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NEW OSCILLATION CIRITERIA OF FIRST ORDER DELAY DIFFERENTIAL EQUATIONS

Abstract. In this paper we shall consider the first order delay differential equations with variable coefficients. Some new sufficient conditions for oscillation of all solutions are obtained. Our results based on the analysis of the generalized charachterestic equation. The results partially improve some previously known results in the literature. Some examples are considered to illustrate our main results.

1. Introduction

In recent years there has been much research activity concerning the oscillatory behavior of solutions of delay differential equations. To a large extent, this is due to the fact that, the delay differential equations are important in applications. New applications which involve delay differential equations continue to arise with increasing frequency in the modelling of diverse phenomena in Physics [8], Biology [18], Ecology [20], Physiology [10], and Spread of Infectious Diseases [5].

In this paper we shall consider the following first delay differential equation,

$$x'(t) + P(t)x(t-\tau) = 0, \quad t \geq t_0 \tag{1.1}$$

with

$$P(t) \in C[[t_0,\infty), R^+] \text{ and } \tau \in (0,\infty) \tag{h$_1$}$$

and the more general equation,

$$x'(t) + \sum_{i=1}^{n} P_i(t)x(t-\tau_i) = 0, \quad t \geq t_0 \tag{1.2}$$

with

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(h$_2$) $P_i(t) \in C[[t_0,\infty), R^+]$ and $\tau_i \in (0,\infty)$, for $i = 1,\dots,n$.

By a solution of the equation (1.1) we mean a function $x(t) \in C^1([t_0 - \tau), R)$, for some t_0, and satisfying the equation(1.1). Also by a solution of Eq.(1.2) we mean a function $x(t) \in C^1([t_0-\rho), R)$ for some t_0, and satisfying the equation(1.2), where $\rho = \{\max_{1\le i\le n} \tau_i\}$. As usual a function x(t) is called oscillatory if it has arbitrarily large zeros. Otherwise the solution is called non-oscillatory. The equation will be called oscillatory if every solution of this equation is oscillatory.

Many authors have considered the delay differential equation (1.1) and established some sufficient conditions for oscillation. The first systematic study for oscillation of all solutions of Eq.(1.1) was made by Myshkis [19]. Ladas, Lakshmikantham and Papadakis [15] obtained the well-known oscillation criterion for Eq.(1.1)

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} P(s)ds > 1. \tag{1.3}$$

Ladas [8] and, in 1982, Kopltadaze and Canturija [13] improved (1.3) and proved that every solution of Eq.(1.1) oscillates if

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} P(s)ds > \frac{1}{e}. \tag{1.4}$$

The conditions (1.3) and (1.4) are extended to Eq.(1.2) (see [17], [9]). Every solution of Eq.(1.2) oscillates if

$$\limsup_{t\to\infty} \int_{t-\tau_{\max}}^{t} \sum_{i=1}^{n} P_i(s)ds > 1 \tag{1.5}$$

or

$$\liminf_{t\to\infty} \int_{t-\tau_{\min}}^{t} \sum_{i=1}^{n} P_i(s)ds > \frac{1}{e}. \tag{1.6}$$

It is obvious that there is a gap between the conditions (1.3) and (1.4) for the oscillation of Eq.(1.1) when the limit,

$$\lim_{t\to\infty} \int_{t-\tau}^{t} P(s)ds$$

does not exist. How to fill this gap is an interesting problem which has been recently investigated by several authors (see [7], [4], [14], [11] and [12]). They have established some new finite sufficient conditions for oscillation of all solutions of Eq.(1.1) which extended and improved the conditions (1.3) and (1.4). The techniques that are used in [7], [4], [14], [11] and [12] are not applied to Eq.(1.2) to fill the gap between (1.5) and (1.6). For the oscillation of various functional differential equations we refer to the monographs [1, 2, 3, 6, 9, 17].

Our aim in this paper is to analyze the generalized characteristic equations to obtain some new sufficient conditions for oscillation of all solutions of Eqns. (1.1) and (1.2). Our results indicate that the conditions (1.3), (1.4) and (1.5), (1.6) are no longer necessaries for oscillation of all solutions of Eq.(1.1) and (1.2) recpectivelly. Then our results improve the results of [13], [17] and [9], and simply for verification throughout examples than of the results in [7], [4], [14], [11] and [12].

In the sequel, when we write a functional inequality we will assume that it holds for all sufficiently large values of t.

2. Main results

In this Section we establish some oscillation criteria for oscillation of Eq. (1.1) and Eq. (1.2).

THEOREM 2.1. *Assume that* (h_1) *holds,*

$$0 < d \leq \liminf_{t\to\infty} \int_t^{t+\tau} P(s)ds, \tag{2.1}$$

and

$$\int_{t_0}^{\infty} P(t) \ln\Big(\int_t^{t+\tau} P(s)ds + 1\Big)dt = \infty. \tag{h_3}$$

Then every solution of Eq. (1.1) *oscillates.*

Proof. Assume for the sake of contradiction, that the equation (1.1) has an eventually positive solution x(t). Let $\lambda(t) = -x'(t)(t)$, then $\lambda(t)$ is nonnegative and continuous, and there exists $t_1 \geq t_0$ such that $x(t_1) > 0$and $x(t) = x(t_1)\exp(-\int_{t_1}^t \lambda(s)ds)$. Furthermore, $\lambda(t)$ satisfies the generalized characteristic equation,

$$\lambda(t) = P(t)\exp\Big(\int_{t-\tau}^{t} \lambda(s)ds\Big).$$

As one can show that

$$e^{xr} \geq x + \frac{\ln(r+1)}{r} \quad \text{for all real } x \text{ and } r > 0. \tag{2.2}$$

Let A(t)=$\int_t^{t+\tau} P(s)ds$, then

$$\lambda(t) = P(t)\exp\Big[\frac{1}{A(t)}A(t)\int_{t-\tau}^{t}\lambda(s)ds\Big].$$

By using (2.2) we find that,

$$A(t)\lambda(t) - P(t)\int_{t-\tau}^{t}\lambda(s)ds \geq P(t)[\ln(A(t)+1)].$$

Then for $N > T$,

$$\int_T^N \lambda(t)A(t)dt - \int_T^N P(t)\int_{t-\tau}^{t}\lambda(s)dsdt \geq \int_T^N P(t)[\ln(A(t)+1)]dt. \tag{2.3}$$

By interchanging the order of integration, we find that

$$\int_T^N P(t)\Big(\int_{t-\tau}^{t}\lambda(s)ds\Big)dt \geq \int_T^{N-\tau}\lambda(t)\Big(\int_t^{t+\tau}P(s)ds\Big)dt.$$

Hence

$$\begin{aligned}\int_T^N \lambda(t)A(t)dt - \int_T^{N-\tau}\lambda(t)\Big(\int_t^{t+\tau}P(s)ds\Big)dt \\ \geq \int_T^N \lambda(t)A(t)dt - \int_T^N P(t)\int_{t-\tau}^{t}\lambda(s)dsdt.\end{aligned} \tag{2.4}$$

From (2.3) and (2.4), it follows that

$$\int_T^N \lambda(t)A(t)dt - \int_T^{N-\tau}\lambda(t)\Big(\int_t^{t+\tau}P(s)ds\Big)dt \geq \int_T^N P(t)[\ln(A(t)+1)]dt. \tag{2.5}$$

Since $x(t)$ is positive and decreasing, then integrating (1.1) from t to $t+\tau$, we get

$$x(t+\tau) - x(t) + \int_t^{t+\tau} P(s)x(s-\tau)ds = 0,$$

which gives

$$x(t) > \int_t^{t+\tau} P(s)x(t-\tau)ds > x(t)\int_t^{t+\tau} P(s)ds$$

and hence

$$\int_t^{t+\tau} P(s)ds < 1.$$

Thus, we have

$$d \le A(t) = \int_t^{t+\tau} P(s)ds < 1. \tag{2.6}$$

Combining (2.5) and (2.6), we find

$$\int_T^N \lambda(t)A(t)dt + \int_{N-\tau}^T \lambda(t)A(t)dt \ge \int_T^N P(t)[\ln(A(t)+1)]dt.$$

Hence

$$\int_{N-\tau}^N \lambda(t)dt \ge \int_T^N P(t)[\ln(A(t)+1)]dt$$

or

$$\ln \frac{x(N-\tau)}{x(N)} \ge \int_T^N P(t)[\ln(A(t)+1)]dt.$$

In view of (h_3)

$$\lim_{t\to\infty} \frac{x(t-\tau)}{x(t)} = \infty. \tag{2.7}$$

Because of $d < \int_t^{t+\tau} P(s)ds$ there exists a sequence $\{t_k\}$, $t_k \to \infty$ as $k \to \infty$ and there exists $\zeta_k \in (t_k, t_k+\tau)$ for every k such that

$$\int_{t_k}^{\zeta_k} P(s)ds \ge \frac{d}{2} \quad \text{and} \quad \int_{\zeta_k}^{t_k+\tau} P(s)ds \ge \frac{d}{2}. \tag{2.8}$$

By integrating the both sides of (1.1) over the intervals $[t_k, \zeta_k]$ and $[\zeta_k, t_k+\tau]$, we have

$$x(\zeta_k) - x(t_k) + \int_{t_k}^{\zeta_k} P(s)x(s-\tau)ds = 0 \tag{2.9}$$

and

$$x(t_k+\tau) - x(\zeta_k) + \int_{\zeta_k}^{t_k+\tau} P(s)x(s-\tau)ds = 0. \tag{2.10}$$

From (2.8), (2.9) and (2.10) we have,

$$-x(t_k) + \frac{d}{2}x(\zeta_k-\tau) \le 0 \quad \text{and} \quad -x(\zeta_k) + \frac{d}{2}x(t_k) \le 0.$$

Then

$$\frac{x(\zeta_k-\tau)}{x(\zeta_k)} \le \left(\frac{2}{d}\right)^2$$

which contradicts (2.7), and this completes the present proof. Therefore, every solution of Eq.(1.1) oscillates. ∎

EXAMPLE 2.1. Consider the delay differential equation

$$x'(t) + \frac{1}{4}((\sqrt{2} + \frac{1}{e})\frac{2}{\pi} + \cos t)x\left(t - \frac{\pi}{2}\right) = 0, \quad t \geq 0 \tag{2.11}$$

$$P(t) = \frac{1}{4}\left(\left(\sqrt{2} + \frac{1}{e}\right)\frac{2}{\pi} + \cos t\right) > 0 \quad \text{for } t \geq 0$$

and

$$\int_{t-\frac{\pi}{2}}^{t} P(s)ds = \frac{1}{4}\left(\int_{t-\frac{\pi}{2}}^{t} \left(\sqrt{2} + \frac{1}{e}\right)\frac{2}{\pi} + \cos s\right)ds$$
$$= \frac{1}{4}(\sqrt{2} + \frac{1}{e} + \sin t + \cos t).$$

It is clear that

$$\liminf_{t\to\infty} \int_{t-\frac{\pi}{2}}^{t} P(s)ds < \frac{1}{e}$$

and

$$\limsup_{t\to\infty} \int_{t-\frac{\pi}{2}}^{t} P(s)ds < 1.$$

Then the conditions (1.3) and (1.4) do not hold, but one can prove by Theorem 2.1 that every solution of Eq. (2.11) oscillates.

EXAMPLE 2.2. Consider the delay differential equation

$$x'(t) + \frac{0.6}{\alpha\pi + \sqrt{2}}(2\alpha + \cos t)x\left(t - \frac{\pi}{2}\right) = 0, \quad t \geq 0 \tag{2.12}$$

where $\alpha = \frac{\sqrt{2}(0.6e+1)}{\pi(0.6e-1)}$

$$P(t) = \frac{0.6}{\alpha\pi + \sqrt{2}}(2\alpha + \cos t) > 0 \quad \text{for } t \geq 0$$

and

$$\int_{t-\frac{\pi}{2}}^{t} P(s)ds = \int_{t-\frac{\pi}{2}}^{t} \frac{0.6}{\alpha\pi + \sqrt{2}}(2\alpha + \cos s)ds.$$

Hence

$$\liminf_{t\to\infty} \int_{t-\frac{\pi}{2}}^{t} P(s)ds = \frac{1}{e}, \quad \limsup_{t\to\infty} \int_{t-\frac{\pi}{2}}^{t} P(s)ds = 0.6.$$

This shows that (1.3) and (1.4) are failed to apply on the Eq. (2.12), but one can see by Theorem 2.1 that every solution of Eq. (2.12) oscillates.

EXAMPLE 2.3. Consider the delay differential equation

$$x'(t) + \left(\frac{1}{e} + \frac{1}{(t+1)}\right)x(t-1) = 0, \quad t \geq 0 \tag{2.13}$$

$$P(t) = \left(\frac{1}{e} + \frac{1}{(t+1)}\right) \quad \text{for } t \geq 0$$

and

$$\int_{t-1}^{t} P(s)ds = \int_{t-1}^{t} \left(\frac{1}{e} + \frac{1}{s+1}\right)ds = \log\frac{t+1}{t} + \frac{1}{e}.$$

Hence

$$\liminf_{t\to\infty} \int_{t-1}^{t} \overline{P}(s)ds = \frac{1}{e}.$$

Then the condition (1.3) is failed to apply on the Eq. (2.13), but one can see that for $T > 1$

$$\int_1^T \overline{P}(t)\ln\Big(\int_t^{t+1} P(s)ds + 1\Big)dt = \int_1^T \left(\frac{1}{e} + \frac{1}{t+1}\right)\ln\left(\ln\frac{t+2}{t+1} + \frac{1}{e} + 1\right)dt$$

$$\geq \int_1^T \frac{1}{e}\ln\left(\frac{t+2}{t+1}\right)dt \to \infty \ \text{ as } T \to \infty.$$

Then by Theorem 2.1 every solution of Eq. (2.13) oscillates.

EXAMPLE 2.4. Consider the delay differential equation

$$x'(t) + \frac{1}{8e}\left(1 - \frac{1}{2}\cos t\right)x(t-2\pi) = 0, \quad t \geq 0. \tag{2.14}$$

Clearly, for t$\geq$ 0,

$$\liminf_{t\to\infty} \int_{t-2\pi}^{t} P(s)ds = \int_{t-2\pi}^{t} \frac{1}{8e}\left(1 - \frac{1}{2}\cos(s)\right)ds < \frac{1}{e}$$

and

$$\limsup_{t\to\infty} \int_{t-2\pi}^{t} P(s)ds = \int_{t-2\pi}^{t} \frac{1}{8e}\left(1 - \frac{1}{2}\cos(s)\right)ds < 1.$$

Also

$$0 < \liminf_{t\to\infty} \int_{t}^{t+2\pi} P(s)ds = \liminf_{t\to\infty} \int_{t}^{t+2\pi} \frac{1}{8e}\left(1 - \frac{1}{2}\cos(s)\right)ds < \frac{1}{e}.$$

But one can prove easily that

$$\int_0^\infty P(t)\Big[\ln\Big\{\int_t^{t+2\pi} P(s)ds+1\Big\}\Big]dt=\infty$$

and then every solution of Eq.(2.14) oscillates. For instance

$$x(t)=\frac{1+2\cos^3 t}{\sin t-\frac{1}{8e}\cos t+\frac{1}{16e}\cos^2 t}$$

is such a solution.

In the following Theorems we give an infinite integral sufficient conditions for oscillation of all solutions of Eq. (1.2).

THEOREM 2.2. *Assume that,* (h$_2$) *holds* $\tau_n=\max\{\tau_i\}$, *for* $i=1,\ldots,n$,

$$0<d\le\liminf_{t\to\infty}\sum_{i=1}^{n}\int_t^{t+\tau_i}P_i(s)ds$$

and

(h$_4$) $\int_{t_0}^{\infty}\sum\limits_{i=1}^{n}P_i(t)\ln\Big(\sum\limits_{i=1}^{n}\int_t^{t+\tau_i}P_i(s)ds+1\Big)dt=\infty.$

Then every solution of Eq. (1.2) *oscillates.*

Proof. Assume for the sake of contradiction, that Eq.(1.2) has an eventually positive solution x(t). Let $\lambda(t)=-x'(t)(t)$, then $\lambda(t)$ is a nonnegative and continuous, and there exists $t_1\ge t_0$ with $x(t_1)>0$ such that $x(t)=x(t_1)\exp\left(-\int_{t_1}^{t}\lambda(s)ds\right)$. Furthermore, $\lambda(t)$ satisfies the generalized characteristic equation

$$\lambda(t)=\sum_{i=1}^{n}P_i(t)\exp\Big(\int_{t-\tau_i}^{t}\lambda(s)ds\Big). \tag{2.15}$$

Define B(t)=$\sum\limits_{i=1}^{n}\int_t^{t+\tau_i}P_i(s)ds$, and using (2.2) we find that

$$\lambda(t)B(t)-\sum_{i=1}^{n}P_i(t)\int_{t-\tau_i}^{t}\lambda(s)ds\ge\sum_{i=1}^{n}P_i(t)[\ln(B(t)+1)].$$

Then for $N>T$,

$$\int_T^N\lambda(t)B(t)dt-\sum_{i=1}^{n}\int_T^N P_i(t)\int_{t-\tau_i}^{t}\lambda(s)dsdt\ge\int_T^N\sum_{i=1}^{n}P_i(t)[\ln(B(t)+1)]dt. \tag{2.16}$$

By interchanging the order of integration, we find that

$$\int_T^N \sum_{i=1}^n P_i(t) \int_{t-\tau_i}^t \lambda(s)dsdt \geq \sum_{i=1}^n \int_T^{N-\tau_i} \lambda(t) \int_t^{t+\tau_i} P_i(s)dsdt.$$

Also as we choose $\tau_n \geq \tau_i$ for $i = 1, \dots, n$, then

$$\sum_{i=1}^n \int_T^{N-\tau_i} \lambda(t)\Big(\int_{t-\tau_i}^t P_i(s)ds\Big)dt \geq \int_T^{N-\tau_n} \lambda(t) \sum_{i=1}^n \int_t^{t+\tau_i} P_i(s)ds)dt. \tag{2.17}$$

From (2.16) and (2.17) we have,

$$\int_T^N \lambda B(t)(t)dt - \int_T^{N-\tau_n} \lambda(t)\sum_{i=1}^n \int_t^{t+\tau_i} P_i(s)dsdt \geq \int_T^N \sum_{i=1}^n P_i(t)[\ln(B(t)+1)]dt. \tag{2.18}$$

Hence

$$\int_T^N \lambda B(t)(t)dt + \int_{N-\tau_n}^T \lambda(t)\sum_{i=1}^n \int_t^{t+\tau_i} P_i(s)dsdt \geq \int_T^N \sum_{i=1}^n P_i(t)[\ln(B(t)+1)]dt$$

and therefore

$$\int_T^N \lambda(t)B(t)dt + \int_{N-\tau_n}^T \lambda(t)B(t)dt \geq \int_T^N \sum_{i=1}^n P_i(t)[\ln(B(t)+1)]dt. \tag{2.19}$$

On the other hand, as in Theorem 2.1 one can prove that

$$B(t) = \sum_{i=1}^n \int_t^{t+\tau_i} P_i(s)ds < 1. \tag{2.20}$$

Then by (2.19) and (2.20), we find that

$$\int_{N-\tau_n}^N \lambda(t)dt \geq \int_T^N \sum_{i=1}^n P_i(t)[\ln(B(t)+1)]dt$$

or

$$\ln \frac{x(N-\tau_n)}{x(N)} \geq \int_T^N \sum_{i=1}^n P_i(t)[\ln(B(t)+1)]dt.$$

In view of (h_4) we have

$$\lim_{t\to\infty} \frac{x(t-\tau_n)}{x(t)} = \infty. \tag{2.21}$$

However from Eq. (1.2) we have

$$x'(t) + P_n(t)x(t-\tau_n) \leq 0, t \geq t_0. \tag{2.22}$$

As in Theorem 2.1 one can prove that from Eq.(2.22)

$$\liminf_{t\to\infty} \frac{x(t-\tau_n)}{x(t)} < \infty$$

which contradicts (2.21) and this completes the present proof. Therefore, every solution of Eq. (1.2) oscillates. ∎

EXAMPLE 2.5: Consider the equation

$$x'(t) + \frac{1}{3e}(1+\cot s)x(t-\pi) + \frac{1}{15e}(1+\sin t)x(t-2\pi) = 0, \quad t \geq 0. \tag{2.23}$$

It is clear that

$$\begin{aligned}
&\lim_{t\to\infty} \inf \int_{t-\pi}^{t} (P_1(s)+P_2(s))ds \\
&\qquad = \lim_{t\to\infty} \inf \int_{t-\pi}^{t} \left(\frac{1}{3e}(1+\cos s) + \frac{1}{15e}(1+\sin s)\right)ds \\
&\qquad = \lim_{t\to\infty} \inf \left(\frac{2\pi}{5e} + \frac{2}{3e}\sin t - \frac{2}{15e}\cos t\right) \\
&\qquad = \frac{1}{15e}(6\pi - \sqrt{26}) < \frac{1}{e}.
\end{aligned}$$

This shows that (1.6) does not hold. Also

$$\lim_{t\to\infty} \sup \int_{t-\pi}^{t} (P_1(s)+P_2(s))ds < 1.$$

Then (1.5) does not hold. But one can prove by Theorem 2.2 that every solution of Eq.(2.23) oscillates.

Note that the results in Theorems 2.1 and 2.2 can be extended to the equation

$$\left(\frac{1}{r(t)}y(t)\right)' + R(t)y(t-\tau) = 0, t \geq t_0, \tag{2.24}$$

where $r(t)$, $P(t) \in C[[t_0,\infty), R^+]$ and $\tau \in (0,\infty)$, and to the more general equation,

$$\left(\frac{1}{r(t)}y(t)\right)' + \sum_{i=1}^{n} R_i(t)y(t-\tau_i) = 0, t \geq t_0, \tag{2.25}$$

where $r(t)$, P_i(t)$\in C[[t_0,\infty),R^+]$ and $\tau_i \in (0,\infty)$, for $i=1,\ldots,n$, by using the transformation

$$y(t)=x(t)e^{\int_{t_0}^{t}\frac{r'(s)}{r(s)}ds}$$

one can reduce Eq. (2.24) and (2.25) to the Eq. (1.1) and Eq. (1.2) respectively with P(t)=R(t)r(t-τ) and P$_i(t)$ =R$_i$(t)r(t-τ_i) and we obtain some sufficient conditions for oscillation of all solutions of (2.24) and (2.25).

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