DEMONSTRATIO MATHEMATICA
Vol. XXXV No 2 2002

Jan Bochenek

AN ABSTRACT SECOND ORDER CAUCHY PROBLEM WITH NON-DENSELY DEFINED OPERATOR, II

Abstract. By using the theory of the extrapolation space X_{-1} associated with an operator A which is non-densely defined in a Banach space X, the existence and uniqueness of solutions of the semilinear second order differential initial value problem (1) is proved.

1. Introduction

We continue the study of abstract semilinear second order initial value problem

$$(1)\qquad \begin{cases} \dfrac{d^2u}{dt^2} = Au + f(t, u, \dfrac{du}{dt}), & t \in (0,T] \\ u(0) = u_0, \dfrac{du}{dt}(0) = u_1, & u_0, u_1 \in X. \end{cases}$$

In (1) X is a Banach space, u is a mapping from $\mathbb{R}$ to X, f is a nonlinear mapping from $\mathbb{R} \times X \times X$ into X. In the preceding paper [2] we have discuses the problem of existence uniqueness and smoothness of solutions of the linear problem corresponding to (1) when the operator A is non-densely defined. The present paper is devoted to investigate the semilinear problem (1). Recall that a solution of (1) is defined as usual, as a function $u : [0,T] \to X$ twice continuously differentiable in $(0,T]$ and once continuously differentiable in $[0,T]$ such that $u(t) \in D(A)$ for $t \in [0,T]$ and (1) holds.

Our main tools in this paper are the theory of strongly continuous cosine families of linear operators in a Banach space,a certain weak continuous cosine family and some extrapolation spaces associated with a linear operator A.

1991 *Mathematics Subject Classification*: 34K30, 34G20.

Key words and phrases: semilinear initial value problems, extrapolation space, strongly continuous cosine family.

2. Preliminaries

Let the operator A in Section 1 be closed such that its resolvent set $\rho(A)$ contains $\{\lambda^2 : \lambda > \omega\}$, and

$$\|\frac{d^n}{d\lambda^n}[\lambda R(\lambda^2, A)]\| \leq \frac{Mn!}{(\lambda-\omega)^{n+1}} \text{ for } \lambda > \omega, n \in N \tag{2}$$

and some $M \geq 1,\ \omega \in R$.

We do not recall the definition or properties of the cosine family generated by the operator A satisfying (2). For this we refer (e.g. [1, 2, 4 and 7]). We recall only the definition and some properties of extrapolation spaces from [2], see also [3],[5] and [6].

Let A be a closed linear operator on the Banach space X with non-empty resolvent set $\rho(A)$. We do not assume that A is densely defined. We define (see [6])

$$X^{-1} := (X \times X)/G_A, \tag{3}$$

where G_A denote the graph of the operator A. Note that G_A is a closed linear subspace of $X \times X$ since A is closed. Let us define

$$i : X \ni x \to ix := (0, x) \in X^{-1}. \tag{4}$$

The function (4) maps the space X onto the linear subspace iX of X^{-1}. This allows us to identify X with iX. We also define a linear operator A^{-1} on X^{-1} by

$$D(A^{-1}) := iX, \tag{5}$$

$$A^{-1}(0, x) := (-x, 0) \text{ for } x \in X. \tag{6}$$

Note that, if $x \in D(A)$ then $(-x, 0) = (0, Ax)$. The operator A^{-1} should not be confused with the inverse of A if this inverse exists. If we identify iX with X, we may regard A^{-1} as a bounded linear operator $X \to X^{-1}$. In fact if $x \in D(A)$ then $A^{-1}x = A^{-1}(ix) := A^{-1}(0, x) = iAx = Ax$, so A^{-1} is an extension of A. In the space X^{-1} it may be defined an equivalent norm by formula

$$|(x, y)|_\mu := \|AR(\mu, A)x - R(\mu, A)y\| \tag{7}$$

for each $\mu \in \rho(A)$ and $(x, y) \in X^{-1}$.

THEOREM 1 ([6; Th.3.1.6]). *The space X is dense in X^{-1} if and only if A is densely defined, i.e. $\overline{D(A)} = X$.*

If the operator A is closed with nonempty resolvent set, we define the space X_{-1} as the closure of X in the norm of X^{-1}. From this and Theorem 1 follows that if A is densely defined, then $X_{-1} = X^{-1}$.

Let us denote by A_{-1} the part of A^{-1} in X_{-1} and by A_0 the part of A in $X_0 := \overline{D(A)}$. Clearly, A_{-1} is an extension of A.

We have the following

THEOREM 2 ([6; Prop. 3.1.9]). *If A is closed and $\lambda \in \rho(A)$, then*

(i) $D(A_{-1}) = X_0$ *and* $\lambda - A_{-1} : X_0 \to X_{-1}$ *is an isomorphism*

(ii) A *is the part of* A_{-1} *in* X*: if* $\lambda \in \rho(A)$*, then* $\lambda \in \rho(A_{-1})$ *and* $R(\lambda, A) = R(\lambda, A_{-1})|_X$.

In the sequel we shall need the following theorem which is analogous to Theorem 3.1.10 in [6].

THEOREM 3. *Let A be a closed linear operator on X which resolvent $R(\lambda^2, A)$ exists for $\lambda > \omega$ and which satisfies the inequality* (2). *Then:*

(i) A_0 *generates a cosine family* $\{C_0(t); t \in \mathbb{R}\}$ *on* X_0 *and* $R(\lambda^2, A_0) = R(\lambda^2, A)|_{X_0}$

(ii) X_0 *is* X_{-1} *dense in* X *and* $(X_0)_{-1}$ *is isomorphic to* X_{-1},

(iii) *under the identification* $(X_0)_{-1} = X_{-1}$ *we have* $(A_0)_{-1} = A_{-1}$.

The proof of this theorem is given in [2].

THEOREM 4 ([6. Th.3.1.11 and 2, Th.7]). *Under the assumptions of Theorem 3, the cosine family* $\{C_0(t) : t \in \mathbb{R}\}$ *generated by* A_0 *on* X_0 *extends to a cosine family* $\{C_{-1}(t); t \in \mathbb{R}\}$ *on* X_{-1} *whose generator is the operator* A_{-1}.

Let A be a closed linear operator on a Banach space X with non-empty resolvent set $\rho(A)$, satisfying to (2).

We denote

$$X_0^{\odot} = (X_0)^{\odot} := \{x^\star \in X^\star : \lim_{\lambda \to \infty} \|\lambda R(\lambda, A)^\star x^\star - x^\star\| = 0 \tag{8}$$

(cf. [6, Lemma 3.1.12]). Since the restriction of $R(\lambda, A)^\star$ to $X_0^{\odot}$ form the resolvent of some densely defined operator on $X_0^{\odot}$, then we define

$$A_0^{\odot} : X_0^{\odot} \to X_0^{\odot}$$

such that $R(\lambda, A_0^{\odot}) := R(\lambda, A)^\star|_{X_0^{\odot}}$. From this and from (2) follows that the operator $A_0^{\odot}$ is the generator of a cosine family on $X_0^{\odot}$. This cosine family we denote by $\{C_0^{\odot}(t) : t \in \mathbb{R}\}$ and the statement is that $C_0^{\odot}(t) = C_0^\star(t)|_{X_0^{\odot}}$. We define

$$X^{\odot\times} := \{x^{\odot\star} \in X^{\odot\star} : R(\lambda, A^{\odot\star})x^{\odot\star} \in jX\}, \tag{9}$$

where the map j is the natural embedding of X into $X^{\odot\star}$.

The subspace $X^{\odot\times} \subset X^{\odot\star}$ is closed and $C^{\odot\star}(t)$ invariant. If the operator A satisfies the inequality (2) and is non-densely defined we define

$$X_0^{\odot\times} := (X_0)^{\odot\times}. \tag{10}$$

We have the following

PROPOSITION 1 ([6, Prop. 4.3.1]). *If A is the generator of the cosine family $\{C(t), t \in \mathbb{R}\}$ on the space X, then*

$$X = \{x^{\odot\times} \in X^{\odot\times} : \lim_{t\to 0} \|C^{\odot\times}(t)x^{\odot\times} - x^{\odot\times}\| = 0\}. \tag{11}$$

If A satisfies (2) *and is non-densely defined, then*

$$X \subset X_0^{\odot\times} \tag{12}$$

and this inclusion is continuous.

3. Existence and uniqueness of solutions of problem (1)

Following [6,Ch.4] we shall use the space $X_0^{\odot\times}$ to obtain a method of solving the problem (1) for an arbitrary operator A satisfying (2). The basic idea of this method comes from [6;Ch.4]. For a given operator A satisfying (2) we use that $X \subset X_0^{\odot\times}$ (see(12)) and first we study the problem (1) in the space $X_0^{\odot\times}$. Solutions to this problem which lie in X_0 are likely to be also solutions to the problem (1) (cf.[2] and [6,Ch.4]).

We have

LEMMA 1 ([2, Lemma 1]). *If:*

1^0 *$A : X \supset D(A) \to X$ is a linear operator satisfying* (2),

2^0 *$f : [0, T] \to X$ is continuous,*

then $s \to C_{-1}(t-s)f(s)$ is Bochner integrable in X_{-1} and the mapping

$$[0,T] \ni t \to v(t) := \int_0^t C_{-1}(t-s)f(s)ds \text{ is a norm continuous } X_0 \tag{13}$$

valued function such that

$$\|v(t)\|_{X_0} \le Mt\|f\|_{C([0,T],X_0^{\odot\times})}, \tag{14}$$

where $X_0 := \overline{D(A)}, M := sup\{\|C_0(t)\| : t \in [0,T]\}$, $\{C_0(t), t \in \mathbb{R}\}$ is a cosine family generated by A_0 and $\{C_{-1}(t); t \in \mathbb{R}\}$ is a cosine family on X_{-1} which is an extension of the cosine family $\{C_0(t); t \in \mathbb{R}\}$ (cf. Th.4).

Now we turn to study the problem (1). We recall the following

DEFINITION 1. A function $u : [0, T] \to X_0$ is said to be a (classical) solution of the problem (1) if:

(i) $u \in C^1([0, T]) \cap C^2((0, T])$,

(ii) $u(0) = u_0$ and $u'(0) = u_1$,

(iii) $u''(t) = Au(t) + f(t, u(t), u'(t))$ for $t \in (0, T]$.

Now we shall prove

THEOREM 5. *If:*

1^0 $A : X \supset D(A) \to X$ *is a closed linear non-densely defined operator satisfying (2),*

2^0 $f : [0,T] \times X_0 \times X_0 \to X$ *is continuous,*

3^0 $u_0, u_1 \in X_0$

then every classical solution of problem (1) *is a solution of the following integral equation*

$$u(t) = \mathcal{C}_0(t)u_0 + \mathcal{S}_0(t)u_1 + \int_0^t \mathcal{S}_{-1}(t-s)f(s,u(s),u'(s))ds, \tag{16}$$

where $\mathcal{S}_0(t)x := \int_o^t \mathcal{C}_0(s)xds$ *for* $t \in \mathbb{R}, x \in X_0$ *and by extending the family* $\{\mathcal{S}_0(t) : t \in \mathbb{R}\}$ *on the space* X_{-1} *we get* $\{\mathcal{S}_{-1}(t) : t \in \mathbb{R}\}$.

Proof. According to [6.Ch.4] and [2,Th.11], we first study the following problem in X_{-1}

$$\begin{cases} \frac{d^2u}{dt^2} = A_{-1}u + f(t,u,u'(t)), \ t \in (0,T], \\ u(0) = u_0, \ u'(0) = u_1. \end{cases} \tag{17}$$

Since A_{-1} is the generator of the cosine family $\{C_{-1}(t); t \in \mathbb{R}\}$ on X_{-1} and the function $f : [0,T] \times X_0 \times X_0 \to X$ which is continuous is also continuous as $f : [0,T] \times X_0 \times X_0 \to X_{-1}$, standard arguments of cosine function theory (see e.g.[7]), show that every classical solution of (17) must be a solution of the following integral equation

$$u(t) = C_{-1}(t)u_o + \mathcal{S}_{-1}(t)u_1 + \int_0^t S_{-1}(t-s)f(s,u(s),u'(s))ds. \tag{18}$$

But since $u_0, u_1 \in X_0$ and $C_{-1}(t)|_{X_0} = C_0(t)$ and $\mathcal{S}_{-1}(t)|_{X_0} = S_0(t)$, for $t \in R$, we obtain that the equations (16) and (18) are identical, and so every classical solution of (17) is a solution of (16).

On the other hand every classical solution of (1) is a classical solution of (17). This implies that every classical solution of (1) is a solution of (16).

DEFINITION 2. Every function $u \in C^1([0,T], X_0)$ which satisfies the integral equation (16) is said to be a mild solution of the problem (1).

THEOREM 6. *Let the assumptions* 1^0 *and* 2^0 *of Theorem* 5 *hold. Suppose that there exists* $L > 0$ *such that*

$\|f(t,x,y) - f(t,u,v)\| \leq L(\|x-u\| + \|y-v\|)$ *for* $t \in [0,T], \ x,y,u,v \in X_0$.

Then for any $u_0 \in E_0$ *and* $u_1 \in X_0$ *there exists exactly one solution of the integral equation* (16) *belonging to* $C^1([0,T], X_0)$, *where*

$$E_0 := \{x \in X_0 : C_0(t)x \text{ is once continuously differentiable in } t\}.$$

Proof. Note first that since $f : [0,T] \times X_0 \times X_0 \to X$ is continuous the map $\tilde{f} : [0,T] \to X$ is also continuous, where $\tilde{f}(t) := f(t, u(t), u'(t))$ for $t \in [0,T]$ and $u \in C^1([0,T], X_0)$.A necessary condition that the solution of (18) be of class C^1 is that $u_0 \in E_0$ and $u_1 \in X_0$. From this, by Lemma 1, the mappings:

$$[0,T] \ni t \to \int_0^t C_{-1}(t-s) f(s, u(s), u'(s)) ds \quad \text{and}$$

$$[0,T] \ni t \to \int_0^t \mathcal{S}_{-1}(t-s) f(s, u(s), u'(s)) ds$$

are norm continuous X_0 -valued functions.

Therefore the mapping G, defined by

$$(19)\quad (Gu)(t) := C_0(t)u_0 + \mathcal{S}_0(t)u_1 + \int_0^t \mathcal{S}_{-1}(t-s) f(s, u(s), u'(s)) ds,\ t \in [0,T],$$

is a mapping from $C^1([0,T], X_0)$ into itself.

Now a standard contraction mapping argument (cf. e.g. [1, Th. 4]) shows that G has a unique fixed point u, which is obviously a mild solution of (1). On the other hand, every mild solution of (1) is a fixed point of G. Therefore u is unique.

THEOREM 7. *Suppose that*

1^0 $A : X \supset D(A) \to X$ *is a closed linear non-densely defined operator satisfying* (2),

2^0 $f : [0,T] \times X_0 \times X_0 \to X$ *is of class* C^1,

3^0 $u_0 \in D(A) \cap E_0$, $u_1 \in E_0$,

4^0 $Au_0 + f(0, u_0, u_1) \in X_0$,

then any $u \in C^1([0,T], X_0)$, *which is a solution of the equation (16), is a classical solution of problem (1).*

Proof. At first we prove that, under the assumptions of this theorem, each solution of (16) is twice continuously differentiable in $(0,T]$. Indeed, u satisfying (16), by definition, $u \in C^1([0,T], X_0)$.

Differentiating (16) we get

$$v(t) := u'(t) = A_0 \mathcal{S}_0(t) u_0 + C_0(t) u_1 + \int_0^t C_{-1}(t-s) f(s, u(s), v(s)) ds$$

and so

$$v(t) = A_0 S_0(t)u_0 + C_0(t)u_1 + \int_0^t C_{-1}(s) f(t-s, u(t-s), v(t-s))ds.$$

Let $t, t+h \in (0, T]$, where t is fixed and $h \neq 0$. We have

$$\begin{aligned} v(t+h) - v(t) = {} & A_0[S_0(t+h) - S_0(t)]u_0 + [C_0(t+h) - C_0(t)]u_1 \\ & + \int_0^{t+h} C_{-1}(s)[f(t+h-s, u(t+h-s), v(t+h-s))ds \\ & - \int_0^t C_{-1}(s) f(t-s, u(t-s), v(t-s))ds. \end{aligned}$$

Since A_0 is the part of A we get

$$\begin{aligned} A_0[S_0(t+h) - S_0(t)]u_0 &= A[S_0(t+h) - S_0(t)]u_0 \\ &= [S_{-1}(t+h) - S_{-1}(t)]Au_0 = \int_0^h C_{-1}(t+s)Au_0 ds, \end{aligned}$$

$$[C_0(t+h) - C_0(t)]u_1 = hA_0 S_0(t)u_1 + \omega_1(t,h)h$$

and so

$$\begin{aligned} v(t+h) - v(t) = {} & \int_0^h C_{-1}(t+s)Au_0 ds + [A_0 S_0(t)u_1 + \omega_1(t,h)]h \\ & + \int_0^{t+h} C_{-1}(s) f(t+h-s, u(t+h-s), v(t+h-s))ds \\ & - \int_0^t C_{-1}(s) f(t-s, u(t-s), v(t-s))ds. \end{aligned}$$

From this we obtain after some rearrangements

$$\begin{aligned} (20) \qquad \frac{1}{h}[v(t+h) - v(t)] = {} & \frac{1}{h}\int_0^h C_{-1}(t+s)[Au_0 + f(0, u_0, u_1)]ds \\ & + A_o S_0(t)u_1 + \omega_1(t,h) \\ & + \frac{1}{h}\int_0^t C_{-1}(t-s)[f(s+h, u(s+h), v(s+h) - f(s, u(s), v(s))]ds \\ & + \frac{1}{h}\int_0^h C_{-1}(t+s)[f(h-s, u(h-s), v(h-s)) - f(0, u_0, u_1)]ds. \end{aligned}$$

From the assumption on f we have

$$
\begin{aligned}
(21)\qquad & f(s+h,u(s+h),v(s+h))-f(s,u(s),v(s)) \\
&= \frac{\partial f}{\partial s}(s,u(s+h),v(s+h))h+\omega_2(s,h)h \\
&\quad+\frac{\partial f}{\partial u}(s,u(s),v(s+h))u'(s)h+\omega_3(s,h)h \\
&\quad+\frac{\partial f}{\partial v}(s,u(s),v(s))[v(s+h)-v(s)]+\omega_4(s,h)h,
\end{aligned}
$$

where $\|\omega_i(s,h)\| \to 0$ when $h \to 0$ for $i = 1,\dots,4$, uniformly in $s \in [0,T]$.

We employ the following notation:

$$
\begin{aligned}
B(s) &:= \frac{\partial f}{\partial v}(s,u(s),v(s)), \\
g(t) &:= C_0(t)x_0 + A_0S_0(t)u_1 \\
&\quad+\int_0^t C_{-1}(t-s)[\frac{\partial f}{\partial s}(s,u(s),v(s))+\frac{\partial f}{\partial u}(s,u(s))u'(s)]ds
\end{aligned}
$$

where $x_0 := Au_0 + f(0,u_0,u_1) \in X_0$.

By Lemma 1, $g \in C([0,T],X)$. Moreover, the function $F(t,w) := B(t)w$ is continuous in t, and satisfies the Lipschitz condition in w uniformly in t,since the mapping $[0,T] \ni t \to B(t) \in B(X)$ is continuous. Let $w \in C([0,T],X)$ be a solution of

$$
w(t) = g(t) + \int_0^t C_{-1}(t-s)B(s)w(s)ds.
$$

The existence of such a solution follows from the assumptions on F, g and from Lemma 1. Denoting by

$$
(22)\qquad w_h(t) := h^{-1}[v(t+h)-v(t)] - w(t),
$$

we obtain from (20)–(22)

$$
\begin{aligned}
(23)\qquad w_h(t) &= \omega_1(t,h) \\
&+\frac{1}{h}\int_0^h C_{-1}(t+s)[f(h-s),u(h-s),v(h-s))-f(0,u(0),v(0))]ds \\
&+\int_0^t C_{-1}(t-s)\left[\frac{\partial f}{\partial s}(s,u(s+h),v(s+h))-\frac{\partial f}{\partial s}(s,u(s),v(s))\right]ds \\
&+\int_0^t C_{-1}(t-s)[\frac{\partial f}{\partial u}(s,u(s),v(s+h))-\frac{\partial f}{\partial u}(s,u(s),v(s))]u'(s)ds
\end{aligned}
$$

$$+\int_0^t C_{-1}(t-s)[\omega_2(s,h)+\omega_3(s,h)+\omega_4(s,h)]ds$$

$$+[\frac{1}{h}\int_0^h C_{-1}(t+s)(Au_0+f(0,u_0,u_1))ds - C_0(t)x_0]$$

$$+\int_0^t C_{-1}(t-s)B(s)w_h(s)ds.$$

Since the inclusion $X \subset X_0^{\odot\times}$ is continuous, by assumption 2^0, we see that all integrands are *weak** - continuous $X_0^{\odot\star}$ - valued functions.

Hence the integrals, in (23), can be interpreted as weak *- integrals in $X_0^{\odot\star}$. Therefore,by inequality (14) in Lemma 1, the first four integrals on the right side of (23) tend to zero as $h \to 0$ in the norm of $X_0^{\odot\star}$, hence in X. Let us remark that

$$\lim_{h\to 0}\left\|\frac{1}{h}\int_0^h C_{-1}(t+s)(Au_0+f(0,u_0,u_1))ds - C_0(t)x_0\right\| = 0$$

in X_0 because $Au_0 + f(0,u_0,u_1) = x_0 \in X_0$.

From this and from definition and properties of $\omega_i(t,h)$, for $i = 1,\dots,4$, it follows that the norm of each one of the six first terms on the right hand of the equality (23) tends to zero as $h \to 0$. Therefore from (23), by Gronwall's inequality, we have

$$\lim_{h\to 0} w_h(t) = 0 \text{ in norm of } X_0. \tag{24}$$

The equality (24) means that there exists $v'(t) = u''(t)$ and $u''(t) = w(t)$ for $t \in (0,T]$. Since $w \in C([0,T],X)$ so $u \in C^2((0,T],X)$.

Now we may to differentiate twice the equality (16) in X_{-1}. This and (24) show that

$$u''(t) = A_{-1}u(t) + f(t,u(t),u'(t))$$

for $t \in (0,T]$. It remains to show that $u(t) \in D(A)$ for $t \in (0,T]$. But since $u(t) \in X_0$ and u is twice differentiable in X we get that $u'(t) \in X_0$ and that $u''(t)$ belongs to X. We obtain

$$A_{-1}u(t) = u''(t) - f(t,u(t),u'(t)) \in X.$$

Since A is the part of A_{-1} in X, it follows that $u(t) \in D(A)$, which completes the proof.

As a consequence of Theorems 6 and 7 we get

THEOREM 8. *Under the assumptions of Theorem* 7 *the problem* (1) *has a unique classical solution which is the unique solution of the integral equation* (16).

REMARK. In this paper and in the previous paper [2] we study the Cauchy problem for the second order equation directly using the theory of the operator cosine function generated by the operator A. In my opinion this method is natural.

It is known the other (standard) method to reduce the second order equation to a system of first order equations.Unfortunately this method needs an additional assumptions on the operator A (see for example [7, Prop. 2.7.]).

References

[1] J. Bochenek, *An abstract nonlinear second order differential equation*, Ann. Polon. Math. 54 (1991), 155–166.
[2] —, *An abstract second order Cauchy problem with non-densely defined operator, I*, Demonstratio Math. 33 (2000), 489–501.
[3] G. Da Prato, P. Grisvard, *Maximal regularity for evolution equations by interpolation and extrapolation*, J. Func. Anal. 58 (1984), 285–344.
[4] H. O. Fattorini, *Second order linear differential equations in Banach spaces*, North-Holland, New York, 1985.
[5] R. Nagel, *Sobolev spaces and semigroups*, Semesterberichte Funktionalanalysis 4 (1983), 1–20.
[6] J. van Neerven, *The adjoint of a semigroup of linear operators LNM 1529*, Springer-Verlag, 1992.
[7] C. C. Travis and G. F. Webb, *Cosine families and abstract nonlinear second order differential equations*, Acta Math. Acad. Sci. Hungar 32 (1978), 75–96.

INSTITUTE OF MATHEMATICS
CRACOW UNIVERSITY OF TECHNOLOGY
Warszawska 24,
31-155 KRAKÓW, POLAND

Received October 9, 2000; revised version December 12, 2001.