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DISPROOF OF AN INCLUSION RELATION OF CLASSES RELATED TO SPIRALLIKE FUNCTIONS

Abstract. An inclusion relation between classes of functions, defined by Ruscheweyh derivative and related to spirallike functions, given by S.S. Bhoosnurmath and M. V. Devadas [2] is disproved and some new results are given.

1. Introduction

Let H denote the class of functions f which are analytic in the unit disk $\Delta = \{z : |z| < 1\}$ and normalized by $f(0) = 0 = f'(0) - 1$. An analytic function f on Δ is said to be subordinate to an analytic function g on Δ (written $f \prec g$) if $f(z) = g(w(z))$, $z \in \Delta$ for some analytic function w with $w(0) = 0$ and$|\ w(z)\ |< 1$ in Δ. The Hadamard product (convolution) of two power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=0}^{\infty} b_k z^k,$$

is defined as the power series

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k, \qquad z \in \Delta.$$

Denote by $D^n : H \to H$ the operator defined by

$$\begin{aligned} D^n f(z) &= \frac{z}{(1-z)^{n+1}} * f, \quad n \in N_o = \{0, 1, 2, \ldots\} \\ &= z + \sum_{k=2}^{\infty} \frac{(k+n-1)!}{n!(k-1)!} a_k z^k. \end{aligned}$$

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Note that $D^0 f(z) = f(z)$ and $D^1 f(z) = zf'(z)$. The symbol $D^n f$ was named the Ruscheweyh derivative by Al-Amiri [1].

The class $P[A, B]$ introduced by Janowski [4], is defined as follows. For real numbers A and $B, -1 \leq B < A \leq 1, p \in P[A, B]$ if, and only if, p is analytic on Δ, $p(0) = 1$ and

$$p(z) \prec \frac{1 + Az}{1 + Bz}, \quad z \in \Delta.$$

We have following:

DEFINITION 1.1. A function $f \in H$ is said to belong to the class $Sp_n^\lambda[A, B]$ if and only if

$$e^{i\lambda} \frac{z(D^n f(z))'}{D^n f(z)} \prec \cos\lambda \frac{1 + Az}{1 + Bz} + i \sin\lambda, \quad z \in \Delta,$$

where A and B are arbitrary fixed numbers, $-1 \leq B < A \leq 1$, and $|\lambda| < \frac{\pi}{2}$.

From the definition of subordination we get $f \in Sp_n^\lambda[A, B]$ if, and only if,

$$e^{i\lambda} \frac{z(D^n f(z)'}{D^n f(z)} = \cos\lambda \frac{1 + A\omega(z)}{1 + B\omega(z)} + i \sin\lambda, \quad z \in \Delta.$$

By giving specific values to n, A, B and λ, we obtain the following known classes:

1. For $n = 0$ and $\lambda = 0$, $Sp_0^0[A, B] \equiv S^*[A, B]$, the subclasses of starlike functions, defined by Janowski [4].

2. For $n = 0, A = 1$ and $B = -1$, $Sp^\lambda[1, -1] \equiv Sp^\lambda$, the class of $\lambda -$ *spirallike* univalent functions, introduced by Špaček [6].

3. For $n = 1$, $A = 1$, $B = -1$, $Sp_1^\lambda[1, -1] \equiv C^\lambda$, the class of Robertson functions f for which $zf' \in Sp^\lambda$,introduced by Robertson [5].

4. For $n = 0$, $Sp_0^\lambda[A, B] \equiv Sp^\lambda[A, B]$, the subclasses of spirallike functions defined by Dashrath and Shukla[3].

From Definition 1.1 we observe that

$$f \in Sp_n^\lambda[A, B] \text{ if and only if } D^n f \in Sp^\lambda[A, B].$$

2. Counter example

S. S. Bhoosnurmath and M. V.Devadas [2] considered the classes $Sp_n^\lambda[A, B]$, where A and B are complex numbers with $|A| \leq 1$, $|B| \leq 1$ and $A \neq B$. Bhoosnurmath and Devads [2] (Theorem 4) stated the following

$$Sp_{n+1}^\lambda[A, B] \subset Sp_n^\lambda[A, B] \tag{2.1}$$

for $n \in N_0$. In this paper we show that (2.1) is not true.

COUNTER-EXAMPLE. Let $n = 0$. Then $Sp_1^\lambda[A,B] \not\subseteq Sp_0^\lambda[A,B]$. Consider the function.

(2.2) $$g(z) = \frac{1}{\mu}\left[(1-z)^{-\mu} - 1\right], \quad \mu + 1 = |\mu+1|e^{-i\lambda} \quad \left(-\frac{\pi}{2} < \lambda < \frac{\pi}{2}\right).$$

Robertson [5] showed that g belongs to $C_\lambda \equiv Sp_1[1,-1]$, and is not univalent on Δ, when μ lies in the set R defined by the inequalities $|\mu| \le 1$, $|\mu+1| > 1$, $|\mu-1| > 1$ (see figure 1). Hence g belongs to $Sp_1^\lambda[1,-1]$ and not to $Sp_0^\lambda[1,-1] \equiv Sp^\lambda$, when μ lies in the set R defined above.

3. Other properties of the class $Sp_n^\lambda[A,B]$

In the following we show how to move through different classes of $Sp_n^\lambda[A,B]$, for different n, first through an integral transformation and the second by using convolution with hypergeometric functions.

THEOREM 3.1. *Let $J : H \to H$ be defined by*

(3.1) $$J(f) = \frac{n+1}{z^n}\int_0^z t^{n-1} f(t)dt.$$

Then $f \in Sp_n^\lambda[A,B]$ if and only if $J(f) \in Sp_{n+1}^\lambda[A,B]$.

Proof. We have to show that $D^n f = D^{n+1}J(f)$, from (3.1) we get

(3.2) $$D^n J(f(z)) = \frac{n+1}{z^n}\int_0^z t^{n-1} D^n f(t)dt.$$

Differentiating both sides of (3.2) gives

(3.3) $$\frac{n}{n+1}D^n J(f(z)) + \frac{z}{n+1}(D^n J(f(z))' = D^n f(z).$$

From the well-known identity

$$z(D^n f(z))' = (n+1)D^{n+1}f(z) - n(D^n f(z)),$$

and (3.3), we get the required result. ∎

Recall the generalized hypergeometric function

$${}_mF_n(\alpha_1, \alpha_2..\alpha_m;\ \beta_1, \beta_2, \dots, \beta_n; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k \dots (\alpha_m)_k}{(\beta_1)_k(\beta_2)_k \dots (\beta_n)_k k!} z^k$$

where $(\alpha)_0 = 1$ and $(a)_k = a(a+1)\dots(a+k-1)$ for $k \ge 1$.

THEOREM 3.1. *Let*

$$L(z) = {}_{m+1}F_n(n+1, \dots, n+1, 1; n+2, \dots, n+2, \dots, n+2; z),$$

be hypergeometric function. Then $f \in Sp_n^\lambda[A,B]$ if and only if $(f * zL) \in Sp_{n+m}^\lambda[A,B]$.

Proof. Let $f \in Sp_n^\lambda[A,B]$. For $f(z)=\sum_{k=0}^{\infty} a_k\, z^k$, $a_1=1$ we have

$$\begin{aligned} J(f(z)) &= \sum_{k=1}^{\infty} \frac{n+1}{n+k} a_k z^k \\ &= \left(z \sum_{k=0}^{\infty} \frac{n+1}{n+k+1} z^k \right) * \sum_{k=1}^{\infty} a_k z^k \\ &= \left(z \sum_{k=0}^{\infty} \frac{(n+1)_k\,(1)_k}{(n+2)_k\, k!} z^k \right) * f(z), \end{aligned}$$

belongs to $Sp_{n+1}^\lambda[A,B]$ from Theorem 3.1. Applying Theorem 3.1 m times we get the required result. ∎

Using the relation (1.1) and the coefficient bounds for the class $Sp_0^\lambda[A,B]$ given in [3], we have the following.

THEOREM 3.3. *If* $f(z)=z+\sum_{k=2}^{\infty} a_k\, z^k \in Sp_n^\lambda[A,B]$, *then*

$$|a_2| \le (A-B)\cos\lambda$$

for $A-2B \le \sqrt{\{1+(1-B^2)\tan^2\lambda\}}$, $k\ge 3$,

$$|a_k| \le \frac{n1(k-1)!}{(k+n-1)!}\frac{(A-B)\cos\lambda}{(k-1)},$$

and for

$$A-(k-1)B > (k-2)\sqrt{\{1+(1-B^2)\tan^2\lambda\}}, \qquad k\ge 3,$$

and

$$|a_k| \le \frac{n1(k-1)!}{(k+n-1)!}\prod_{j=0}^{k-2}\left|\frac{(A-B)e^{-i\lambda}\cos\lambda - jB}{j+1}\right|,$$

for $z\in\Delta, -1\le B<A\le 1$, $\lambda\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. *This result is sharp as seen by the function*

$$D^n f(z) = \begin{cases} z\,(1+Bz)^{\frac{(A-B)e^{-i\lambda}\cos\lambda}{B}}, & B\ne 0 \\ z\exp\left(Ae^{-i\lambda}\cos\lambda z\right), & B=0. \end{cases} \tag{3.4}$$

The following Lemma can be used to get the radius of being in Sp_n^β for functions in $\mathrm{Sp}_n^\lambda[A,B]$.

LEMMA 3.1 [7]. *If* $f\in Sp_n^\lambda[A,B]$, *then* f *is* β*-spirallike in* $|z|<R_1(\lambda,\beta,A,B)$, *where* $R_1(\lambda,\beta,A,B)$ *is the smallest positive root of the equation*

$$B[A\cos\lambda\cos(\lambda+\beta)+B\sin\lambda\sin(\lambda-\beta)]r^2+(A-B)\cos\lambda r-\cos\beta=0. \tag{3.5}$$

This result is sharp.

Using Lemma 3.1 and the relation (1.1) we immediately have

THEOREM 3.4. *Let* $f(z) = z+\sum_{k=2}^{\infty} a_k\, z^k$ *belong to* $Sp_n^{\lambda}[A, B]$. *Then* $f \in Sp_n^{\beta}$ *in* $|z| < R_1(\lambda, \beta, A, B)$, *where* $R_1(\lambda, \beta, A, B)$ *is the smallest positive root of the equation* (3.5). *This result is sharp as can be seen from the function* f *given in* (3.4).

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