bochner integral of f. We shall show that if the function $f : \Omega \to X_\rho$ is strongly measurable and Bochner integrable, then f is also ρ-measurable and ρ-Bochner integrable on Ω.

Let us suppose that the function f is Bochner integrable. Then there exists a sequence (f_n) of simple functions, $f_n : \Omega \to X_\rho$, such that

(5) $$\lim_{n\to\infty} \|f_n(x) - f(x)\|_\rho = 0 \quad \text{for almost all} \quad x \in \Omega,$$

and

(6) $$\lim_{n\to\infty} \int_\Omega \|f_n(x) - f(x)\|_\rho d\mu = 0.$$

In virtue of (5),

(7) $$\rho(\lambda(f_n(x) - f(x))) \to 0 \quad \text{almost everywhere in } \Omega \text{ for every } \lambda > 0.$$

Let $0 < \epsilon \leq 1$. By (5) we have $\|\lambda(f_n(x) - f(x))\|_\rho < \epsilon \leq 1$ almost everywhere in Ω for every $\lambda > 0$ and $n > N(\lambda, x)$. Hence

$$\rho(\lambda(f_n(x) - f(x))) \leq \|\lambda(f_n(x) - f(x))\|_\rho$$

almost everywhere in Ω for sufficiently large n. Integrating above inequality over Ω and applying (6) we obtain

$$\lim_{n\to\infty} \int_\Omega \rho(\lambda(f_n) - f(x)))d\mu = 0 \quad \text{for every} \quad \lambda > 0.$$

In order to prove (2) let us first show that for every strongly measurable function f there exists a constant $d > 0$ such that $\rho(df(x)) < \infty$ for almost all $x \in \Omega$. Let $g : \Omega \to X_\rho$ be a simple function, $g(x) = \sum_{i=1}^{k} c_i \chi_{B_i}(x)$, where $\{c_1, c_2, \dots, c_k\} \subset X_\rho$. For every i, $i = 1, 2, , \dots, k$, there exists $\lambda_i > 0$ such that $\rho(\lambda_i c_i) < \infty$. Let $a = \min\{\lambda_1, \lambda_2, \dots, \lambda_k\}$. Then

$$\rho(ac_i) < \infty \quad \text{for} \quad i = 1, 2, \dots, k. \tag{9}$$

Hence $\int_\Omega \rho(ag(x))d\mu = \sum_{i=1}^{k} \rho(ac_i)\mu(B_i) < \infty$. We conclude: there exists a constant $a > 0$, independent on x, such that the real function $\rho(ag(x))$ is μ-integrable on Ω.

For arbitrary $n_0 \in N$, by (9), we have that

$$\rho(cf_{n_0}(x)) < \infty \quad \text{for some} \quad c > 0 \quad \text{and every} \quad x \in \Omega. \tag{10}$$

Thus, by (7) and (10), we obtain

$$\rho\left(\frac{1}{2}cf(x)\right) \leq \frac{1}{2}\rho(c(f(x) - f_{n_0}(x))) + \frac{1}{2}\rho(cf_{n_0}(x)) < \infty \tag{11}$$

for allmost all $x \in \Omega$. Therefore, putting $d = \frac{1}{2}c$, we have

$$\rho(df(x)) < \infty \quad \text{for almost all} \quad x \in \Omega. \tag{12}$$

In virtue of *Proposition*, by (7) and (12), we obtain

$$\rho\left(\frac{1}{8}cf_n(x)\right) \to \rho\left(\frac{1}{8}cf(x)\right) \text{ as } n \to \infty, \text{ almost everywhere in } \Omega. \tag{13}$$

Thus f is ρ-measurable

Replacing the constant $\frac{1}{2}c$ by $\frac{1}{8}c$ in (11) and integrating this inequality over Ω, we have by (8)

$$\int_\Omega \rho\left(\frac{1}{8}cf(x)\right) d\mu \leq \int_\Omega \rho\left(\frac{1}{4}c(f(x) - f_{n_0}(x))\right) d\mu + \int_\Omega \rho\left(\frac{1}{4}cf_{n_0}(x)\right) d\mu < \infty.$$

Thus the real function $\rho\left(\frac{1}{8}cf(x)\right)$ is μ-integrable on Ω. Hence, by *Theorem 1*, f is ρ-Bochner integrable on Ω.

Now, let us take into account the following example. Let us consider the Lebesgue space $L^p(0,1)$, $p \geq 1$ as the Orlicz space L^ϕ where $\phi(s) = \frac{1}{p}s^p$, $p \geq 1$. Then obviously $L^\phi(0,1)$, $p \geq 1$ is a modular space generated by the modular $\rho(u) = \frac{1}{p}\int_0^1 |u(t)|^p dt$. Let $\| \ \|_\rho$ be a Luxemburg norm generated by ρ. Then we can express the norm $\| \ \|_\rho$ in the following form $\|u\|_\rho = \left(\frac{1}{p}\right)^{\frac{1}{p}} \|u\|_p$, where $\|u\|_p = \left(\int_0^1 |u(t)|^p dt\right)^{\frac{1}{p}}$. If $f : \Omega \to L^p(0,1)$ is ρ-integrable, then $\rho(f_n(x) - f(x)) \to 0$ and $\int_\Omega \rho(f_n(x) - f(x))d\mu \to 0$ as $n \to \infty$ almost everywhere in Ω for some sequence (f_n) of simple functions. The first condition is equivalent to $\|f_n(x) - f(x)\|_\rho \to 0$ as $n \to \infty$ a.e. in Ω and for the second of them we have

$$\int_\Omega \rho(f_n(x) - f(x))d\mu = \int_\Omega \|f_n(x) - f(x)\|_\rho^p d\mu.$$

This equality implies that

$$\int_\Omega \|f_n(x) - f(x)\|_\rho d\mu \to 0 \quad \text{as} \quad n \to \infty \text{ a.e. in } \Omega.$$

Thus f is a strongly measurable and Bochner integrable.

Further elementary facts about the ρ-Bochner integral are collected bellow.

THEOREM 2. *If f is a ρ-Bochner integrable function, then*

(i) $\lim\limits_{\mu(A)\to 0} \int_A f(x)d\mu = 0,$

(ii) *if (A_i) is a sequence of pairwise disjoint members of Σ and $A = \cup_{i=1}^\infty A_i$, then* $\int_A f(x)d\mu = \sum_{i=1}^\infty \int_{A_i} f(x)d\mu.$

P r o o f. (i) Let (f_n) be a sequence of simple functions. Let us denote

$$\nu(A) = \int_A f(x)d\mu \quad \text{and} \quad \nu_n(A) = \int_A f_n(x)d\mu \quad \text{for any} \quad A \in \Sigma.$$

There exists the constant $a_1 > 0$ such that for every $\epsilon > 0$

$$\int_\Omega \left(\frac{1}{2}a_1(f_n(x) - f(x))\right) d\mu < \epsilon \quad \text{for} \quad n > n_0.$$

We can find the constant $a_2 > 0$ such that the real functions $\rho(a_2 f_n(x))$, $n = 1, 2, \ldots, n_0$ are μ-integrable. Thus, we can choose $\delta > 0$, that

$$\int_A \rho(a_2 f_n(x))d\mu < \epsilon \quad \text{for every} \quad A \in \Sigma \quad \text{with} \quad \mu(A) < \delta \tag{14}$$

and $n = 1, 2, \ldots, n_0$. Taking $a = \min\{a_1, a_2\}$, we have that for $n > n_0$

$$\int_A \left(\frac{1}{4} a f_n(x)\right) d\mu \leq \frac{1}{2} \int_A \rho\left(\frac{1}{2} a(f_n(x) - f_{n_0}(x))\right) d\mu + \frac{1}{2} \int_A \rho\left(\frac{1}{2} a f_{n_0}(x)\right) d\mu < \epsilon \tag{15}$$

provided $\mu(A) < \delta$. Combining (14) and (15) we obtain that for every n

$$\int_A \rho\left(\frac{1}{4} a f_n(x)\right) d\mu < \epsilon \quad \text{provided} \quad \mu(A) < \delta. \tag{16}$$

Since, for $c = \frac{a}{4\mu(\Omega)}$, the following inequality

$$\rho(c\nu_n(A)) \leq \frac{1}{\mu(\Omega)} \int_A \rho\left(\frac{1}{4} a f_n(x)\right) d\mu$$

holds. Then, in virtue of (16), the functions ν_n are uniformly absolutely continuous.

From the definition of ρ-Bochner integral and the inequality

$$\rho(\nu(A)) \leq \frac{1}{2}\rho\left(2\left(\int_A f(x)d\mu - \int_A f_n(x)d\mu\right)\right) + \frac{1}{2}\rho(2\nu_n(A))$$

it follows that there exists a constant $b > 0$ such that $\rho(b\nu(A)) < \epsilon$ provided $\mu(A) < \delta$.

(ii) Let us show first that for any fixed n, we have

$$\nu_n\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \nu_n(A_i). \tag{17}$$

For arbitrary set $B \in \Sigma$ and $\epsilon > 0$ by Lemma 2 and (3) we have

$$\rho\left(\frac{\lambda_3}{\mu(\Omega)}(\nu_n(B) - \nu_m(B))\right) \leq \frac{1}{\mu(\Omega)} \int_B \rho(\lambda_3(f_n(x - f_m(x))))d\mu < \epsilon$$

for $n, m > n_0$. Hence, by completeness of the space X_ρ, the sequence $(\nu_n(B))$ is uniformly convergent to $\nu(B)$ with respect to $B \in \Sigma$. Thus, there exists a constant $a > 0$ such that

$$\rho(a(\nu(A) - \nu_n(A))) < \epsilon \quad \text{and} \quad \rho\left(a\left(\nu_n\left(\bigcup_{i=1}^{k} A_i\right) - \nu\left(\bigcup_{i=1}^{k} A_i\right)\right)\right) < \epsilon \tag{18}$$

for sufficiently large n.

Arguing in a similar manner as in the first part of the proof of the thesis (i), we obtain that there exists a constant $c > 0$ such that

$$\int_A \rho(cf_n(x))d\mu < \infty$$

for every n. Therefore the series $\sum_{i=1}^{\infty} \int_{A_i} \rho(cf_n(x))d\mu$ is convergent and consequently

$$\sum_{i=k+1}^{\infty} \int_{A_i} \rho(cf_n(x))d\mu \to 0 \quad \text{as} \quad k \to \infty. \tag{19}$$

It follows from (17) that

$$\rho\Big(\nu_n(A) - \sum_{i=1}^{k} \nu_n(A_i)\Big) \leq \frac{1}{\mu(\Omega)} \sum_{i=k+1}^{\infty} \int_{A_i} \rho(\mu(\Omega) f_n(x))d\mu.$$

Hence, by (19), we get that for $c_1 = \frac{c}{\mu(\Omega)}$

$$\rho\Big(c_1\Big(\nu_n(A) - \sum_{i=1}^{k} \nu_n(A_i)\Big)\Big) < \epsilon \quad \text{for sufficiently large } k. \tag{20}$$

By convexity of ρ, we have that for every pair of positive integers k and n

$$\begin{aligned}\rho\Big(\nu(A) - \sum_{i=1}^{k} \nu(A_i)\Big) \leq &\frac{1}{3}\rho(3(\nu(A) - \nu_n(A))) \\ &+ \frac{1}{3}\rho\Big(3\Big(\nu_n(A) - \sum_{i=1}^{k} \nu_n(A_i)\Big)\Big) \\ &+ \frac{1}{3}\rho\Big(3\Big(\sum_{i=1}^{k} \nu_n(A_i) - \sum_{i=1}^{k} \nu(A_i)\Big)\Big).\end{aligned}$$

Finally, let $b = \frac{1}{3}\min\{a, c_1\}$. Then, by (18) and (20), we have, for sufficiently large positive integers k

$$\rho\Big(b\Big(\nu(A) - \sum_{i=1}^{k} \nu(A_i)\Big)\Big) < \epsilon.$$

This completes the proof of the theorem. ■

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