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ON MATRICES OF MAZUR TYPE ON RATE SPACES

1. Preliminaries

Mazur [6] showed that Cesáro metrices of all positive orders are of type M. Later Hill [5] established that under certain conditions the Nörlund, Hausdorff and Weighted Mean are of type M. Chandrasekhara Rao proved that the Abel and the Borel methods are also of type M. General properties of methods of type M are discussed in Wilansky's book [8]. The present paper is devoted to matrix methods of type M in respect of rate spaces and related topics.

A sequence with n-th term x_n is denoted by (x_n) or x.

Let $\mathbb{C}$ denote the set of all complex numbers.

Let $\pi = (\pi_n)$ be a sequence of positive numbers. Let

$$m_\pi = \left\{(x_n) : \left(\frac{x_n}{\pi_n}\right) \text{is bounded}\right\}; \quad m = \{\text{all bounded sequences}\};$$

$$c_\pi = \left\{(x_n) : \lim_{n\to\infty} \frac{x_n}{\pi_n} \text{exists}\right\}; \quad c = \{\text{all convergent sequences}\};$$

$$c_{0\pi} = \left\{(x_n) : \lim_{n\to\infty} \frac{x_n}{\pi_n} = 0\right\}; \quad c_0 = \{\text{all null sequences}\};$$

$$L = \left\{\text{all those sequences } \{x_n\} \text{ such that } \sum_{n=1}^{\infty} |x_n| < \infty\right\}.$$

Note that m_π, c_π and $c_{0\pi}$ are BK-spaces with the norm

$$\|x\| = \sup_{(n)} |x_n \ \pi_n|.$$

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Also $c_{0\pi} \subset c_\pi \subset m_\pi$. All continuous linear functionals on c_π will be denoted by c_π^*.

THEOREM 1 ([3] and [4]). *$f \in c_\pi^*$ if and only if*

$$f(x) = \sum_{n=1}^{\infty} t_n x_n + \alpha \lim_\pi x$$

where $t_\pi = (t_n/\pi_n) \in \ell$, $\alpha \in \mathbb{C}$ and

$$\lim_\pi x = \lim_{n\to\infty} \frac{x_n}{\pi_n}.$$

In what follows, let $A = (a_{nk})$, $n, k = 1, 2, \ldots$ be an infinite matrix with complex entries.

THEOREM 2 ([3] and [4]). *Let $c_{A\pi} = \{x : Ax \in c_\pi\}$. Then $c_{\pi A}$ is an FK - space with seminorms*

$$p_0(x) = \sup_{(n)} (1/\pi_n) \Big| \sum_{k=1}^{\infty} a_{nk} x_k \Big|;$$
$$p_{2n}(x) = |x_n|, n = (1, 2, \ldots); \text{ and}$$
$$p_{2n-1}(x) = \sup_{(m)} \Big| \sum_{k=1}^{m} a_{nk} x_k \Big|; \quad (n = 1, 2, \ldots).$$

THEOREM 3 ([3] and [4]). *$f \in c_{\pi A}^*$ if and only if*

$$f(x) = \sum_{k=1}^{\infty} \beta_k x_k + \sum_{n=1}^{\infty} t_n \sum_{k=1}^{\infty} a_{nk} x_k + \alpha \lim_{\pi A} x,$$

where $(t_n \pi_n) \in \ell$, $\alpha \in \mathbb{C}$, $(\beta_n) \in c_{\pi A}^\beta$, the β- dual of $c_{\pi A}$, and

$$\lim_{\pi A} x = \lim_{n\to\infty} (1/\pi_n) \sum_{k=1}^{\infty} a_{nk} x_k.$$

THEOREM 4 ([3] and [4]). *$A \in (c_\varrho : c_\pi)$ if and only if*

(1) $$\lim_{n\to\infty} a_{nk}/\pi_n = a_k^\pi \quad \text{exists for all } k;$$

(2) $$\lim_{n\to\infty} (1/\pi_n) \sum_{k=1}^{\infty} a_{nk} \varrho_k = a^{\pi\varrho} \quad \text{exists};$$

(3) $$\sum_{k=1}^{\infty} |a_{nk}| (\varrho_k/\pi_k) \le M,$$

where M is a constant independent of n and k.

THEOREM 5 ([3] and [4]). *$A \in (c_\varrho : c)$ if and only if*

$$\lim_{n\to\infty} a_{nk} = a_k \quad \textit{exists;} \tag{4}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}\varrho_k = a^{\varrho 1} \quad \textit{exists;} \tag{5}$$

$$\sum_{k=1}^{\infty} |a_{nk}|\varrho_k = O(1) \text{ for all } n. \tag{6}$$

DEFINITION [7]. Let $A = (a_{nk})$ and $B = (b_{nk})$, $(n, k = 1, 2, \ldots)$ be sequence —to—sequence matrix transformations. Their convolution $C = A \cdot B$ is defined by $C = (c_{nk})$, where

$$c_{nk} = a_{n1}b_{n,k-1} + a_{n2}b_{n,k-2} + \ldots + a_{n,k-1}b_{n1}.$$

2. Properties and Theorems

THEOREM 6. *Let $A \in (c_\varrho : c)$ and $B \in (c_\varrho : c)$. Then their convolution $A \cdot B$ belongs to $(c_\varrho : c)$.*

Proof. $a_{nk} \to a_k$ as $n \to \infty$, $b_{nk} \to b_k$ as $n \to \infty$. Hence $c_{nk} \to a_1 b_{k-1} + a_2 b_{k-2} + \ldots + a_{k-1}b_1$. Also $\sum_{k=1}^{\infty} a_{nk}\varrho_k \to a^{\varrho 1}$ as $n \to \infty$,

$$\sum_{k=1}^{\infty} b_{nk}\varrho_k \to b^{\varrho 1} \mathrm{as} n \to \infty.$$

But then $\sum_{k=1}^{\infty} c_{nk}\varrho_k = (\sum_{k=1}^{\infty} a_{nk}\varrho_k)(\sum_{k=1}^{\infty} b_{nk}\varrho_k)$ and so $\sum_{k=1}^{\infty} c_{nk}\varrho_k \to a^{\varrho 1}b^{\varrho 1}$. This completes the proof.

DEFINITION. A matrix A is called c_π—reversible if for every $y \in c_\pi$, there exists a unique x such that $Ax = y$.

THEOREM 7. *If A is c_π—reversible, then $c_{\pi A}$ is a Banach space with the norm*

$$p_0(x) = \sup_{(n)}(1/\pi_n)\Big|\sum_{k=1}^{\infty} a_{nk}x_k\Big|.$$

Also, every $f \in (c_{\pi A})^*$ has a representation $f(x) = \sum_{n=1}^{\infty} t_n \sum_{k=1}^{\infty} a_{nk}x_k + \mu \lim_{\pi A} x$, where $(t_n\pi_n) \in \ell$, $\mu \in \mathbb{C}$ and $\lim_{n\to\infty}(1/\pi_n)\sum_{k=1}^{\infty} a_{nk}x_k$ exists.

THEOREM 8. *We have $\chi(f) = \mu\chi(A)$ where*

$$\chi(f) = f(1^*) - \sum_{k=1}^{\infty} f(\delta^k),$$

$$\chi(A) = a^\pi - \sum_{k=1}^{\infty} a_k^\pi \text{ with } a_k^\pi = \lim_{n\to\infty} \frac{a_{nk}}{\pi_n}$$

and

$$a^\pi = \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{a_{nk}}{\pi_n}.$$

Proof. We have that $f \in (c_{A\pi})^*$ has the representation

$$f(x) = \sum_{n=1}^{\infty} t_n \sum_{k=1}^{\infty} a_{nk} x_k + \mu \lim\nolimits_{\pi A} x. \tag{7}$$

Take $x = \delta^k$, with $\delta^k = (0, \ldots, 1, 0, \ldots)$, 1 in the k^{th} place and zeros elsewhere. Then we have

$$\sum_{k=1}^{\infty} f(\delta^k) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} t_n a_{nk} + \mu \sum_{n=1}^{\infty} a_k^\pi.$$

Take $1^* = (1, 1, 1, \ldots)$ in place of the sequence $x = x_k$ in (7).
We obtain

$$f(1^*) = \sum_{n=1}^{\infty} t_n \sum_{k=1}^{\infty} a_{nk} + \mu a^\pi.$$

Hence $f(1^*) - \sum_{k=1}^{\infty} f(\delta^k) = \mu(a^\pi - \sum_{k=1}^{\infty} a_k^\pi)$ which implies $\chi(f) = \mu\chi(A)$, where $\chi(A) = a^\pi - \sum_{k=1}^{\infty} a_k^\pi$. This establishes the result.

THEOREM 9. *Let $f \in c_{\pi A}^*$. Then for $x \in c_{\pi A}$*

$$f(x) - \sum_{k=1}^{\infty} x_k f(\delta^k) = \alpha \wedge (x) + t(Ax) - (tA)x,$$

where

$$\wedge(x) = \lim\nolimits_{\pi A} x - \sum_{k=1}^{\infty} a_k^\pi x_k$$

for $x \in I = \{x \in c_{\pi A} : \sum_{k=1}^{\infty} a_k^\pi x_k$ converges$\}$.

Proof. Take $x = \delta^k$ in (7). Then we have

$$f(\delta^k) = \beta_k + \sum_{n=1}^{\infty} t_n a_{nk} + \alpha a_k^\pi. \tag{8}$$

Hence $\beta_k = f(\delta^k) - \sum_{n=1}^{\infty} t_n a_{nk} - \alpha a_k^{\pi}$ Using this in (8) we obtain

$$f(x) = \sum_{k=1}^{\infty}[f(\delta^k) - \sum_{n=1}^{\infty} t_n a_{nk} + \alpha a_k^{\pi}]x_k + \sum_{n=1}^{\infty} t_n \sum_{k=1}^{\infty} a_{nk}x_k + \alpha \lim_{\pi A} x.$$

Therefore

$$f(x) - \sum_{k=1}^{\infty} x_k f(\delta^k) = \alpha \wedge (x) + t(Ax) - (tA)x.$$

Hence the result is proved.

DEFINITION. We define

$$L = \Big\{x \in c_{\pi A} : (tA)x = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} t_n a_{nk} x_k \text{ exists}\Big\}.$$

We recall $I = \{x \in c_{\pi A} : \sum_{k=1}^{\infty} a_k^{\pi} x_k \text{ converges}\}$.

We define $F = \{x \in c_{\pi A} : \sum_{k=1}^{\infty} x_k f(\delta^k) \text{ converges for all } f \in c_{\pi A}^*\}$.

THEOREM 10. *If c_0 is not dense in L, then $L = I$. Consequently $L = F$ and $L \neq W$.*

Proof. For $x \in L$, we have $t(Ax) = (tA)x$. Consequently $f \in c_{\pi A}^*$, we have

$$f(x) - \sum_{k=1}^{\infty} x_k f(\delta^k) = \alpha \wedge (x). \tag{9}$$

If $f = 0$ on c_0 then $f(\delta^k) = 0$. Hence (9) reduces to $f(x) = \alpha \wedge (x)$. If $f \neq 0$ on L, then $\alpha \neq 0$. So, $x \in I$. Thus $L \subset I$. But always $I \subset L$. Hence $L = I$. Now $F = L \cap I$. Hence $F = L$ since $I = L$. Therefore $f = \mu\wedge$ and $\wedge \neq O$ on L.

This means that

$$L \not\subset \wedge^{\perp} = \{x \in c_{\pi A} : \wedge(x) = O\}.$$

But $W = F \cap \wedge^{\perp} = L \cap \wedge^{\perp}$. Since $L \not\subset \wedge^{\perp}$, we get $L \neq W$.

This proves the result.

DEFINITION. Let $A \in (c_{\pi} : c_{\pi})$. Then $A = (a_{nk})$, $n, k = 1, 2, \ldots$, is said to be of type $M(c_{\pi} : c_{\pi})$ if $\sum_{n=1}^{\infty} t_n a_{nk} = 0$, $\sum_{n=1}^{\infty} |t_n \pi_n| < \infty$ imply that $t_n = 0$ for all n.

THEOREM 11. *Let A belong to $(c_{\pi} : c_{\pi})$. Let A be c_{π}—reversible and multiplicative. Then A is of type $M(c_{\pi} : c_{\pi})$ if and only if $\overline{c}_0$ is a maximal subspace of $c_{\pi A}$. Here $\overline{c}_0$ is the closure of c_0 in $c_{\pi A}$.*

Proof. (Step 1). Suppose that A is of Type $M(c_\pi : c_\pi)$. Assume that $f = 0$ on c_0. Since A is reversible we have $f(x) = \alpha \lim_{\pi A} x + t(Ax)$ where

$$\sum_{n=1}^{\infty} |t_n \pi_n| < \infty. \tag{10}$$

Also $f(\delta^k) = 0$. Consequently $\alpha a_k^\pi + \sum_{n=1}^{\infty} t_n a_{nk} = 0$. Since A is multiplicative, we have $a_k^\pi = 0$ for $k = 1, 2, \ldots$.

Therefore, we obtain

$$\sum_{n=1}^{\infty} t_n a_{nk} = 0 \text{ with } \sum_{n=1}^{\infty} |t_n \pi_n| < \infty.$$

But A is of type $M(c_\pi : c_\pi)$. Hence $t_n = 0$ for all $n = 1, 2, \ldots$.

That is, $t = 0$, the zero sequence. Using this in (10), we get $f = \alpha \lim_{\pi A}$ which implies that, either $\overline{c}_0 = c_{\pi A}$, or $\overline{c}_0$ is a maximal linear subspace of $c_{\pi A}$.

However $\overline{c}_0 \neq c_{\pi A}$. Hence $\overline{c}_0$ is a maximal linear subspace of $c_{\pi A}$.

(Step 2). Suppose that $\overline{c}_0$ is a maximal linear subspace of $c_{\pi A}$. Assume that A is not of type $M(c_\pi : c_\pi)$. Let

$$\sum_{n=1}^{\infty} |t_n \pi_n| < \infty, \quad t_n \neq 0 \text{ for all } n$$

and

$$\sum_{n=1}^{\infty} t_n a_{nk} = 0 \text{ for } k = 1, 2, \ldots \tag{11}$$

Take $f(x) = t(Ax)$. Always $\lim_{\pi A}$ and $\sum_{k=1}^{\infty} t_k a_k^\pi$ are linearly independent, because A is c_π - reversible. Therefore, both f and $\lim_{\pi A}$ vanish on $\overline{c}_0$. Hence $\overline{c}_0$ is not a maximal linear subspace of $c_{\pi A}$. This contradicts our present hypothesis. This contradiction shows that $t = 0$ and so A is of type $M(c_\pi : c_\pi)$.

DEFINITION. We define $P = \{x : c_{\pi A} : t(Ax) = (tA)x, \pi t \in \ell\}$. Note that $L \subset P$.

THEOREM 12.
(i) $\overline{c} \subset P$ *and* P *is closed.*
(ii) *If* A *is coregular, then* $P = \overline{c}$.

Proof. (Step 1). Let $\pi t \in \ell$ have the property that $(tA)x$ exists for all x in $c_{\pi A}$. Define f_t, by $ft(x) = (tA)x - t(Ax)$. But then $f_t^\perp = \{x : f_t(x) = 0\}$.

By the Banach–Steinhaus theorem, f_t, is continuous. Hence $f_t^{\perp}$ is closed. But $P = \bigcap_t f_t^{\perp}$. So, P is closed in $c_{\pi A}$. But always $c \subset P$ and consequently

$$\bar{c} \subset P. \tag{12}$$

This proves (i).

(Step 2). Suppose that A is coregular. Let $f \in c_{\pi A}^*$ satisfy that $f = 0$ on c. But then

$$f^{\perp} = c = \bar{c} \tag{13}$$

where $f^{\perp} = \{x : f(x) = 0\}$.

With $1 = (1, 1, 1, \ldots)$. We have

$$f(1) = \alpha \lim_{\pi_A} 1 + t(A1) + \sum_{k=1}^{\infty} \Big[f(\delta^k) - \alpha a_k^{\pi} - \sum_{n=1}^{\infty} t_n a_{nk} \Big] 1.$$

Therefore

$$0 = f(1) - \sum_{k=1}^{\infty} f(\delta^k) = \alpha[\lim_{\pi_A} 1 - \sum_{k=1}^{\infty} a_k^{\pi}] + t(A1) - (tA)1.$$

$$0 = \chi(f) = \alpha\chi(A) + t(A1) - t(A)1.$$

Note that $t(A1) - t(A)1 = 0$. Therefore $\alpha\chi(A) = 0$. Since A is coregular, $\chi(A) = 0$. Hence $\alpha = 0$ and so $f(x) = t(Ax) - (tA)x = 0$ on P. Therefore

$$P \subset f^{\perp}. \tag{14}$$

From (13) and (14) we obtain

$$P \subset \bar{c}. \tag{15}$$

From (12) and (15) we have $P = \bar{c}$. This proves the result (ii).

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