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GLOBAL CONTINUA OF POSITIVE SOLUTIONS FOR SOME BOUNDARY VALUE PROBLEMS

I. Introduction

In this paper we shall apply some general results on the existence of solutions in cone, in combining with the special properties of concave real functions to show the existence of a continua of positive solutions for the following differential equations

$$x'' + \lambda a(t) f(x) = 0, \quad 0 < t < 1, \tag{1}$$

$$x'' + \lambda^2 f\left(x, \frac{1}{\lambda} x'\right) = 0, \quad 0 < t < 1 \tag{2}$$

with the boundary condition

$$x(0) = x(1) = 0. \tag{3}$$

Boundary value problem (1)–(3) describes some phenomena in the mathematical sciences and has been studied by many authors (see [3] and references therein). Boundary value problem (2)–(3) was considered by Krasnosel'skii for finding $2\lambda-$ periodic solution of the autonomous differential equation $x'' + f(x, x') = 0$, arising in celestial mechanics [5]. Obtained results in our paper will extend the studying in [3], [5] on problems (1)–(3) and (2)–(3).

Let us recall some preliminaries about ordered Banach spaces and fixed point theorems in them. Let X be a real Banach space and $K \subset X$ be a cone (i.e K is nonempty closed convex subset such that $tK \subset K$ for all $t \geq 0$ and $K \cap (-K) = \{0\}$). We define a partial ordering by $x \leq y$ iff $y - x \in K$.

Together with the cone K, defining an ordering in X, we consider another cone $P \subset K$, in which we want to seek solution of the equation

$$x = F(\lambda, x), \tag{4}$$

where $F : (0, \infty) \times P \longrightarrow P$ is a completely continuous operator.

We denote by $\sum$ the solution set of the equation (4), that is,

$$\sum = \{(\lambda, x) \in (0,\infty) \times P \mid x = F(\lambda, x)\},$$

and we set

$$S = \{x \in P \mid \exists \lambda \in (0,\infty) : x = F(\lambda, x)\}.$$

If operator F is differentiable at 0 or has a homogeneous monotone minorant, then the existence of an unbounded subcontinuum in $\sum$ can be studied by using general theorems of Dancer, Amann and others [1, 2]. In our work, considered operators are not required to be either differentiable or to have a monotone minorant and instead of the solution set $\sum$ we shall study its projection S onto x−space. We use the following definition of Krasnosel'skii [5].

DEFINITION. We say that S is an unbounded continuous branch, emanating from 0 if $S \cap \partial G \neq \emptyset$ for every bounded open subset $G \ni 0$.

The next theorem will be fundamental in this paper, it can be proved by using the fixed point index [1, 6].

THEOREM A. *Let $F : (0,\infty) \times P \longrightarrow P$ be a completely continuous operator and G be a bounded open neighborhood of zero. Assume that there are numbers λ_1, λ_2 in $(0,\infty)$ and an element $u \in P \setminus \{0\}$ such that*

i) $\mu x \neq F(\lambda_1, x)$ for $x \in P \cap \partial G$ and $\mu \geq 1$,

ii) $x - \mu x \neq F(\lambda_2, x)$ for $x \in P \cap \partial G$ and $\mu \geq 0$.

Then, $S \cap \partial G \neq \emptyset$.

To determine values of λ, for which there exist solutions in $P \setminus \{0\}$ of the equation (4), we can apply the following theorem of Krasnosel'skii [5].

THEOREM B. *Let $F(\lambda, x) = \lambda G(x)$, where $G : P \longrightarrow P$ is completely continuous and let for each $x \in S$, $\lambda(x)$ be a positive number such that $(\lambda(x), x) \in \sum$. Assume that the set S of the equation (4) is bounded continuous, emanating from 0 and either*

$$\lim_{\|x\|\to 0} \sup \lambda(x) = \lambda_0 < \lambda_\infty = \lim_{\|x\|\to\infty} \inf \lambda(x)$$

or

$$\lim_{\|x\|\to \infty} \sup \lambda(x) = \lambda_\infty < \lambda_0 = \lim_{\|x\|\to 0} \inf \lambda(x).$$

Then for every $\lambda \in (\lambda_0, \lambda_\infty)$ (respectively, for every $\lambda \in (\lambda_\infty, \lambda_0)$) the equation (4) has a solution in $P \setminus \{0\}$.

The paper is organized as follows. In Section 2 we present some properties of concave functions and estimates for spectral radius of some linear integral operators. These technical results will be used in Section 3 to prove the existence of a continua of positive solutions for boundary value problems (1)–(3) and (2)–(3).

2. Technical lemmes

Throughout this section we denote by $X = C_{[0,1]}$ with norm $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$. Let X be ordered by the cone K of all nonegative functions and let P be the cone of all concave functions $x \in K$ such that $x(0) = x(1) = 0$.

2.1. First we present some properties of functions from P.

LEMMA 1. *1) Every function $x \in P$ is differentiable a.e (almost everywhere) on $(0,1)$ and satisfies*

$$x(t) \geq \|x\|.t(1-t), \quad t \in [0,1], \tag{6}$$

$$|x'(t)| \leq \frac{x(t)}{t(1-t)} \quad a.e \ \ on\ (0,1). \tag{7}$$

2) If sequence $\{x_n\} \subset P$ converges in $C_{[0,1]}$ to a function x, then some subsequence $\{x'_{n_k}\}$ converges a.e on $(0,1)$ to function x'.

Proof.

1) Let $\|x\| = x(t_0)$ for some $t_0 \in (0,1)$. We have by the concavity of x that

$$x(t) \geq \frac{t}{t_0} x(t_0) \quad \text{for } t \in [0, t_0]$$

$$x(t) \geq \frac{1-t}{1-t_0} x(t_0) \quad \text{for } t \in [t_0, 1],$$

so (6) follows.

It is well-known that, every concave on $[0,1]$ function x is differentiable a.e on $(0,1)$. If x is differentiable at some $t \in (0,1)$, then

$$x(t) \geq x(s) + x'(t)(t-s) \text{ for } s \in [0,1].$$

Putting $s = 0$ and $s = 1$ we obtain

$$x(t) \geq x'(t)t, \quad x(t) \geq x'(t)(t-1),$$

which proves (7).

2) It follows from the concavity of x_n that x'_n is nonincreasing on its domain. We define for $n = 1, 2, \ldots$, and $t \in (0,1)$

$$y_n(t) = \inf\{x'_n(s) |\ s \in [0,t],\ x'_n(s) \text{ exists}\}.$$

The sequence $\{y_n\}$ of nonincreasing functions is uniformly bounded on every interval $[a,b] \subset (0,1)$ by (7). Hence, some subsequence is convergent at every $t \in [a,b]$. Using the argument on diagonal sequence, we conclude that some subsequence $\{y_{n_k}\}$ converges to a function y at every $t \in (0,1)$. Since $y_n(t) = x'_n(t)$ a.e. on $(0,1)$, we have $\lim x'_{n_k}(t) = y(t)$ a.e. on $(0,1)$. It remains to show that $y(t) = x'(t)$ a.e on $(0,1)$. Consider a fixed but arbitrary

$[s,t] \subset (0,1)$, we have by absolute continuity of x_{n_k} that [7]

$$x_{n_k}(t) - x_{n_k}(s) = \int_s^t x'_{n_k}(u)du.$$

By letting $k \longrightarrow \infty$ we get by the Dominated Convergence Theorem

$$x(t) - x(s) = \int_s^t y(u)du.$$

Thus, $x'(t) = y(t)$ a.e on $(0,1)$. The lemma is proved.

2.2. Now we will obtain some estimates for the spectral radius of linear integral operators.

Let $G : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ be the Green's function for the boundary value problem $-x'' = 0$ in $(0,1)$, $x(0) = x(1) = 0$, so that

$$G(t,s) = \begin{cases} t(1-s) \text{ if } 0 \le t \le s \le 1 \\ s(1-t) \text{ if } 0 \le s \le t \le 1. \end{cases} \tag{8}$$

Assume $a : [0,1] \longrightarrow [0,\infty)$ is a continuous function, not vanishing identically on $[0,1]$ and define $a_\varepsilon : [0,1] \longrightarrow [0,\infty)$ such that $a_\varepsilon(t) = a(t)$ on $(\varepsilon, 1-\varepsilon)$, $a_\varepsilon(t) = 0$ on $[0,\varepsilon] \cup [1-\varepsilon, 1]$. Consider the following linear integral operators

$$Bx(t) = \int_0^1 G(t,s)a(s)x(s)ds, \tag{9}$$

$$B_\varepsilon x(t) = \int_0^1 G(t,s)a_\varepsilon(s)x(s)ds, \tag{10}$$

and denote by $r(B), r(B_\varepsilon)$ the spectral radius of B, B_ε.

LEMMA 2.

1) $\lim_{\varepsilon \to 0} r(B_\varepsilon) = r(B)$.

2) $r(B)$ is an eigenvalue of B with an eigenfunction from P.

3) If $\alpha x \le B_\varepsilon x$ for some $x \in P \setminus \{0\}$ then $\alpha \le r(B_\varepsilon)$.

If $B_\varepsilon x \le \beta x$ for some $x \in P \setminus \{0\}$ then $r(B_\varepsilon) \le \beta$.

The analogous assertions are valid also for operator B.

Proof. Assertion 1) follows from $\lim_{\varepsilon \to 0} B_\varepsilon = B$ in $L(X)$ and that the operator $A \longmapsto r(A)$ from $L(X)$ to $\mathbb{R}$ is continuous.

It can be verified that

$$t(1-t)s(1-s) \le G(t,s) \le t(1-t) \quad \text{on} \quad [0,1] \times [0,1].$$

From these inequalities we easily prove that operator B is u_0-bounded and operator B_ε is u_0-bounded above in the sense of [5, 6], where $u_0(t) =$

$t(1-t)$. Therefore, assertions 2) and 3) of the lemma are consequences of general results on positive u_0–bounded linear operators in ordered Banach spaces [5, 6].

3. Global continua of positive solutions for boundary value problems

In this section we still keep definitions of X, K and P as in the section 2.

3.1. First we study boundary value problem (1)–(3) under following hypotheses:

(A) $f : [0,\infty) \longrightarrow [0,\infty)$ is continuous and does not vanish identically on any subinterval,

(B) $a : [0,1] \longrightarrow [0,\infty)$ is continuous and $a(t)$ does not vanish identically,

(C) $f_0 = \lim\limits_{x\to 0} \dfrac{f(x)}{x}$ and $f_\infty = \lim\limits_{x\to\infty} \dfrac{f(x)}{x}$ exist (can be equal to ∞) and $f_0 \neq f_\infty$.

These conditions are the same as in [3] with a difference that we allow function a to be zero on a set of positive measure. We shall prove that the problem (1)–(3) has a global continua of positive solutions and that the set of $\lambda' s$, for which (1)–(3) has positive solutions, contains an explicit interval. This interval is larger than the interval, obtained in [3].

The boundary value problem (1)–(3) is equivalent to the eigenvalue problem

$$x(t) = \lambda \int_0^1 G(t,s)a(s)f[x(s)]ds, \tag{11}$$

where, function G is defined in (8). If we denote by F the operator in the right-hand side of (11) then $F : P \longrightarrow P$ is completely continuous.

THEOREM 1. *Assume that conditions (A), (B) are satisfied. Then the set S from* (5) *for equation* (11) *is an unbounded continuous branch, emanating from 0.*

Proof. Let G be a bounded open subset containing 0. We set

$$m = \inf\{\|x\| \mid x \in P \cap \partial G\}, \quad M = \sup\{\|F(x)\| \mid x \in P \cap \partial G\}.$$

If $\mu x = \lambda F x$ for $\mu > 0$, $\lambda > 0$ and $x \in P \cap \partial G$ then $\mu m \leq \lambda M$. Hence, the condition i) in the theorem A will be satisfied if λ_1 is sufficiently small.

Now, we shall show that condition ii) in the theorem A holds for sufficiently large λ_2 and $u(t) = t(1-t)$. To do this, we suppose the contradiction that

$$x_n - \mu_n u = \lambda_n F x_n, \quad n = 1, 2, \ldots$$

for $x_n \in P \cap \partial G$, $\mu_n \geq 0$ and $\lambda_n \to \infty$ as $n \to \infty$.

From inequality (7), sequence $\{x'_n\}$ is uniformly bounded on every interval $[a,b] \subset (0,1)$. Therefore, we can use the Ascoli's theorem to take from sequence $\{x_n\}$ a subsequence, still denoted by $\{x_n\}$, which converges at every point $t \in (0,1)$ to some continuous on $(0,1)$ function x such that $x(t) \geq mt(1-t)$ on $(0,1)$. Passing to the limit in inequality

$$\frac{x_n(t)}{\lambda_n} \geq Fx_n = \int_0^1 G(t,s)a(s)f(x_n(s))ds$$

by the Fatou Lemma, we get

$$0 \geq \int_0^1 G(t,s)a(s)f(x(s))ds,$$

which contradicts with the condition (A).

Thus, the equation (11) has a solution on $P \cap \partial G$ by the theorem A. Theorem is proved.

THEOREM 2. *Assume that conditions (A), (B) and (C) are satisfied and let* λ_1 *be the smallest eigenvalue of the following boundary value problem*

$$x'' + \lambda a(t)x = 0 \quad in\ (0,1), \quad x(0) = x(1) = 0.$$

Then for each λ *satisfying*

$$\min\left\{\frac{\lambda_1}{f_0}, \frac{\lambda_1}{f_\infty}\right\} < \lambda < \max\left\{\frac{\lambda_1}{f_0}, \frac{\lambda_1}{f_\infty}\right\},$$

there exists at least one solution of (11) in $P \setminus \{0\}$.

Here we impose that $\lambda_1/0 = \infty$, $\lambda_1/\infty = 0$.

Proof. We provide the argument only for the case $f_0 < f_\infty$. The case $f_\infty < f_0$ is treated in a completely similar way. We will prove that

$$\liminf_{\|x\|\to 0} \lambda(x) \geq \frac{\lambda_1}{f_0}, \tag{12}$$

$$\limsup_{\|x\|\to 0} \lambda(x) \leq \frac{\lambda_1}{f_\infty}. \tag{13}$$

Then, the assertion of the theorem will follow from Theorem B.

Consider a number m such that $m < \lambda_1/f_0$ and choose a positive number r so that $f(x) < \lambda_1 x/m$ for $x < r$. If $x \in S$, $x \neq 0$, $\|x\| < r$, then

$$x = \lambda(x)Fx \leq \frac{\lambda(x)\lambda_1}{m}Bx,$$

where B is the linear operator, defined in (9). Hence, by using Lemma 2 we have $m/\lambda(x)\lambda_1 \leq r(B)$. Taking account that $r(B) = 1/\lambda_1$ we get $\lambda(x) \geq m$. Thus $\liminf_{\|x\|\to 0} \lambda(x) \geq m$. Since m can be arbitrarily closed to λ_1/f_0, then (12) is proved.

To prove (13) we consider arbitrary numbers m, k such that $\lambda_1/f_\infty < m < k$ and choose r satisfying $f(x) > \lambda_1 x/m$ for $x > r$. Then we choose by Lemma 2 a number $\varepsilon > 0$ so small that

$$r(B_\varepsilon) > \frac{m}{k} r(B) = \frac{m}{k\lambda_1}. \tag{14}$$

If $x \in S$, $\|x\| > r/\varepsilon^2$ then we have from (6)

$$x(t) \geq \|x\| t(1-t) \geq \|x\|\varepsilon^2 > r \quad \text{for } t \in [\varepsilon, 1-\varepsilon].$$

Hence,

$$\begin{aligned} x(t) &= \lambda(x) \int_0^1 G(t,s)a(s)f(x(s))ds \\ &\geq \lambda(x) \int_0^1 G(t,s)a_\varepsilon(s)\frac{\lambda_1}{m}x(s)ds = \frac{\lambda(x)\lambda_1}{m} B_\varepsilon(x), \end{aligned}$$

where function a_ε and operator B_ε are defined in section 2.2. Applying Lemma 2 we have

$$\frac{m}{\lambda(x)\lambda_1} \geq r(B_\varepsilon). \tag{15}$$

Combining (14) and (15) we get $\lambda(x) \leq k$. Since k can be arbitrarily close to λ_1/f_∞, then (13) is established. This completes the proof.

3.2. Now we consider boundary value problem (2)–(3). This problem arises when we want to find a 2λ–periodic solution of the autonomous differential equation

$$y'' + f(y, y') = 0. \tag{16}$$

Indeed, if we assume that the function $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is odd on y then by a solution (λ, x) of (2)–(3) we can construct by [5] a 2λ–periodic solution of (16) as follows. We first extend x on $[-1, 0]$ by setting $x(t) = -x(-t)$, then we periodically extend x to obtain a 2λ–periodic function on $\mathbb{R}$. Now, the function $y(t) = x(t/\lambda)$ will be a 2λ–periodic solution for (16).

To study (2)–(3) we impose the following hypothesis

(A) $f : [0, \infty) \times \mathbb{R} \longrightarrow [0, \infty)$ is continuous such that

$$g(x) \leq f(x, x') \leq h(x)(1 + |x'|^r),$$

where $r \in (0, 1)$, the functions h, g are nonegative continuous and g does not vanish identically on any interval.

We reduce problem (2)–(3) to the following integro-differential equation

$$x(t) = \lambda^2 \int_0^1 G(t,s) f[x(s), \frac{1}{\lambda}x'(s)]ds, \tag{17}$$

where G is Green's function from (8).

We denote by $F(\lambda, x)$ the operator in right-hand side in (17). From the upper restriction for f in the hypothesis (A) and the inequality (7) we see that the operator F is acting from $(0, \infty) \times P$ into P and takes every subset $[a, b] \times \{x \in P, \|x\| \le r\}$ into relatively compact subset. It follows easily from Lemma 1 and Dominated Convergence Theorem that, if a sequence (λ_n, x_n) converges to $(\lambda, x) \in (0, \infty) \times P$, then a subsequence of $\{F(\lambda_n, x_n)\}$ converges to $F(\lambda, x)$. Hence, F is continuous.

THEOREM 3. *Assume that the condition (A) is satisfied. Then the equation* (17) *has an unbounded continuous branch of positive solutions, emanating from* 0.

Proof. Consider an arbitrary bounded open subset $G \ni 0$. Arguing as in the proof of Theorem 1 we see that the condition ii) in Theorem A is satisfied for equation (17) if λ_2 is sufficiently large. To verify the condition i) in Theorem A we assume that $\mu x = F(\lambda, x)$ for some $x \in P \cap \partial G$, $\mu > 0, \lambda > 0$.

From the upper restriction for f and (7) we have

$$\mu\|x\| \le \lambda^2 \int_0^1 h(x(s)) \left(1 + \frac{|x'(s)|^r}{\lambda^r}\right) ds$$

$$\le C. \int_0^1 \left(\lambda^2 + \frac{\|x\|^r . \lambda^{2-r}}{s^r (1-s)^r}\right) ds,$$

where $C = \sup\{h(x(t)) \mid x \in P \cap \partial G\}$. Since $\inf\{\|x\| \mid x \in P \cap \partial G\} > 0$, we conclude that $\mu < 1$ if λ sufficiently small.

Thus, by using Theorem A we see that the equation (17) has a solution on $P \cap \partial G$. The Theorem is proved.

REMARK. The paper [4] has proved the existence of a global continua of positive solutions for problem (2)–(3) under hypothesis, different from (A), when the upper restriction for f is weaker, but the lower restriction is stronger than those assumed in (A).

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