DEMONSTRATIO MATHEMATICA
Vol. XXXV No 2 2002

Teodor Bulboacă

CLASSES OF FIRST-ORDER DIFFERENTIAL SUPERORDINATIONS

Abstract. Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ be an analytic function in a domain $\Delta \subset \mathbb{C}^2$, let p an analytic function in the unit disc U such that $\varphi(p(z), zp'(z))$ is univalent in U and suppose that p satisfies the first-order differential superordination

$$h(z) \prec \varphi(p(z), zp'(z)).$$

In the case when $\varphi(p(z), zp'(z)) = \alpha(p(z)) + \beta(p(z))\,\gamma(zp'(z))$ we determine conditions on h, α, β and γ so that the above subordination implies $q(z) \prec p(z)$, where q is the *largest* function so that $q(z) \prec p(z)$ for all p functions satisfying the first-order differential superordination. The concept of differential superordination was introduced by S. S. Miller and P. T. Mocanu in [2] like a dual problem of differential subordination [1].

1. Introduction

Let denote by $H(\mathrm{U})$ the class of analytical functions in the unit disc $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$ and for $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ let

$$H[a,n] = \left\{ f \in H(\mathrm{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots ,\ z \in \mathrm{U} \right\}.$$

If $f,\ F \in H(\mathrm{U})$ and F is univalent in U we say that the function f is *subordinate* to F, or F is *superordinate* to f, written $f(z) \prec F(z)$, if $f(0) = F(0)$ and $f(\mathrm{U}) \subseteq F(\mathrm{U})$.

Let $\varphi : \mathbb{C}^3 \times \overline{\mathrm{U}} \to \mathbb{C}$, let $h \in H(\mathrm{U})$ and $q \in H[a,n]$. In [2] the authors determined conditions on φ such that

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z)$$

implies $q(z) \prec p(z)$, for all p functions that satisfies the above superordination. Moreover, they found sufficient conditions so that the q function is the *largest* function with this property, called the *best subordinant* of this subordination.

1991 *Mathematics Subject Classification*: Primary 30C80; Secondary 30C45, 30A20.
Key words and phrases: differential subordination, differential superordination, convex, starlike, univalent function.

The present paper deals with the case when

$$\varphi(p(z), zp'(z)) = \alpha(p(z)) + \beta(p(z))\,\gamma(zp'(z)).$$

We determine conditions on h, α, β and γ so that this subordination implies $q(z) \prec p(z)$, and we find its *best subordinant* q.

2. Preliminaries

In order to prove our main results we will need to use the next definitions and lemmas.

Let $h \in H(\mathrm{U})$ with $h(0) = 0$, $h'(0) \neq 0$. We say that h is a *starlike* (univalent) function if

$$\operatorname{Re}\frac{zh'(z)}{h(z)} > 0,\ z \in \mathrm{U}.$$

A function $h \in H(\mathrm{U})$ is called to be *convex* (univalent) if

$$\operatorname{Re}\frac{zh''(z)}{h'(z)} + 1 > 0,\ z \in \mathrm{U}.$$

We say that the function $L : \mathrm{U} \times [0, +\infty) \to \mathbb{C}$ is a *subordination (or Loewner) chain* if $L(\cdot, t)$ is analytic and univalent in U for all $t \geq 0$, $L(z, \cdot)$ is continuously differentiable on $[0, +\infty)$ for all $z \in \mathrm{U}$, and $L(z, s) \prec L(z, t)$ when $0 \leq s \leq t$. The following lemma gives us sufficient conditions for L to be a subordination chain.

LEMMA 2.1 ([3, p. 159]). *The function* $L(z, t) = a_1(t)z + a_2(t)z^2 + \ldots$, *with* $a_1(t) \neq 0$ *for* $t \geq 0$ *and* $\lim_{t\to+\infty} |a_1(t)| = +\infty$, *is a subordination chain if*

$$\operatorname{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] > 0,\ z \in \mathrm{U},\ t \geq 0. \tag{2.1}$$

For the next lemma that we will use to prove our result we need the following definition.

DEFINITION 2.1 ([2]). We denote by Q the set of functions h that are analytic and injective on $\overline{\mathrm{U}} \setminus E(h)$, where

$$E(h) = \left\{\zeta \in \partial\mathrm{U} : \lim_{z\to\zeta} h(z) = \infty\right\},$$

and such that $h'(\zeta) \neq 0$ for $\zeta \in \partial\mathrm{U} \setminus E(h)$.

LEMMA 2.2 ([2, Theorem 7]). *Let* $q \in H[a, 1]$, *let* $\varphi : \mathbb{C}^2 \to \mathbb{C}$ *and set* $\varphi(q(z), zq'(z)) \equiv h(z)$. *If* $L(z, t) = \varphi(q(z), tzq'(z))$ *is a subordination chain and* $p \in H[a, 1] \cap Q$, *then*

$$h(z) \prec \varphi(p(z), zp'(z)) \quad \Rightarrow \quad q(z) \prec p(z).$$

Furthermore, if $\varphi(q(z), zq'(z)) = h(z)$ has a univalent solution $q \in Q$, then q is the best subordinant.

3. Main results

THEOREM 3.1. *Let q be a convex (univalent) function in the unit disc* U, *let α, $\beta \in H(D)$, where $D \supset q(\mathrm{U})$ is a domain, and let $\gamma \in H(\mathbb{C})$. Suppose that*

$$\text{(3.1)} \qquad \operatorname{Re} \frac{\alpha'(q(z)) + \beta'(q(z))\,\gamma(tzq'(z))}{\beta(q(z))\,\gamma'(tzq'(z))} > 0,\ \forall z \in \mathrm{U} \text{ and } \forall t \geq 0.$$

If $p \in H[q(0), 1] \cap Q$, with $p(\mathrm{U}) \subset D$, and $\alpha(p(z)) + \beta(p(z))\,\gamma(zp'(z))$ is univalent in U, *then*

$$\alpha(q(z)) + \beta(q(z))\,\gamma(zq'(z)) = h(z) \prec \alpha(p(z)) + \beta(p(z))\,\gamma(zp'(z)) \Rightarrow q(z) \prec p(z),$$

and q is the best subordinant.

Proof. Setting $\varphi(p(z), zp'(z)) = \alpha(p(z)) + \beta(p(z))\,\gamma(zp'(z))$, by the assumption we have $h(z) \prec \varphi(p(z), zp'(z))$ and $\varphi(p(z), zp'(z))$ is univalent in U.

If we let

$$L(z,t) = \alpha(q(z)) + \beta(q(z))\,\gamma(tzq'(z)) = a_1(t)z + \dots,$$

then

$$\frac{\partial L(0,t)}{\partial z} = \beta(q(0))\gamma'(0)q'(0)\left[t + \frac{\alpha'(q(0)) + \beta'(q(0))\,\gamma(0)}{\beta(q(0))\,\gamma'(0)}\right].$$

From the univalence of q we have $q'(0) \neq 0$ and by using (3.1) for $z = 0$ we deduce that

$$a_1(t) = \frac{\partial L(0,t)}{\partial z} \neq 0,\ \forall t \geq 0, \quad \text{and} \quad \lim_{t \to +\infty} |a_1(t)| = +\infty.$$

A simple computation shows that

$$\operatorname{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] = \operatorname{Re}\left[\frac{\alpha'(q(z)) + \beta'(q(z))\,\gamma(tzq'(z))}{\beta(q(z))\,\gamma'(tzq'(z))} + t\left(1 + \frac{zq''(z)}{q'(z)}\right)\right].$$

According to (3.1) and using the fact that q is a convex (univalent) function in U we obtain

$$\operatorname{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] > 0,\ z \in \mathrm{U},\ t \geq 0,$$

and by Lemma 2.1 we conclude that L is a subordination chain. Now, applying Lemma 2.2 we obtain our result. ∎

Taking $\beta(w) = 1$ in the above Theorem we get the next corollary:

COROLLARY 3.1. *Let q be a convex (univalent) function in the unit disc* U, $\alpha \in H(D)$, *where* $D \supset q(\mathrm{U})$ *is a domain, and let* $\gamma \in H(\mathbb{C})$. *Suppose that*

$$\mathrm{Re}\,\frac{\alpha'(q(z))}{\gamma'(tzq'(z))} > 0,\ \forall z \in \mathrm{U}\ \textit{and}\ \forall t \geq 0.$$

If $p \in H[q(0),1] \cap Q$, *with* $p(\mathrm{U}) \subset D$, *and* $\alpha(p(z)) + \gamma(zp'(z))$ *is univalent in* U, *then*

$$\alpha(q(z)) + \gamma(zq'(z)) \prec \alpha(p(z)) + \gamma(zp'(z)) \ \Rightarrow\ q(z) \prec p(z),$$

and q is the best subordinant.

For the particular case when $\gamma(w) = w$, using a similar proof as in Theorem 3.1 we obtain:

COROLLARY 3.2. *Let q be a univalent function in the unit disc* U *and let* α, $\beta \in H(D)$, *where* $D \supset q(\mathrm{U})$ *is a domain. Suppose that*

(i) $\mathrm{Re}\,\dfrac{\alpha'(q(z))}{\beta(q(z))} > 0,\ \forall z \in \mathrm{U}$,

(ii) $Q(z) = zq'(z)\beta(q(z))$ *is a starlike (univalent) function in* U.

If $p \in H[q(0),1] \cap Q$, *with* $p(\mathrm{U}) \subset D$, *and* $\alpha(p(z)) + zp'(z)\beta(p(z))$ *is univalent in* U, *then*

$$\alpha(q(z)) + zq'(z)\,\beta(q(z)) \prec \alpha(p(z)) + zp'(z)\beta(p(z)) \ \Rightarrow\ q(z) \prec p(z),$$

and q is the best subordinant.

For the case $\beta(w) = 1$, using the fact that the function $Q(z) = zq'(z)$ is starlike (univalent) in U if and only if q is convex (univalent) in U, Corollary 3.2 becomes:

COROLLARY 3.3. *Let q be a convex (univalent) function in the unit disc* U *and let* $\alpha \in H(D)$, *where* $D \supset q(\mathrm{U})$ *is a domain. Suppose that*

$$\mathrm{Re}\,\alpha'(q(z)) > 0,\ \forall z \in \mathrm{U}. \tag{3.2}$$

If $p \in H[q(0),1] \cap Q$, *with* $p(\mathrm{U}) \subset D$, *and* $\alpha(p(z)) + zp'(z)$ *is univalent in* U, *then*

$$\alpha(q(z)) + zq'(z) \prec \alpha(p(z)) + zp'(z) \ \Rightarrow\ q(z) \prec p(z),$$

and q is the best subordinant.

Next we will give some particular cases of the above results obtained for appropriate choices of the q, α and β functions.

EXAMPLE 3.1. [2, Theorem 8] Let q be a convex (univalent) function in the unit disc U and let $\gamma \in \mathbb{C}$, with $\mathrm{Re}\,\gamma > 0$. If $p \in H[q(0),1] \cap Q$ and

$p(z) + \dfrac{zp'(z)}{\gamma}$ is univalent in U, then

$$q(z) + \frac{zq'(z)}{\gamma} \prec p(z) + \frac{zp'(z)}{\gamma} \ \Rightarrow \ q(z) \prec p(z),$$

and q is the best subordinant.

Proof. Taking $\alpha(w) = w$ and $\beta(w) = 1/\gamma$, $\operatorname{Re}\gamma > 0$, in Corollary 3.2, condition (i) holds if $\operatorname{Re}\gamma > 0$ and (ii) holds if and only if q is a convex (univalent) function in U. ∎

EXAMPLE 3.2. Let q be a convex (univalent) function in the unit disc U and suppose that

$$|\operatorname{Im} q(z)| < \frac{\pi}{2},\ z \in \mathrm{U}. \tag{3.3}$$

If $p \in H[q(0), 1] \cap Q$ and $e^{p(z)} + zp'(z)$ is univalent in U, then

$$e^{q(z)} + zq'(z) \prec e^{p(z)} + zp'(z) \ \Rightarrow \ q(z) \prec p(z),$$

and q is the best subordinant.

Proof. Considering in Corollary 3.3 the case $\alpha(w) = e^w$, then condition (3.2) becomes

$$\operatorname{Re}\alpha'(q(z)) = e^{\operatorname{Re} q(z)} \cos(\operatorname{Im} q(z)) > 0,\ z \in \mathrm{U},$$

that is equivalent with (3.3). ∎

REMARK 3.1. Taking $q(z) = \lambda z$, $|\lambda| \leq \pi/2$ in Example 3.2 we have the next result:

If $p \in H[0, 1] \cap Q$ such that $e^{p(z)} + zp'(z)$ is univalent in U and $|\lambda| \leq \pi/2$, then

$$e^{\lambda z} + \lambda z \prec e^{p(z)} + zp'(z) \ \Rightarrow \ \lambda z \prec p(z),$$

and λz is the best subordinant.

EXAMPLE 3.3. Let q be a convex (univalent) function in the unit disc U and suppose that

$$\operatorname{Re} q(z) > \beta,\ z \in \mathrm{U}. \tag{3.4}$$

If $p \in H[q(0), 1] \cap Q$ and $\dfrac{p^2(z)}{2} - \beta p(z) + zp'(z)$ is univalent in U, then

$$\frac{q^2(z)}{2} - \beta q(z) + zq'(z) \prec \frac{p^2(z)}{2} - \beta p(z) + zp'(z) \ \Rightarrow \ q(z) \prec p(z),$$

and q is the best subordinant.

Proof. If we consider in Corollary 3.3 the case $\alpha(w) = \dfrac{w^2}{2} - \beta w$, then we may easily see that (3.2) is equivalent with (3.4). ∎

REMARK 3.2. 1. The function $q(z) = e^{\lambda z}$ is convex (univalent) in U if and only if $|\lambda| \leq 1$, and under this assumption $\operatorname{Re} q(z) > 0$, $z \in$ U. Using the result of Example 3.3 for this choice we obtain:

If $p \in H[1,1] \cap Q$ such that $\frac{p^2(z)}{2} + zp'(z)$ is univalent in U and $|\lambda| \leq 1$, then

$$\frac{e^{2\lambda z}}{2} + \lambda z e^{\lambda z} \prec \frac{p^2(z)}{2} + zp'(z) \Rightarrow e^{\lambda z} \prec p(z),$$

and $e^{\lambda z}$ is the best subordinant.

2. The function $q(z) = \frac{1 + (2\beta - 1)z}{1 + z}$, $\beta < 1$, is convex (univalent) in U and $\operatorname{Re} q(z) > \beta$, $z \in$ U. Hence, by using Example 3.3 we have:

If $p \in H[1,1] \cap Q$ such that $\frac{p^2(z)}{2} - \beta p(z) + zp'(z)$ is univalent in U and $\beta < 1$, then

$$\frac{1 - 2\beta - 2(2\beta^2 - 4\beta + 3)z + (1 - 2\beta)z^2}{2(1+z)^2} \prec \frac{p^2(z)}{2} - \beta p(z) + zp'(z)$$
$$\Rightarrow \frac{1 + (2\beta - 1)z}{1 + z} \prec p(z),$$

and $\frac{1 + (2\beta - 1)z}{1 + z}$ is the best subordinant.

References

[1] S. S. Miller and P. T. Mocanu, *Differential Subordinations. Theory and Applications*, Dekker, New York, 1999

[2] —, *Subordinants of differential superordinants*, to appear in Proc. Amer. Math. Soc.

[3] Ch. Pommerenke, *Univalent Functions*, Vanderhoeck and Ruprecht, Göttingen, 1975.

DEPARTMENT OF FUNCTION THEORY
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
BABEŞ-BOLYAI UNIVERSITY
3400 CLUJ-NAPOCA, ROMANIA
E-mail: bulboaca@math.ubbcluj.ro

Received April 4, 2001.