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OSCILLATORY PROPERTIES OF THIRD ORDER NEUTRAL DELAY DIFFERENCE EQUATIONS

Abstract. Some new sufficient criteria for the oscillation of all solutions of the neutral difference equation

$$\Delta(c_n\Delta(d_n\Delta(y_n + p_n y_{n-k}))) + q_n f(y_{n-m}) = e_n$$

are obtained. Existence of nonoscillatory solution and its asymptotic behavior are also discussed. Examples illustrating the results are inserted.

1. Introduction

Consider a third order neutral delay difference equation of the form

$$\Delta(c_n\Delta(d_n\Delta(y_n + p_n y_{n-k}))) + q_n f(y_{n-m}) = e_n \tag{E}$$

where $n \in \mathbb{N}(n_0) = \{n_0, n_0+1, n_0+2, \cdots\}$, n_0 is a nonnegative integer, Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$, k and m are nonnegative integers such that $k \le m$ and the real sequences $\{c_n\}, \{d_n\}, \{p_n\}$, $\{q_n\}$, $\{e_n\}$ and the function f satisfy the following conditions:

(H_1) $\{c_n\}$ and $\{d_n\}$ are positive real sequences such that $\sum\limits_{n=n_0}^{\infty} \frac{1}{c_n} = \sum\limits_{n=n_0}^{\infty} \frac{1}{d_n} = \infty$;

(H_2) $0 \le p_n < 1$, $\{p_n\}$ is nondecreasing, $q_n \ge 0$ and $q_n \not\equiv 0$ for infinitely many values of $n \in \mathbb{N}(n_0)$;

(H_3) $f : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing such that $uf(u) > 0$ for all $u \ne 0$.

By a solution of equation (E) we mean a real sequence $\{y_n\}$ that is defined for $n \ge n_0 - m$ and satisfies equation (E) for all $n \in \mathbb{N}(n_0)$. Clearly if $y_n = A_n$ for $n = n_0 - m, n_0 - m + 1, \cdots, n_0 - 1$ are given then equation (E) has a unique solution satisfying the above initial conditions. A solution of (E) is said to be oscillatory if the terms y_n of the sequence $\{y_n\}$ are

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neither eventually all positive nor eventually all negative and nonoscillatory otherwise. Most of the results established the equation (E) state that the solutions of equation (E) are either oscillatory or tends to zero monotonically as, $n \to \infty$ see for example [1, 6, 8, 9, 10, 11, 12], and the references cited therein. Motivated by this observation, in this paper we establish conditions under which all solutions of equation (E) are oscillatory. Existence of nonoscillatory solutions and its asymptotic behavior will also be discussed. Examples illustrating the results are included in the text of the paper.

2. Some preliminary lemmas

In this section we state and prove some lemmas, which are useful in establishing our main results.

LEMMA 2.1. *Let $e_n \equiv 0$. If $\{y_n\}$ is an eventually positive solution of equation (E), then there are only the following two cases for $n \in \mathbb{N}(n_0)$ sufficiently large:*

(I) $z_n > 0,\ \Delta z_n > 0,\ \Delta(d_n \Delta z_n) > 0;$
(II) $z_n > 0,\ \Delta z_n < 0,\ \Delta(d_n \Delta z_n) > 0;$

where $z_n = y_n + p_n y_{n-k}$.

Proof. Let $\{y_n\}$ be an eventually positive solution of equation (E). Then there exists a $n_1 \in \mathbb{N}(n_0)$ such that $y_{n-k} > 0$ and $y_{n-m} > 0$ for $n \geq n_1$. From the definition of z_n and (H_2), it is clear that $z_n > 0$ for $n \geq n_1$ and from equation (E), $\Delta(c_n)\Delta(d_n \Delta z_n)) \leq 0$ for $n \geq n_1$. Thus $\{z_n\}$,$\{\Delta z_n\}$ and $\{\Delta(d_n \Delta z_n\}$ are monotone and eventually one signed. We claim that there is an $n_2 \in \mathbb{N}(n_1)$ such that for $n \geq n_2$, $\Delta(d_n \Delta z_n) > 0$. Suppose $\Delta(d_n \Delta z_n) \leq 0$. Since $c_n > 0$, it is clear that there is an integer $n_3 > n_2$ such that $c_{n_3} \Delta(d_{n_3} \Delta z_{n_3}) < 0$. Thus for $n \geq n_3$, we have

$$c_n \Delta(d_n \Delta z_n) \leq c_{n_3} \Delta(d_{n_3} \Delta z_{n_3}) < 0. \tag{1}$$

Dividing (1) by c_n and then summing from n_3 to $n-1$, we obtain

$$d_n \Delta z_n - d_{n_3} \Delta z_{n_3} < c_{n_3} \Delta(d_{n_3} \Delta z_{n_3}) \sum_{s=n_3}^{n-1} \frac{1}{c_s}. \tag{2}$$

Letting $n \to \infty$ in (2) then $d_n \Delta z_n \to -\infty$ by (H_1). Thus, there is an integer $n_4 \geq n_3$ such that for $n \geq n_4$,

$$d_n \Delta z_n \leq d_{n_4} \Delta z_{n_4} < 0. \tag{3}$$

Dividing (3) by d_n and then summing from n_4 to $n-1$, we obtain

$$z_n - z_{n_4} \leq d_{n_4} \sum_{s=n_4}^{n-1} \frac{1}{d_s},$$

which implies $z_n \to -\infty$ as $n \to \infty$ by (H_1), a contradiction. Thus $\Delta(d_n \Delta z_n) > 0$. This completes the proof.

LEMMA 2.2. *Let $\{y_n\}$ be an eventually positive solution of equation* (E) *and suppose Case* (I) *of Lemma* 2.1 *holds. Then there exists an integer $N \in \mathbb{N}(n_0)$ such that*

$$y_n \geq (1-p_n)z_n \geq (1-p)z_n, \tag{4}$$

for $n \geq N$.

LEMMA 2.3. *Let $\{y_n\}$ be an eventually positive solution of equation* (E) *and suppose Case* (II) *of Lemma* 2.1 *holds. Then there exists an integer $N \in \mathbb{N}(n_0)$ such that*

$$y_{n-k} \geq \frac{z_n}{1+p_n} \geq \frac{z_n}{1+p}, \tag{5}$$

for $n \geq N$.

LEMMA 2.4. *Let $\{y_n\}$ be an eventually solution of equation* (E) *and assume that $\lim_{n\to\infty} p_n = p^* \in [0,1)$ and $\lim_{n\to\infty} z_n = c \neq 0$. Then $\lim_{n\to\infty} y_n = \dfrac{c}{1+p^*}$.*

The proof of Lemmas 2.2 to 2.4 can be found in [7] and [4].

LEMMA 2.5. *Let $\{y_n\}$ be an eventually positive solution of equation* (E) *and suppose Case* (I) *of Lemma* 2.1 *holds. Then there exists an integer $N \in \mathbb{N}(n_0)$ such that for $n \geq N$,*

$$\Delta z_{n-m} \geq \frac{\rho_{n-m} c_n \Delta(d_n \Delta z_n)}{d_{n-m}}, \tag{6}$$

where $\rho_n = \sum_{s=n_0}^{n-1} \dfrac{1}{c_s}$.

Proof. From Case (I) of Lemma 2.1 and from equation (E), we have $d_n \Delta z_n > 0$, $c_n \Delta(d_n \Delta z_n) > 0$ and $\Delta(c_n \Delta(d_n \Delta z_n)) \leq 0$ for $n \geq N$. Hence

$$\begin{aligned} d_n \Delta z_n &= d_N \Delta z_N + \sum_{s=N}^{n-1} \frac{c_s \Delta(d_s \Delta z_s)}{c_s} \\ &\geq c_n \Delta(d_n \Delta z_n) \rho_n, \quad n \geq N. \end{aligned}$$

Since $\Delta(c_n \Delta(d_n \Delta z_n)) \leq 0$, we have $\Delta z_{n-m} \geq \dfrac{c_n \Delta(d_n \Delta z_n) \rho_{n-m}}{d_{n-m}}$, for $n \geq N$. The proof is now complete.

REMARK 2.1. Clearly, the inequalities parallel to those established in Lemmas 2.1-2.5 hold for an eventually negative solution of (E).

3. Oscillation results

In this section we establish conditions for the oscillation of all solutions of equation (E). We begin with the following theorem.

THEOREM 3.1. *Assume $f(u) \equiv u$ and $e_n \equiv 0$ in equation* (E). *Assume that there exist a double sequence $\{\mathcal{H}_{n,s}\}, n, s \in \mathbb{N}(n_0)$ such that*

(i) $\mathcal{H}_{n,n} = 0$ *for* $n \in \mathbb{N}(n_0)$ *and* $\mathcal{H}_{n,s} > 0$, *for* $n > s \in \mathbb{N}(n_0)$,
(ii) $\Delta_2 \mathcal{H}_{n,s} = \mathcal{H}_{n,s+1} - \mathcal{H}_{n,s} \le 0$ *for* $n > s \in \mathbb{N}(n_0)$.

Suppose that $\{h_{n,s} | n > s \in \mathbb{N}(n_0)\}$ is a double sequence with $\Delta_2 \mathcal{H}_{n,s} = -h_{n,s}\sqrt{\mathcal{H}_{n,s}}$ for $n > s \in \mathbb{N}(n_0)$. If there exists a positive real sequence $\{\varphi_n\}$ such that

$$\limsup_{n\to\infty} \frac{1}{\mathcal{H}_{n,n_0}} \sum_{s=n_0}^{n-1} \left[\mathcal{H}_{n,s}\Psi_s - \frac{d_{s-m}\varphi_{s+1}^2}{4\rho_{s-m}\varphi_s} h_{n,s}^2 \right] = \infty \tag{7}$$

where

$$\Psi_n = \varphi_n \left\{ q_n(1 - p_{n-m}) + \frac{\rho_{n-m} c_{n+1-m}^2 \alpha_n^2}{d_{n-m}} - \Delta(c_{n-m}\alpha_{n-1}) \right\},$$

$$\alpha_n = -\frac{d_{n-m}\Delta\varphi_n}{2\rho_{n-m}c_{n+1-m}},$$

and

$$\sum_{t=n}^{n+m-k} \left[\sum_{s=n}^{t} \frac{1}{d_s} \left(\sum_{j=s}^{t} \frac{1}{c_j} \right) \right] q_t > 1 + p, \tag{8}$$

then all solutions of equation (E) *are oscillatory.*

Proof. Let $\{y_n\}$ be a nonoscillatory solution of equation (E). Without loss of generality we may assume that $y_{n-m} > 0$ for $n \in \mathbb{N}(N_0)$, where N_0 is chosen so large that Lemmas 2.1 to 2.3 and Lemma 2.5 hold for $n \in \mathbb{N}(N_0)$.
Case (I): From Lemma 2.2 and equation (E), we have

$$\Delta(c_n \Delta(d_n \Delta z_n)) + q_n(1 - p_{n-m}) z_{n-m} \le 0, \quad n \in \mathbb{N}(N_0). \tag{9}$$

Define

$$\mathcal{W}_n = \varphi_n \left(\frac{c_n \Delta(d_n \Delta z_n)}{z_{n-m}} + c_{n-m}\alpha_{n-1} \right), \tag{10}$$

for $n \ge N_1 \in \mathbb{N}(N_0 + m)$, then using Lemma 2.5, we have

$$\begin{aligned}
\Delta\mathcal{W}_n &\le \frac{\Delta\varphi_n}{\varphi_{n+1}} \mathcal{W}_{n+1} \\
&+ \varphi_n \left\{ -q_n(1 - p_{n-m}) - \frac{\rho_{n-m}}{d_{n-m}} \left(\frac{\mathcal{W}_{n+1}}{\varphi_{n+1}} - c_{n+1-m}\alpha_n \right)^2 + \Delta(c_{n-m}\alpha_{n-1}) \right\} \\
&= -\Psi_n - \frac{\rho_{n-m}}{d_{n-m}} \frac{\varphi_n}{\varphi_{n+1}^2} \mathcal{W}_{n+1}^2, \quad \text{for } n \in \mathbb{N}(N_0).
\end{aligned}$$

Hence for all $n \geq N \in \mathbb{N}(N_1)$, we have

$$\sum_{s=N}^{n-1} \mathcal{H}_{n,s}\Psi_s \leq \mathcal{H}_{n,N}\mathcal{W}_N - \sum_{s=N}^{n-1} \left\{ (-\Delta_2 \mathcal{H}_{n,s})\,\mathcal{W}_{s+1} + \mathcal{H}_{n,s}\frac{\rho_{s-m}}{d_{s-m}}\frac{\varphi_s}{\varphi_{s+1}^2}\mathcal{W}_{s+1}^2 \right\}$$

$$= \mathcal{H}_{n,N}\mathcal{W}_N - \sum_{s=N}^{n-1} \frac{\rho_{s-m}}{d_{s-m}}\frac{\varphi_s}{\varphi_{s+1}^2}\left\{ \mathcal{W}_{s+1}\sqrt{\mathcal{H}_{n,s}} + \frac{h_{n,s}d_{s-m}\varphi_{s+1}^2}{2\rho_{s-m}\varphi_s} \right\}^2 + \frac{1}{4}\sum_{s=N}^{n-1} \frac{d_{s-m}\varphi_{s+1}^2}{\rho_{s-m}\varphi_s}h_{n,s}^2.$$

Thus, for all $n \in \mathbb{N}(N)$, we obtain

$$\sum_{s=N}^{n-1} \left[\mathcal{H}_{n,s}\Psi_s - \frac{1}{4}\frac{d_{s-m}\varphi_{s+1}^2}{\rho_{s-m}\varphi_s}h_{n,s}^2 \right] \leq \mathcal{H}_{n,N}\mathcal{W}_N,$$

which implies for $n \geq N_1$

$$\sum_{s=N_1}^{n-1} \left[\mathcal{H}_{n,s}\Psi_s - \frac{1}{4}\frac{d_{s-m}\varphi_{s+1}^2}{\rho_{s-m}\varphi_s}h_{n,s}^2 \right] \leq \mathcal{H}_{n,N_1}\mathcal{W}_{N_1} \leq \mathcal{H}_{n,n_0}|\mathcal{W}_{N_1}|.$$

Therefore,

$$\begin{aligned}
&\sum_{s=n_0}^{n-1} \left[\mathcal{H}_{n,s}\Psi_s - \frac{1}{4}\frac{d_{s-m}\varphi_{s+1}^2}{\rho_{s-m}\varphi_s}h_{n,s}^2 \right] \\
&\leq \sum_{s=n_0}^{N_1-1} \left[\mathcal{H}_{n,s}\Psi_s - \frac{1}{4}\frac{d_{s-m}\varphi_{s+1}^2}{\rho_{s-m}\varphi_s}h_{n,s}^2 \right] + \mathcal{H}_{n,n_0}|\mathcal{W}_{N_1}| \\
&\leq \mathcal{H}_{n,n_0}\sum_{s=n_0}^{N_1-1} |\Psi_s| + \mathcal{H}_{n,n_0}|\mathcal{W}_{N_1}| \\
&= \mathcal{H}_{n,n_0}\left\{ \sum_{s=n_0}^{N_1-1} |\Psi_s| + |\mathcal{W}_{N_1}| \right\},
\end{aligned}$$

for all $n \in \mathbb{N}(n_0)$. The latter gives

$$\limsup_{n\to\infty} \frac{1}{\mathcal{H}_{n,n_0}} \sum_{s=n_0}^{n-1} \left[\mathcal{H}_{n,s}\Psi_s - \frac{d_{s-m}\varphi_{s+1}^2}{4\rho_{s-m}\varphi_s}h_{n,s}^2 \right] \leq \sum_{s=n_0}^{N_1-1} |\Psi_s| + |\mathcal{W}_{N_1}|,$$

which contradicts with condition (7).

Case (II): Let $s \in \mathbb{N}(n_0)$ be fixed and summing the equations (E) from s to $n-1$, we obtain

$$c_n\Delta(d_n\Delta z_n) - c_s\Delta(d_s\Delta z_s) + \sum_{t=s}^{n-1} q_t y_{t-m} = 0,$$

or

$$-\Delta(d_n\Delta z_n) + \frac{1}{c_n}\sum_{t=n}^{\infty} q_t y_{t-m} \leq 0.$$

Summing again from s to n and rearranging, we obtain

$$d_n\Delta z_n + \sum_{t=n}^{\infty}\Big(\sum_{s=n}^{t}\frac{1}{c_s}\Big) q_t y_{t-m} \leq 0.$$

A final summation of the last inequality divided by d_n from s to n gives

$$\sum_{t=n}^{\infty}\Big[\sum_{s=n}^{t}\frac{1}{d_s}\Big(\sum_{j=s}^{t}\frac{1}{c_j}\Big)\Big] q_t y_{t-m} \leq z_n.$$

In view of (5), the last inequality implies

$$\sum_{t=n}^{n+m-k}\Big[\sum_{s=n}^{t}\frac{1}{d_s}\Big(\sum_{j=s}^{t}\frac{1}{c_j}\Big)\Big] q_t \frac{z_{t+k-m}}{1+p} \leq z_n.$$

Since $\{z_n\}$ is decreasing, we obtain

$$\sum_{t=n}^{n+m-k}\Big[\sum_{s=n}^{t}\frac{1}{d_s}\Big(\sum_{j=s}^{t}\frac{1}{c_j}\Big)\Big] q_t \leq 1+p,$$

which contradicts with (8). This completes the proof of the theorem.

REMARK 3.1. Let $p_n = 0$ in the equation (E), then Theorem 3.1 improves Theorem 1 of Graef and Thandapani [3] in the sense that we do not require that $\Delta c_n \geq 0$ for $n \in \mathbb{N}(n_0)$.

We have used a general class of double sequence $\{\mathcal{H}_{n,s}\}$ as the parameter sequence in Theorem 3.1. By choosing various specific double sequence $\{\mathcal{H}_{n,s}\}$ we can derive several oscillation criteria.

First, let us consider a double sequence $\{\mathcal{H}_{n,s}\}$ defined by

$$\{\mathcal{H}_{n,s}\} = (n-s)^{\lambda}, \quad n \geq s \in \mathbb{N}(n_0),$$

where $\lambda \geq 1$ is a constant. Then $\mathcal{H}_{n,n} = 0$ for $n \in \mathbb{N}(n_0)$, $\mathcal{H}_{n,s} > 0$ for $n > s \in \mathbb{N}(n_0)$ and $\Delta_2\mathcal{H}_{n,s} \leq 0$ for $n > s \in \mathbb{N}(n_0)$. Thus, we have the following corollary.

COROLLARY 3.2. *In addition to the condition* (8) *assume that*

$$\lim_{n\to\infty}\sup\frac{1}{(n-n_0)^{\lambda}}\sum_{s=n_0}^{n-1}\left((n-s)^{\lambda}\Psi_s - \frac{\lambda^2 d_{s-m}\varphi_{s+1}^2(n-s)^{\lambda-2}}{4\rho_{s-m}\varphi_s}\right) = \infty,$$

for some $\lambda \geq 1$. *Then every solution of* (E) *is oscillatory.*

EXAMPLE 3.1. Consider the following neutral delay difference equation

$$(\mathrm{E}_1) \qquad \Delta^3\Big(y_n + \frac{1}{2}y_{n-1}\Big) + \frac{4n+10}{n-2}y_{n-2} = 0, \quad n \geq 3.$$

Let $\lambda = 2$ and $\varphi_n = 1$. Then all conditions of Corollary 3.2 are satisfied and therefore every solution (E_1) is oscillatory. In fact $\{y_n\} = \{(-1)^n n\}$ is such a solution of equation (E_1).

Next, consider the double sequence $\{\mathcal{H}_{n,s}\}$defined by

$$\mathcal{H}_{n,s} = \Big(\log\frac{n+1}{s+1}\Big)^{\lambda}, \quad n \geq s \in \mathbb{N}(n_0),$$

where $\lambda \geq 1$ is a constant. Then $\mathcal{H}_{n,n} = 0$ for $n \in \mathbb{N}(n_0), \mathcal{H}_{n,s} > 0$ for $n > s \in \mathbb{N}(n_0)$ and $\Delta_2\mathcal{H}_{n,s} \leq 0$ for $n > s \in \mathbb{N}(n_0)$. Then, we have the corollary.

COROLLARY 3.3. *In addition to the condition* (8) *assume that*

$$\lim_{n\to\infty}\sup[\log(n+1)]^{-\lambda}\sum_{s=n_0}^{n-1}\left[\Big(\log\frac{n+1}{s+1}\Big)^{\lambda}\Psi_s - \frac{\lambda^2 d_{s-m}\varphi_{s+1}^2\left(\log\frac{n+1}{s+1}\right)^{\lambda-2}}{4\rho_{s-m}\varphi_s}\right] = \infty,$$

for some $\lambda \geq 1$. *Then every solution of the equation* (E) *is oscillatory.*

REMARK 3.2. Let $\dfrac{f(u)}{u} \geq M > 0$ for $u \neq 0$ and $e_n \equiv 0$ in the equation (E). Then with suitable modifications in the condition (8), that is

$$\sum_{t=n}^{n+m-k}\left[\sum_{s=n}^{t}\frac{1}{d_s}\Big(\sum_{j=s}^{t}\frac{1}{c_j}\Big)\right]q_t \leq \frac{1+p}{M},$$

the conclusion of Theorem 3.1 holds.

Next theorem deals with the oscillation of (E) when f satisfies the condition

$$(11) \qquad f(u) - f(v) = g(u,v)(u-v) \text{ for } u \neq v \text{ and } g(u,v) \geq \mu,$$

for some $\mu > 0$.

THEOREM 3.4. *In addition to the condition* (11) *assume that* $p_n \equiv p$ *and* $e_n \equiv 0$ *in* (E). *If there exists a positive nondecreasing sequence* $\{\varphi_n\}$ *such that*

$$(12) \qquad \sum_{n=n_0}^{\infty}[\alpha\varphi_n + \varphi_{n+1}(q_n - \alpha)] = \infty$$

for every $\alpha > 0$ *and*

$$(13) \qquad \lim_{n\to\infty}\sup\sum_{t=n}^{n+m-k}q_t\left[\sum_{s=n}^{t}\frac{1}{d_s}\Big(\sum_{j=s}^{t}\frac{1}{c_j}\Big)\right] = \infty,$$

then all solutions of the equation (E) *are oscillatory.*

Proof. Let $\{y_n\}$ be a nonoscillatory solution of the equation (E). Without loss of generality we may assume that $y_{n-m} > 0$ for $n \in \mathbb{N}(N), N \geq n_0$, where N is chosen so large that Lemmas 2.1 to 2.2 and Lemma 2.5 hold for $n \in \mathbb{N}(N)$.

Case (I): From Lemma 2.2 and (11), the equation (E) implies that

$$\Delta(c_n\Delta(d_n\Delta z_n)) + q_n f((1-p)z_{n-m}) \leq 0, \quad n \in \mathbb{N}(N). \tag{14}$$

Define

$$\mathcal{W}_n = \frac{c_n\Delta(d_n\Delta z_n)}{f((1-p)z_{n-m})}\varphi_n, \quad n \geq N.$$

Then $\mathcal{W}_n$ is eventually positive and using(14) we have

$$\begin{aligned}\Delta\mathcal{W}_n &\leq -\varphi_{n+1}q_n + \frac{c_n\Delta(d_n\Delta z_n)}{f((1-p)z_{n-m})}\Delta\varphi_n \\ &\leq -\varphi_{n+1}q_n \frac{c_N\Delta(d_N\Delta z_N)}{f((1-p)z_{N-m})}\Delta\varphi_n \\ &= -\varphi_{n+1}q_n + \alpha\Delta\varphi_n,\end{aligned}$$

where $\alpha = \dfrac{c_N\Delta(d_N\Delta z_N)}{f((1-p)z_{N-m})} > 0$ is a constant. Summing the last inequality, we obtain

$$\sum_{s=N}^{n-1}(\alpha\varphi_n + \varphi_{n+1}(q_n - \alpha)) \leq \mathcal{W}_N,$$

which contradicts with the condition (12) as $n \to \infty$.

Case (II): Summing the equation (E) three times, we obtain

$$\sum_{t=n}^{\infty}\left[\sum_{s=n}^{t}\frac{1}{d_s}\Big(\sum_{j=s}^{t}\frac{1}{c_j}\Big)\right]q_t f(y_{t-m}) \leq z_n.$$

Now, from Lemma 2.3, we obtain

$$\sum_{t=n}^{n+m-k}\left[\sum_{s=n}^{t}\frac{1}{d_s}\Big(\sum_{j=s}^{t}\frac{1}{c_j}\Big)\right]q_t f\Big(\frac{z_{t+k-m}}{1+p}\Big) \leq z_n.$$

Since $\{z_n\}$ is decreasing and f is nondecreasing, we have

$$\sum_{t=n}^{n+m-k}\left[\sum_{s=n}^{t}\frac{1}{d_s}\Big(\sum_{j=s}^{t}\frac{1}{c_j}\Big)\right]q_t \leq \frac{z_n}{f\Big(\dfrac{z_n}{1+p}\Big)}. \tag{15}$$

Clearly $\{z_n\}$ approaches a finite nonnegative number as $n \to \infty$. In view of the condition (13) and using (15) we have $\lim\limits_{n\to\infty} z_n > 0$ what is impossible.

If $\lim_{n\to\infty} z_n = 0$, then

$$\lim_{n\to\infty} \frac{z_n}{f\left(\frac{z_n}{1+p}\right)} = \lim_{n\to\infty} \frac{1}{g\left(\frac{z_n}{1+p}, \frac{z_{n+1}}{1+p}\right)} \le \frac{1+p}{\mu} < \infty,$$

a contradiction with (13). This completes the proof of the theorem.

EXAMPLE 3.2. The difference equation

$$(\mathrm{E}_2) \qquad \Delta\left(n\Delta\left(n\Delta\left(y_n + \frac{1}{2}y_{n-1}\right)\right)\right) + 4n^3\left(y_{n-2}^{\frac{1}{3}} + y_{n-2}\right) = 0,$$

satisfies all conditions of Theorem 3.4 for $\varphi_n = 1$. Hence all solutions of (E_2) are oscillatory.

THEOREM 3.5. *In addition to conditions* (11), (12) *and* (13) *assume that* $\{p_n\}$ *is decreasing such that* $\lim_{n\to\infty} p_n = 0$ *and there exists an oscillatory sequence* $\{h_n\}$ *such that*

$$(16) \qquad \Delta(c_n\Delta(d_n\Delta h_n)) = e_n, \lim_{n\to\infty} h_n = 0.$$

Then every solution of equation (E) *is either oscillatory or tends to zero monotonically as* $n \to \infty$.

Proof. Suppose that there is a nonoscillatory solution $\{y_n\}$ such that $\{y_n\}$ is eventually positive and $\lim_{n\to\infty} y_n \ne 0$. Consider the function $x_n = z_n - h_n$. Then $z_n \ge y_n > 0$ and $x_n > 0$ eventually. If $\{x_n\}$ is eventually negative then $z_n < h_n$, which contradicts with oscillatory character of $\{h_n\}$. Also from equation (E), $\Delta(c_n\Delta(d_n\Delta x_n)) \le 0$. Thus $\{x_n\}$, $\{\Delta x_n\}$ and $\{\Delta(d_n\Delta x_n)\}$ are monotone and eventually one signed. Proceeding as in Lemma 2.1, it follows that there is an $n_1 \in \mathbb{N}(n_0)$ such that for $n \in \mathbb{N}(n_1), \Delta(d_n\Delta x_n) > 0$ and $\Delta(c_n\Delta(d_n\Delta x_n)) \le 0$.

Let $\{\Delta x_n\}$ be eventually positive. Then since $\{x_n\}$ is eventually positive and increasing and $h_n \to 0$ and $p_n \to 0$ as $n \to \infty$, it follows that there is an integer $n_2 \ge n_1$ and $\lambda \in (0,1)$ such that $y_{n-m} \ge \lambda x_{n-m}$ for $n \in \mathbb{N}(n_2)$. Moreover, f is nondecreasing and hence

$$f(y_{n-m}) \ge f(\lambda x_{n-m})$$

for $n \in \mathbb{N}(n_2)$. Define

$$\mathcal{W}_n = \frac{c_n\Delta(d_n\Delta x_n)}{f(\lambda x_{n-m})}\varphi_n,$$

then $\mathcal{W} > 0$ for $n \in \mathbb{N}(n_2)$ and satisfies the inequality

$$\Delta\mathcal{W}_n \le -q_n\varphi_{n+1} + \frac{c_{n_2}\Delta(d_{n_2}\Delta x_{n_2})}{f(\lambda x_{n_2-m})}\varphi_n.$$

Summing the last inequality from n_2 to $n-1$, we obtain

$$\sum_{s=n_2}^{n-1} (\beta\varphi_n + \varphi_{n+1}(q_n - \beta)) \le \mathcal{W}_{n_2}$$

where $\beta = \dfrac{c_{n_2}\Delta(d_{n_2}\Delta x_{n_2})}{f(\lambda x_{n_2-m})}$, a contradiction with (12). Thus $\{\Delta x_n\}$ has to be eventually negative. In that case $\lim\limits_{n\to\infty} x_n = c \ge 0$. Since $\lim\limits_{n\to\infty} h_n = 0$ and $\lim\limits_{n\to\infty} p_n = 0$, we have $\lim\limits_{n\to\infty} y_n = c$. Summing the equation (E) three times we obtain

$$\sum_{t=n}^{\infty}\left[\sum_{s=n}^{t}\frac{1}{d_s}\left(\sum_{j=s}^{t}\frac{1}{c_j}\right)\right]q_t f(y_{t-m}) \le x_n. \tag{17}$$

From (13) and (17), we see that $\lim\limits_{n\to\infty}\inf y_n = 0$. By Lemma 1 of [5], we conclude that $c = 0$. This completes the proof.

EXAMPLE 3.3. The difference equation

$$\Delta\Big((n+1)\Delta\Big((n+1)\Delta\Big(y_n + \frac{3}{4^n}y_{n-1}\Big)\Big)\Big) + n^4\Big(y_{n-3}^3 + y_{n-3}\Big) = \frac{(-1)^{n+1}(8n^3+24n^2+18n+1)}{n(n+1)}, n \ge 1, \tag{E$_3$}$$

satisfies all conditions of Theorem 3.5 with $\varphi_n = 1$ and $\{h_n\} = \left\{\dfrac{(-1)^n}{n}\right\}$. Hence every solution of the equation (E$_3$) is either oscillatory or tends to zero monotonically as $n \to \infty$.

Finally, we obtain a sufficient condition for the existence and asymptotic behavior of nonoscillatory solutions of (E). We do not require $q_n > 0$ for $n \in \mathbb{N}(n_0)$ here. Let $\mathcal{A}_n, \mathcal{B}_n$ and $\mathcal{C}_n$ be defined by $\mathcal{A}_n = \sum\limits_{s=n_0}^{n-1}\dfrac{1}{c_s}$, $\mathcal{B}_n = \sum\limits_{s=n_0}^{n-1}\dfrac{1}{d_s}$ and $\mathcal{C}_n = \sum\limits_{s=n_0}^{n-1}\dfrac{\mathcal{A}_s}{d_s}$.

THEOREM 3.6. *Let $\alpha > 0$ be a constant such that $d_n \ge \alpha$ for all $n \in \mathbb{N}(n_0)$. Suppose that*

$$\sum_{n=n_0}^{\infty} |\,\mathcal{C}_{n+1} + \mathcal{A}_{n+1}\mathcal{B}_{n+1}\,|\, e_{n+1} < \infty \tag{18}$$

and

$$\sum_{n=n_0}^{\infty} |\,\mathcal{C}_{n+1} + \mathcal{A}_{n+1}\mathcal{B}_{n+1}\,|\,|q_n| < \infty. \tag{19}$$

Then equation (E) *has a nonoscillatory solution* $\{y_n\}$ *such that* $\{z_n\}$ *approaches a nonzero limit.*

Proof. Let $c > 0$ and let $N \in \mathbb{N}(n_0)$ be so large that from (18) and (19) we have

$$\sum_{n=N}^{\infty} |\,\mathcal{C}_{n+1} + \mathcal{A}_{n+1}\mathcal{B}_{n+1}\,|\,|e_n| < \frac{c(1-p)}{4},$$

$$\sum_{n=N}^{\infty} |\,\mathcal{C}_{n+1} + \mathcal{A}_{n+1}\mathcal{B}_{n+1}\,|\,|q_n < \frac{c(1-p)}{4f(5c)}$$

and

$$N_0 = N - m \geq n_0.$$

Let $\mathcal{B}_{N_0}$ be the linear space of all real sequences $y = \{y_n\}_{n=N_0}^{\infty}$ such that $||y|| = \sup\limits_{n \geq N_0} |y_n|$. It is not difficult to see that $\mathcal{B}_{N_0}$ endowed with this norm is a Banach space. Consider the set

$$\mathcal{S} = \left\{ y \in \mathcal{B}_{N_0} : \left(\frac{1-p}{2}\right)c \leq y_n \leq 5c \text{ for } n \geq N \text{ and } y_n = y_N \text{ for } N_0 \leq n < N \right\}$$

and define an operator $\mathcal{T} : \mathcal{S} \to \mathcal{B}_{N_0}$ by

$$(\mathcal{T}y)_n \begin{cases} (3+2p)c - p_n y_{n-k} - \sum\limits_{s=n}^{\infty} K(s,n) q_s f(y_{s-m}) - e_s) & \text{for } n \geq N; \\ (\mathcal{T}y)_N, & N_0 \leq n < N, \end{cases}$$

where $K(s,n) = \mathcal{C}_{s+1} - \mathcal{C}_n + \mathcal{A}_{s+1}\mathcal{B}_n - \mathcal{A}_{s+1}\mathcal{B}_{s+1}$. Clearly, $\mathcal{S}$ is a bounded closed and convex subset of $\mathcal{B}_{N_0}$.

First, we will show that $\mathcal{T}$ maps $\mathcal{S}$ into itself. For $y \in \mathcal{S}$ we have

$$\mathcal{T}y_n \geq c\left(3 + 2p - 5p - \frac{1-p}{2}\right) \geq \frac{(1-p)c}{2}$$

and

$$\mathcal{T}y_n \leq c\left(3 + 2p + \frac{1-p}{2}\right) \leq 5c.$$

Thus $\mathcal{T}\mathcal{S} \subset \mathcal{S}$.

Next we let $y = \{y_n\} \in \mathcal{S}$ and for each $i = 1, 2, 3, \cdots$, let $y^i = \{y_n^i\}$ be a sequence in $\mathcal{S}$ such that $\lim\limits_{i \to \infty} ||y^i - y|| = 0$. Then a straightforward argument using the continuity of f shows that $\lim\limits_{i \to \infty} ||(\mathcal{T}y^i)_n - (\mathcal{T}y)_n|| = 0$. Hence $\mathcal{T}$ is continuous.

Finally, in order to apply Schauder Fixed Point Theorem, we need to show that $\mathcal{T}\mathcal{S}$ is relatively compact. In view of a result of Cheng and Patula [2], it suffices to show that $\mathcal{T}\mathcal{S}$ is uniformly Cauchy. To this end, let $y =$

$\{y_n\} \in \mathcal{S}$ and observe that for any $j > n > N$, we have

$$\begin{aligned}|\mathcal{T}y_j - \mathcal{T}y_n| \leq\ & |p_j y_{j-k} - p_n y_{n-k}| \\ & + \sum_{s=n}^{\infty} |\mathcal{C}_{s+1} + \mathcal{A}_{s+1}\mathcal{B}_{s+1}| \left(|q_s| f(5c) + |e_s|\right) \\ \leq\ & 10pc + \frac{1-p}{2}c.\end{aligned}$$

From the hypothesis, it is clear that for a given $\epsilon > 0$, there exists an integer $N_1 \in \mathbb{N}(N)$ such that for all $j > n \geq N_1$, we have $|\mathcal{T}y_j - \mathcal{T}y_n| \leq \epsilon$. Then $\mathcal{TS}$ is uniformly Cauchy and so $\mathcal{TS}$ is relatively compact. Therefore, by Schauder Fixed Point Theorem, there is a fixed point $y \in \mathcal{S}$. It is clear that $y = \{y_n\}$ is a nonoscillatory solution of equation (E) and has the required asymptotic property.

Combining Theorem 3.6 with Lemma 2.4, we have the following corollary:

COROLLARY 3.7. *Assume that the conditions of Lemma* 2.4 *and Theorem* 3.6 *are satisfied. Then equation* (E) *has a nonoscillatory solution that approaches a nonzero limit.*

EXAMPLE 3.4. Consider the neutral difference equation

$$(\mathrm{E}_4) \quad \Delta^2\left(n^3\Delta\left(y_n + \frac{1}{4}y_{n-1}\right)\right) + (-1)^n 3^{-n} y_{n-m}^{\gamma} = (-1)^{n+1}2^{-n}, \quad n \geq 1$$

where γ is the ratio of odd positive integers and m is a positive integer. All conditions of Corollary 3.7 are satisfied so the equation (E_4) has a nonoscillatory solution that approaches a nonzero limit.

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