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SOME PROPERTIES OF A CLASS OF ANALYTIC FUNCTIONS

Abstract. We give, among other results, some criteria for p-valence of functions $f(z) = z^p + a_{p+1}z^{p+1} + \dots$ analytic in the unit disc.

1. Introduction

Let A(p) denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in N = \{1, 2, 3, \dots\}) \tag{1.1}$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$. Further for $\lambda \geq 0$ and $f \in A(p)$, we define a function F_λ by

$$F_\lambda(z) = (1 - \lambda)f(z) + \lambda z f'(z). \tag{1.2}$$

In [11], Saitoh has derived some properties of functions in the class A(p), and of the functions F_λ defined by (1.2). In fact, he proved the following results.

THEOREM A. *If $f \in A(p)$ satisfies the condition*

$$Re\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\} > \alpha \quad \left(0 \leq \alpha < \frac{p!}{(p-j)!}, \quad z \in E\right),$$

then

$$Re\left\{\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right\} > \frac{1}{(p-j+1)!} \quad \frac{(p-j+1)!2\alpha + p!}{2(p-j)+3} \quad (z \in E),$$

where $1 \leq j \leq p$.

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THEOREM B. *Let F_λ be defined by (1.2) for $\lambda \geq 0$ and $f \in A(p)$. If*

$$Re\left\{\frac{F_\lambda^{(j)}(z)}{z^{p-j}}\right\} > \alpha \quad \left(0 \leq \alpha < \frac{p!(1-\lambda+\lambda p)}{(p-j)!}, \quad z \in E\right),$$

then

$$Re\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\} > \frac{(p-j)!2\alpha + p!\lambda}{(p-j)!(2-\lambda+\lambda p)} \qquad (z \in E),$$

where $0 \leq j \leq p$.

In the present paper, we improve the results of Saitoh[11] for functions belonging to the class A(p), and for the functions F_λ for $0 \leq \lambda \leq 1$ and $f \in A(p)$. We also derive certain sufficient conditions for functions in A(p) to be p-valent in E. Some properties of functions in A(p) are also obtained.

2. Preliminaries and main results

To establish our main results, we need the following lemmas.

LEMMA 1 [3]. *Let w be non-constant analytic in E with $w(0) = 0$. If $|w|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, then we have $z_0 w'(z) = kw(z_0)$, where k is real and $k \geq 1$.*

LEMMA 2 [10]. *If q is analytic in E with $q(0)=1$, and if γ is a complex number satisfying $Re(\gamma) \geq 0 (\gamma \neq 0)$, $\alpha < 1$, then*

$$Re\{q(z) + \gamma z q'(z)\} > \alpha \quad (z \in E)$$

implies that

$$Re\{q(z)\} > \alpha + (1-\alpha)(2\rho - 1) \quad (z \in E),$$

where ρ given by

$$\rho = \rho(Re\gamma) = \int_0^1 (1 + t^{Re\gamma})^{-1} dt$$

is an increasing function of Re γ and $(1 + Re\gamma)/(1 + 2Re\gamma) \leq \rho < 1$. The estimate is best possible in the sense that the bound cannot be improved.

LEMMA 3. *Let $f \in A(p)$. If there exists a (p-k+1)-valent starlike function $g(z) = z^{p-k+1} + b_{p-k+2} z^{p-k+2} + \ldots$ that satisfies*

$$Re\{\frac{zf^{(k)}(z)}{g(z)}\} > 0 \quad (z \in E)$$

then f is p-valent in E.

We owe the above lemma to Nunokawa [6].

For real or complex numbers a,b, and $c (c \neq 0, -1, -2, \ldots)$, the hypergeometric series

$$F(a,b;c;z) = 1 + \frac{a \cdot b}{1 \cdot c} + \frac{a(a+1) \cdot b(b+1)}{2!c(c+1)} z^2 + \ldots$$

represents an analytic function in E [1, p. 556]. The following identities are well known [1, p. 556–558].

LEMMA 4. *For real or complex numbers a,b, and $c(c \neq 0, -1, -2, \ldots)$, we have*

$$(2.1) \quad \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt$$

$$= \frac{\Gamma(b).\Gamma(c-b)}{\Gamma(c)} F(a,b;c;z)(Re(c) > Re(b) > 0);$$

$$(2.2) \qquad F(a,b;c;z) = (1-z)^{-a} F\left(a, c-b; c; \frac{z}{z-1}\right);$$

$$(2.3) \qquad F\left(1,1;2;\frac{\delta z}{\delta z+1}\right) = \frac{(\delta z+1) ln(\delta z+1)}{\delta z} \quad (\delta \neq 0);$$

and

$$(2.4) \qquad F(1,1;3;\frac{\delta z}{\delta z+1}) = \frac{2(\delta z+1)}{\delta z}\{1 - \frac{ln(\delta z+1)}{\delta z}\} \quad (\delta \neq 0).$$

REMARK. Putting $z = \frac{1}{\delta}(\delta \neq 0)$ in the identities in (2.3) and (2.4), we get

$$F\left(1,1;2;\frac{1}{2}\right) = 2ln2; F\left(1,1;3;\frac{1}{2}\right) = 4(1-ln2).$$

We now prove

THEOREM 1. *If $f \in A(p)$ satisfies the condition*

$$(2.5) \; Re\left\{(1-\lambda)\left(\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right)^{\mu} + \lambda \frac{f^{(j)}(z)}{z^{p-j+1}}\left(\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right)^{\mu-1}\right\} > \alpha \quad (z \in E)$$

for some $\mu > 0$, $\lambda > 0$ and $\alpha < (1 + \lambda(p-j))(p!/(p-j+1)^{\mu}$, then

$$(2.6) \quad Re\left\{\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right\}^{\mu}$$

$$> \left[\frac{\alpha}{1+\lambda(p-j)} + \left\{\left(\frac{p!}{(p-j+1)!}\right)^{\mu} - \frac{\alpha}{1+\lambda(p-j)}\right\}(2\rho - 1)\right] \quad (z \in E),$$

where $1 \leq j \leq p$, $\rho = F(1,1;\frac{\mu\{1+\lambda(p-j)\}}{\lambda}+1;\frac{1}{2})/2$ and

$$\{\lambda + \mu(1+\lambda(p-j))\}/\{2\lambda + \mu(1+\lambda(p-j))\} \leq \rho < 1.$$

The estimate is best possible in the sense that bound cannot be improved.

Proof. Consider the function q defined in E by

$$q(z) = \left\{ \frac{(p-j+1)!}{p!} \frac{f^{(j-1)}(z)}{z^{p-j+1}} \right\}^{\mu}, \quad 1 \le j \le p. \tag{2.7}$$

We choose the principal branch in (2.7) so that q is analytic with $q(0) = 1$. Differentiating both sides in (2.7) followed by a simple calculation, we get

$$(1-\lambda)\left\{ \frac{f^{(j-1)}(z)}{z^{p-j+1}} \right\}^{\mu} + \lambda \frac{f^{(j)}(z)}{z^{p-j}} \left\{ \frac{f^{(j-1)}(z)}{z^{p-j+1}} \right\}^{\mu-1}$$
$$= \left\{ \frac{p!}{(p-j+1)!} \right\}^{\mu} (1+\lambda(p-j)) \left\{ q(z) + \frac{\lambda}{\mu\{1+\lambda(p-j)\}} zq'(z) \right\}.$$

Using the hypothesis (2.5) in the above equality, we obtain

$$Re\left\{ q(z) + \frac{\lambda}{\mu(1+\lambda(p-j))} .zq'(z) \right\} > \frac{\Big((p-j+1)!\Big)^{\mu} \alpha}{(p!)^{\mu}(1+\lambda(p-j))} \quad (z \in E)$$

from which it follows by Lemma 2

$$Re\left\{ \frac{f^{(j-1)}(z)}{z^{p-j+1}} \right\}$$
$$> \left[\frac{\alpha}{1+\lambda(p-j)} + \left\{ \left(\frac{p!}{(p-j+1)!} \right)^{\mu} - \frac{\alpha}{1+\lambda(p-j)} \right\} (2\rho - 1) \right] \quad (z \in E),$$

where $1 \le j \le p$ *and* ρ is given by

$$\rho = \int_0^1 \left\{ 1 + t^{\frac{\lambda}{\mu(1+\lambda(p-j))}} \right\}^{-1} dt.$$

Now, by change of variable and with the aid of the identities (2.1) and (2.2), we get

$$\rho = \frac{\mu(1+\lambda(p-j))}{\lambda} \int_0^1 u^{\frac{\mu(1+\lambda(p-j))}{\lambda} - 1} (1+u)^{-1} du$$
$$= F\left(1, \frac{\mu(1+\lambda(p-j))}{\lambda}; \frac{\mu(1+\lambda(p-j))}{\lambda} + 1; -1\right)$$
$$= \frac{1}{2} F\left(1, 1; \frac{\mu(1+\lambda(p-j))}{\lambda} + 1; \frac{1}{2}\right).$$

We, further, note that

$$\frac{\lambda + \mu(1+\lambda(p-j))}{2\lambda + \mu(1+\lambda(p-j))} \le \rho < 1, \quad 1 \le j \le p.$$

This completes the proof of Theorem 1. The estimate in (2.6) is best possible as the bound in Lemma 2 is so.

Setting $\mu = \lambda = 1$ in Theorem 1, we get the following result.

COROLLARY 1. *If $f \in A(p)$ satisfies*

$$Re\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\} > \alpha \quad \left(\alpha < \frac{p!}{(p-j)!}; \quad z \in E\right)$$

then

$$Re\left\{\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right\}$$
$$> \left[\frac{\alpha}{p-j+1} + \left\{\frac{p!}{(p-j+1)!} - \frac{\alpha}{p-j+1}\right\}(2\rho - 1)\right] \quad (z \in E),$$

where $1 \leq j \leq p$, $\rho = F(1, 1, p-j+2; \frac{1}{2})/2$ *and* $(p-j+2)/(p-j+3) \leq \rho < 1$. *The estimate is best possible in the sense that the bound cannot be improved.*

COROLLARY 2. *If $f \in A(p)$ satisfies*

$$Re\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\} > \frac{p!(1-2\rho)}{(p-j)!2(1-\rho)} \quad (z \in E)$$

where $2 \leq j \leq p$ and ρ is defined as in Corollary 1, then f is p-valent in E.

Proof. Putting $\alpha = \{p!(1-2\rho)\}/\{(p-j)!2(1-\rho)\}$ in Corollary 1, we get

$$Re\left\{\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right\} > 0 \quad (z \in E)$$

for $2 \leq j \leq p$. Or, equivalently,

$$Re\left\{\frac{zf^{(j-1)}(z)}{z^{p-j+2}}\right\} > 0 \quad (2 \leq j \leq p; \quad z \in E).$$

Since z^{p-j+2} is (p-j+2)-valently starlike in E, in view of Lemma 3, the function f is p-valent in E. This proves Corollary 2.

REMARKS 1. Since for $\mu = \lambda = 1$,

$$\frac{p-j+2}{p-j+3} \leq \rho < 1, \quad 1 \leq j \leq p$$

we deduce that

$$\frac{\alpha}{(p-j+1)} + \left\{\frac{p!}{(p-j+1)!} - \frac{\alpha}{p-j+1}\right\}(2\rho - 1) > \frac{(p-j+1)!2\alpha + p!}{2(p-j)+3}$$

Thus Corollary 1 improves Theorem A of Saitosh [11].

2. Putting $j = p$ in Corollary 2 and using the fact that $F(1, 1; 2; \frac{1}{2}) = 2\ln 2$, we have the following result which was also obtained by Nunokawa [7].

If $f \in A(p)$ satisfies the condition

$$Re\{f^{(p)}(z)\} > \frac{(1-2ln2)}{2(1-ln2)}p! \quad (z \in E)$$

for $p \geq 2$, then f is p-valent in E.

THEOREM 2. *If $f \in A(p)$ satisfies the condition (2.5) for $\mu > 0, \lambda > 0$ and $\alpha < \{(1+\lambda(p-j))(p!)^{\mu}\}/((p-j+1)!)^{\mu}, \quad 1 \leq j \leq p$, then*

$$\left\{\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right\}^{\mu/2} > \left(\frac{p!}{(p-j+1)!}\right)^{\mu/2} \beta \quad (z \in E),$$

where

(2.9)

$$\beta = \frac{\lambda(p!)^{\mu} + \sqrt{\lambda^2(p!)^{2\mu} + 4\mu\alpha((p-j+1)!)^{\mu}(p!)^{\mu}\{\mu+\lambda(\mu(p-j)+1)\}}}{2(p!)^{\mu}\{\mu+\lambda(\mu(p-j)+1)\}}.$$

Proof. Suppose $f \in A(p)$ satisfies (2.5) and let us put

$$(2.10) \qquad \left\{\frac{(p-j+1)!}{p!} . \frac{f^{(j-1)}(z)}{z^{p-j+1}}\right\}^{\mu/2} = \frac{1+(1-2\beta)w(z)}{1-w(z)} \quad (z \in E),$$

where β is defined by (2.9). We choose the principal branch in (2.10) so that w is analytic in E with $w(0) = 0$. On differentiating the expression in (2.10) followed by some simple transformations, we get

$$(2.11) \quad (1-\lambda)\left\{\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right\}^{\mu} + \lambda\frac{f^{(j)}(z)}{z^{p-j}}\left(\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right)^{\mu-1}$$
$$= \left\{\frac{p!}{(p-j+1)!}\right\}^{\mu}(1+\lambda(p-j))\left[\left\{\frac{1+(1-2\beta)w(z)}{1-w(z)}\right\}^2 + \frac{4\lambda(1-\beta)}{\mu(1+\lambda(p-j))}\cdot\frac{1+(1-2\beta)\cdot w(z)}{1-w(z)}\cdot\frac{zw'(z)}{(1-w(z))^2}\right].$$

Suppose that there exists a point $z_0 \in E$ such that

$$max_{|z|\leq|z_0|}|w(z)| = |w(z_0)| = 1 \quad (w(z_0) \neq 1).$$

Then, by using Lemma 1 and writing $w(z_0) = e^{i\theta}$, we have

$$Re\left[(1-\lambda)\left\{\frac{f^{(j-1)}(z)}{z_0^{p-j+1}}\right\}^{\mu} + \lambda\frac{f^{(j)}(z_0)}{z_0^{p-j+1}}\left\{\frac{f^{(j-1)}(z_0)}{z_0^{p-j}}\right\}^{\mu-1}\right]$$

$$= \left\{\frac{p!}{(p-j+1)!}\right\}^{\mu}(1+\lambda(p-j))\, Re\left[\left\{\frac{1+(1-2\beta)e^{i\theta}}{1-e^{i\theta}}\right\}^2 + \frac{4\lambda(1-\beta)k}{\mu(1+\lambda(p-j))} \times \left\{\frac{1+(1-2\beta)e^{i\theta}}{1-e^{i\beta}}\right\}\frac{e^{i\theta}}{(1-e^{i\theta})^2}\right]$$

$$\leq \left\{\frac{p!}{(p-j+1)!}\right\}^{\mu}(1+\lambda(p-j))\left\{\beta^2 + \frac{4\lambda\beta(1-\beta)k}{\mu(1+\lambda(p-j))}\left(-\frac{1}{4sin^2\frac{\theta}{2}}\right)\right\}$$

$$\leq \left\{\frac{p!}{(p-j+1)!}\right\}^{\mu} \cdot (1+\lambda(p-j))\left\{\beta^2 - \frac{\lambda\beta(1-\beta)}{\mu(1+\lambda(p-j))}\right\}$$

$$= \alpha \quad \text{(by using (2.9))},$$

which is a contradiction to the hypothesis (2.5). Thus, $|w(z)| < 1$ for all $z \in E$ and from (2.10), we conclude that

$$Re\left\{\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right\}^{\mu/2} > \left\{\frac{p!}{(p-j+1)!}\right\}^{\mu/2}\beta \quad (z \in E),$$

where $1 \leq j \leq p$. This completes the proof of Theorem 2.

Taking $\mu = \lambda = 1$ in Theorem 2, we obtain

COROLLARY 3. *If $f \in A(p)$ satisfies*

$$Re\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\} > \alpha \ (\alpha < \frac{p!}{(p-j)!}; \quad z \in E),$$

then

$$Re\left\{\sqrt{\frac{f^{(j-1)}(z)}{z^{p-j+1}}}\right\} > \left\{\frac{p!}{(p-j+1)!}\right\}^{\frac{1}{2}}\frac{p! + \sqrt{(p!)^2 + p!(p-j+2)!4\alpha}}{p!2(p-j+2)} \quad (z \in E),$$

where $1 \leq j \leq p$.

With $j = p$, Corollary 3 yields

COROLLARY 4. *If $f \in A(p)$ satisfies*

$$Re\{f^{(p)}(z)\} > \alpha \quad (\alpha < p!,\ z \in E),$$

then

$$Re\left\{\sqrt{\frac{f^{(p-1)}(z)}{z}}\right\} > \frac{p! + \sqrt{(p!)^2 + p!\,8\alpha}}{4\sqrt{p!}} \quad (z \in E).$$

THEOREM 3. *If $f \in A(p)$ satisfies the condition*

$$(2.12)\qquad \left|\beta\,\frac{f^{(j-1)}(z)}{z^{p-j+1}} + \gamma\,\frac{f^{(j)}(z)}{z^{p-j)}} - \frac{p!}{(p-j+1)!}(\beta+\gamma(p-j+1))\right| < \frac{p!}{(p-j+1)!}\{\beta+\gamma(p-j+2)\} \quad (z \in E)$$

for some $\beta \geq 0$, $\gamma \geq 0$, $\beta+\gamma > 0$, then

$$\left|\frac{f^{(j-1)}(z)}{z^{p-j+1}} - \frac{p!}{(p-j+1)!}\right| < \frac{p!}{(p-j+1)!} \quad (z \in E),$$

where $1 \leq j \leq p$.

Proof. Let (2.12) be satisfied and let us put

$$(2.13)\qquad \frac{(p-j+1)!}{p!}\cdot\frac{f^{(j-1)}(z)}{z^{p-j+1}} = 1 + w(z).$$

Then w is analytic in E with $w(0) = 0$. Making differentiation in (2.13) followed by some simple transformation in the resulting equation, we get

$$\frac{f^{(j)}(z)}{z^{p-j}} = \frac{p!}{(p-j+1)!}\{(p-j+1)(1+w(z)) + zw'(z)\}.$$

Using the above the equation and (2.13), we deduce that

$$(2.14)\qquad \left[\beta\frac{f^{(j-1)}(z)}{z^{p-j+1}} + \gamma\frac{f^{(j)}(z)}{z^{p-j}} - \frac{p!}{(p-j+1)!}(\beta+\gamma(p-j+1))\right] = \frac{p!}{(p-j+1)!}\{(\beta+\gamma(p-j+1))w(z) + \gamma zw'(z)\}.$$

Suppose that there exists a point $z_0 \in E$ such that

$$max_{|z|\leq|z_0|}|w(z)| = |w(z_0)| = 1 \quad (w(z_0) \neq 1).$$

Then by writing $w(z_0) = e^{i\theta}$ and using Lemma 1, (2.14) yields

$$\left|\beta\frac{f^{(j-1)}(z_0)}{z_0^{p-j+1}} + \gamma\frac{f^{(j)}(z_0)}{z_0^{p-j}} - (\beta+\gamma(p-j+1)\right| = \frac{p!}{(p-j+1)!}|(\beta+\gamma(p-j+1))e^{i\theta} + \gamma k, e^{i\theta}| \geq \frac{p!}{(p-j+1)!}\{\beta+\gamma(p-j+2)\},$$

which is a contradiction to (2.12). Therefore, $|w(z)| < 1$ for all $z \in E$.

This implies that

$$\left|\frac{f^{(j-1)}(z)}{z^{p-j+1}}-\frac{p!}{(p-j+1)!}\right|<\frac{p!}{(p-j+1)!}\qquad (z\in E).$$

This proves the theorem.

COROLLARY 5. *If $f\in A(p)$ satisfies the condition (2.12) for $2\le j\le p$, then f is p-valent in E.*

Proof. From (2.13) and the inequality $|\omega(z)|<1$ for $z\in E$, which was shown in the proof of Theorem 3, it follows that

$$Re\left\{\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right\}>0\qquad (z\in E),$$

where $2\le j\le p$. Or, equivalently

$$Re\left\{\frac{zf^{(j-1)}(z)}{z^{p-j+2}}\right\}>0\qquad (2\le j\le p,\ z\in E).$$

By using the same argument as in Corollary 2, we conclude that f is p-valent in E.

Putting $j=p$, $\beta=0$ and $\gamma=1$ in Corollary 5, we have

COROLLARY 6. *If $f\in A(p)$ satisfies*

$$|f^{(p)}(z)-p!|<2(p!)\qquad (z\in E)$$

for $p\ge 2$, then f is p-valent in E.

We note that the above result was also obtained by Nunokawa, Kwon and Cho [8].

THEOREM 4. *Let F_λ be defined by (1.2) for $\lambda\ge 0$ and $f\in A(p)$. If*

$$Re\left\{\frac{F_\lambda^{(j)}(z)}{z^{p-j}}\right\}>\alpha\left(\alpha<\frac{p!(1-\lambda+\lambda p)}{(p-j)!};\ z\in E\right)\tag{2.15}$$

then

$$Re\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\}>\left[\frac{\alpha}{1-\lambda+\lambda p}+\left\{\frac{p!}{(p-j)!}-\frac{\alpha}{1-\lambda+\lambda p}\right\}(2\rho-1)\right]\qquad (z\in E),\tag{2.16}$$

where $0\le j\le p$, ρ is given by

$$\rho=\begin{cases}\frac{1}{2}F(1,1;\frac{1+\lambda p}{\lambda};\frac{1}{2}), & \lambda>0\\ \frac{1}{2}, & \lambda=0\end{cases}\tag{2.17}$$

and $(1+\lambda)/(1+\lambda+\lambda p)\le\rho<1$. The estimate in (2.16) is best possible in the sense that the bound cannot be improved.

Proof. Consider the function F_λ defined in E by

$$F_\lambda(z) = (1-\lambda)f(z) + \lambda z f'(z) \quad (\lambda \geq 0,\ f \in A(p)).$$

Differentiating F_λ, we get

$$F_\lambda^{(j)}(z) = (1-\lambda+\lambda j)f^{(j)}(z) + \lambda z f^{(j+1)}(z), \quad 0 \leq j \leq p. \tag{2.18}$$

Let us put

$$q(z) = \frac{(p-j)!}{p!} \cdot \frac{f^{(j)}(z)}{z^{p-j}} \quad (0 \leq j \leq p;\ z \in E). \tag{2.19}$$

Then q is analytic in E with $q(0) = 1$. Making differentiation in (2.19), we obtain

$$zq'(z) = \frac{(p-j)!}{p!}\left\{\frac{f^{(j+1)}(z)}{z^{p-j+1}} - (p-j)\frac{f^{(j)}(z)}{z^{p-j}}\right\}$$

from which it follows that

$$\frac{f^{(j+1)}(z)}{z^{p-j-1}} = \frac{p!}{(p-j)!}\{(p-j)q(z) + zq'(z)\}.$$

By using the above expression and (2.19) in (2.18), we deduce that

$$\frac{F_\lambda^{(j)}(z)}{z^{p-j}} = \frac{p!(1-\lambda+\lambda p)}{(p-j)!}\left\{q(z) + \frac{\lambda}{1-\lambda+\lambda p} zq'(z)\right\}. \tag{2.20}$$

Hence by (2.15), (2.20) yields

$$Re\left\{q(z) + \frac{\lambda}{1-\lambda+\lambda p} \cdot zq'(z)\right\} > \frac{(p-j)!\alpha}{p!(1-\lambda+\lambda p)} \quad (z \in E)$$

which in view of Lemma 2 implies that

$$Re\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\} > \left[\frac{\alpha}{1-\lambda+\lambda p} + \left\{\frac{p!}{(p-j)!} - \frac{\alpha}{1-\lambda+\lambda p}\right\}(2\rho - 1)\right] \quad (z \in E)$$

where $0 \leq j \leq p$, ρ is given by

$$\rho = \int_0^1 \left(1 + t^{\frac{\lambda}{1-\lambda+\lambda p}}\right)^{-1} dt$$

and$(1+\lambda p)/(1+\lambda+\lambda p) \leq \rho < 1$. By following the lines of proof as in Theorem 1, we can show that

$$\rho = \begin{cases} \frac{1}{2}F(1, 1; \frac{1+\lambda p}{\lambda}; \frac{1}{2}), & \lambda > 0 \\ \frac{1}{2}, & \lambda = 0. \end{cases}$$

Hence the theorem is proved.

The estimate in (2.16) is best possible as the bound in Lemma 2 is best possible.

COROLLARY 7. *Let F_λ be defined by* (1.2) *for $\lambda \geq 0$ and $f \in A(p)$. If*

$$Re\left\{\frac{F_\lambda^{(j)}(z)}{z^{p-j}}\right\} > \frac{p!(1+\lambda+\lambda p)(1-2\rho)}{(p-j)!2(1-\rho)} \quad (z \in E),$$

where $1 \leq j \leq p$ and ρ is given by (2.17), then f is p-valent in E.

Proof. Setting $\alpha = \{p!(1-\lambda+\lambda p)(1-2\rho)\}/\{(p-j)!2(1-\rho)\}$ in Theorem 3, we get

$$Re\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\} > 0 \quad (z \in E)$$

for $1 \leq j \leq p$. Or, equivalently

$$Re\left\{\frac{zf^{(j)}(z)}{z^{p-j+1}}\right\} > 0 \quad (z \in E).$$

Since z^{p-j+1} is $(p-j+1)$-valently starlike in E, by using Lemma 3 we conclude that f is p-valent in E.

Putting $j = 0$ in Theorem 4, we get

COROLLARY 8. *Let F_λ be defined by (1.2) for $\lambda \geq 0$ and $f \in A(p)$. If*

$$Re\left\{\frac{F_\lambda(z)}{z^p}\right\} > \alpha \quad (\alpha < 1-\lambda+\lambda p;\ z \in E),$$

then

$$Re\left\{\frac{f(z)}{z^p}\right\} > \left[\frac{\alpha}{1-\lambda+\lambda p} + \left\{1 - \frac{\alpha}{(1-\lambda+\lambda p)}\right\}(2\rho-1)\right] \quad (z \in E)$$

where ρ is given by (2.17). The result is best possible.

Taking $j = 1$ in Theorem 4, we have

COROLLARY 9. *Let F_λ be defined by* (1.2) *for $\lambda \geq 0$ and $f \in A(p)$. If*

$$Re\left\{\frac{F_\lambda'(z)}{z^{p-1}}\right\} > \alpha \quad (\alpha < p(1-\lambda+\lambda p);\ z \in E),$$

then

$$Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > \left[\frac{\alpha}{1-\lambda+\lambda p} + \left\{p - \frac{\alpha}{(1-\lambda+\lambda p)}\right\}(2\rho-1)\right] \quad (z \in E),$$

where ρ is given by (2.17). The result is best possible.

REMARKS 1. Since $(1+\lambda p)/(1+\lambda+\lambda p) \leq \rho < 1$, we have

$$\frac{\alpha}{1-\lambda+\lambda p} + \left\{\frac{p!}{(p-j)!} - \frac{\alpha}{1-\lambda+\lambda p}\right\} \geq \frac{(p-j)!2\alpha + p!\lambda}{(p-j)!(2-\lambda+\lambda p)}$$

for $0 \leq j \leq p$ and $0 \leq \lambda \leq 1$. Thus, Theorem 4 is an improvement of Theorem B for $0 \leq \lambda \leq 1$.

2. In view of the above remark, Corollary 8 and Corollary 9 improve the corresponding results obtained by Owa and Nunokawa [9] for p=1 and $0 \leq \lambda \leq 1$.

Finallay, we prove

THEOREM 5. *If $f \in A(p)$ satisfies*

$$Re\{\frac{f^{(j)}(z)}{z^{p-j}}\} > \alpha \quad (\alpha < \frac{p!}{(p-j)!};\ z \in E), \tag{2.21}$$

then

$$Re\left\{\frac{F_n^{(j)}(z)}{z^{p-j}}\right\} > \left[\alpha + \left\{\frac{p!}{(p-j)!} - \alpha\right\}(2\rho - 1)\right] \quad (z \in E), \tag{2.22}$$

where $0 \leq j \leq p$, $\rho = F(1, 1, n+p+1; \frac{1}{2})/2$ and F_n is given by

$$F_n(z) = \frac{n+p}{z^n}\int_0^z t^{n-1} f(t)dt \quad (n \in N). \tag{2.23}$$

The estimate in (2.22) is best possible in the sense that the bound cannot be improved.

Proof. On differentiating F_n, we obtain

$$zF_n^{(j+1)}(z) + (n+j)F_n^{(j)}(z) = (n+p)f^{(j)}(z), \quad 0 \leq j \leq p. \tag{2.24}$$

Consider the function p defined in E by

$$q(z) = \frac{(p-j)!}{p!}\frac{F_n^{(j)}(z)}{z^{p-j}}. \tag{2.25}$$

Then q is analytic in E with $q(0) = 1$. Again, differentiating the expression in (2.25) and using (2.24) in the resulting equation, we get

$$\frac{f^{(j)}(z)}{z^{p-j}} = \frac{p!}{(p-j)!}\{q(z) + \frac{zq'(z)}{n+p}\}.$$

Now, using the hypothesis (2.21) in the above expression, we get

$$Re\left\{q(z) + \frac{zq'(z)}{n+p}\right\} > \frac{(p-j)!\alpha}{p!} \quad (z \in E)$$

which in view of Lemma 2 implies that

$$Re\left\{\frac{F_n^{(j)}(z)}{z^{p-j}}\right\} > \left[\alpha + \left\{\frac{p!}{(p-j)!} - \alpha\right\}(2\rho - 1)\right] \quad (z \in E),$$

where $0 \leq j \leq p$ ρ is given by

$$\rho = \int_0^1 \left(1 + t^{\frac{1}{n+p}}\right)^{-1} dt$$

and $(n+p+1)/(n+p+2) \leq \rho < 1$. By using the same technique as in Theorem 1, it is easily seen that $\rho = F(1,1,n+p+1;\frac{1}{2})/2$. This proves the assertion (2.22).

The estimate in (2.22) is best possible as the bound in Lemma 2 is so.

COROLLARY 10. *If $f \in A(p)$ satisfies*

$$Re\{\frac{f^{(j)}(z)}{z^{p-j}}\} > \frac{p!(1-2\rho)}{(p-j)!2(1-\rho)} \quad (z \in E),$$

where $0 \leq j \leq p$ and $\rho = F(1,1;n+p+1;\frac{1}{2})/2$, then

$$Re\{\frac{F_n^{(j)}(z)}{z^{p-j}}\} > 0 \quad (n \in N; \quad z \in E).$$

The result is best possible.

Setting $n = p = j = 1$ in Corollary 10 and using fact that $F(1,1;3;\frac{1}{2}) = 4(1 - ln2)$, we derive the following result.

COROLLARY 11. *If $f \in A(p)$ satisfies the condition*

$$Re\{f'(z)\} > \frac{4(\ln 2) - 3}{4(\ln 2) - 2} \simeq -0.29439 \quad (z \in E),$$

then $Re\{F_1'(z)\} > 0$ and hence univalent in E, where F_1 is given by

$$F_1(z) = \frac{2}{z}\int_0^z f(t)\,dt.$$

REMARK. We note that the above result improves an earlier known result

$$Re\{f'(z)\} > -\frac{1}{4} \quad \text{implies} \quad Re\{F_1'(z)\} > 0 \quad (z \in E) \text{ [4]}.$$

Further, it is of special interest as it gives an example of non-univalent function whose Libera transform is univalent.

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