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DIFFERENCE BETWEEN WEIGHTED INTEGRAL MEANS

Abstract. Weighted integral means over $[c,d]$ and $[a,b]$ where $[c,d] \subset [a,b]$ are compared in the current work by determining bounds for their difference in terms of a variety of norms. The bounds are obtained and involve the behaviour of at most the first derivative. Previous work for unweighted integral means is recaptured as particular cases if the weights are taken to be unity.

By a limiting shrinking of the subinterval $[c,d]$ to a single point, weighted Ostrowski type inequalities are shown to be recaptured, under certain conditions as particular instances of the current development.

1. Introduction

Let the difference between two integral means $D(f;a,c,d,b)$ be defined by

$$D(f;a,c,d,b) := \mathfrak{M}(f;a,b) - \mathfrak{M}(f;c,d), \quad a \le c < d \le b \tag{1.1}$$

where

$$\mathfrak{M}(f;a,b) := \frac{1}{b-a}\int_a^b f(t)\,dt =: \frac{\mathfrak{A}(f;a,b)}{b-a}. \tag{1.2}$$

Barnett et al. [2] proved the following theorem demonstrating a number of applications such as in probability theory, information theory and special means.

THEOREM 1. *Let $f:[a,b] \to \mathbb{R}$ be an absolutely continuous mapping with the property that $f' \in L_\infty[a,b]$, i.e.,*

$$\|f'\|_\infty := ess \sup_{t\in[a,b]} |f'(t)| < \infty.$$

Then for $a \le c < d \le b$, we have the inequality

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$$(1.3)\quad |D(f;a,c,d,b)| \le \left\{\frac{1}{4}+\left[\frac{\frac{a+b}{2}-\frac{c+d}{2}}{(b-a)-(d-c)}\right]^2\right\}[(b-a)-(d-c)]\,\|f'\|_\infty \le \frac{1}{2}[(b-a)-(d-c)]\,\|f'\|_\infty,$$

where $D(f;a,c,d,b)$ is as defined by (1.1).

The constant $\frac{1}{4}$ is best possible in the first inequality and $\frac{1}{2}$ is best in the second inequality.

Cerone and Dragomir [3] proved a number of results for bounds on (1.1) assuming various characteristics on the function f. They proved the following three theorems.

THEOREM 2. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping. Then for $a\le c<d\le b$ the inequalities*

$$(1.4)\quad |D(f;a,c,d,b)| \le \begin{cases} \dfrac{b-a}{(q+1)^{\frac{1}{q}}}\left[1+\left(\dfrac{\rho}{1-\rho}\right)^q\right]^{\frac{1}{q}}\left[\nu^{q+1}+\lambda^{q+1}\right]^{\frac{1}{q}}\|f'\|_p, \\ \qquad f'\in L_p[a,b],\ 1<p<\infty,\ \frac{1}{p}+\frac{1}{q}=1; \\ [\nu+\lambda+|\nu-\lambda|]\dfrac{\|f'\|_1}{2},\quad f'\in L_1[a,b], \end{cases}$$

hold where $(b-a)\nu=c-a$, $(b-a)\rho=d-c$, $(b-a)\lambda=b-d$.

The following theorem assumes that f is Hölder continuous.

THEOREM 3. *Assume that the mapping $f:[a,b]\to\mathbb{R}$ is of $r-H-$Hölder type. That is, f satisfies*

$$(1.5)\qquad |f(t)-f(s)|\le H|t-s|^r \quad \text{for all } t,s\in[a,b],$$

where $r\in(0,1]$ and $H>0$ are given.

Then for $a\le c<d\le b$, we have the inequality

$$(1.6)\qquad |D(f;a,c,d,b)|\le\frac{(c-a)^{r+1}+(b-d)^{r+1}}{[(b-a)-(d-c)](r+1)}\cdot H.$$

The inequality (1.6) is best in the sense that we cannot put on the right hand side a constant K less than 1.

The following result holds for f of bounded variation on $[a,b]$.

THEOREM 4. *Let $f:[a,b]\to\mathbb{R}$ be of bounded variation on $[a,b]$. The following bounds hold*

$$(1.7)\quad |D(f;a,c,d,b)| \\ \le \begin{cases} \left[\dfrac{b-a-(d-c)}{2}+\left|\dfrac{c+d}{2}-\dfrac{a+b}{2}\right|\right]\dfrac{\bigvee_a^b(f)}{b-a}, \\ \dfrac{(c-a)^2+(b-d)^2}{2\left[(b-a)-(d-c)\right]}L, & \text{for } f \text{ } L\text{-Lipschitzian;} \\ \left(\dfrac{b-d}{b-a}\right)f(b)-\left(\dfrac{c-a}{b-a}\right)f(a) \\ \qquad +\left[\dfrac{c+d-(a+b)}{b-a}\right]f(s_0), & \text{for } f \text{ monotonic nondecreasing,} \end{cases}$$

where $s_0=\frac{cb-ad}{(b-a)-(d-c)}$ *and* $\bigvee_a^b(f)$ *is the total variation of* f *over* $[a,b]$.

The main focus of the current work is to obtain bounds for the difference of weighted integral means. Let $p(x)$ and $q(x)$ be two positive weight functions on $[a,b]$ and $[c,d]$ respectively with $[c,d]\subset[a,b]$. Further, let $0<P(b)=\int_a^b p(x)\,dx<\infty$ and $0<Q(d)=\int_c^d q(x)\,dx<\infty$, then we may define

$$(1.8)\qquad D(p,q;f;a,c,d,b):=\mathfrak{M}(p;f;a,b)-\mathfrak{M}(q;f;c,d),$$

where

$$(1.9)\qquad \mathfrak{M}(p;f;a,b):=\frac{\int_a^b p(x)f(x)\,dx}{P(b)}.$$

Thus, specifically the current article aims at obtaining bounds on (1.8) with various assumptions regarding $f(t)$ and the weight functions $p(t)$ and $q(t)$.

If $p(t)$ and $q(t)$ are of bounded variation on $[a,b]$ and $[c,d]$ respectively, then the functional, under the more general setting,

$$(1.10)\qquad \Delta(P,Q;f;a,c,d,b):=\mathcal{M}(P;f;a,b)-\mathcal{M}(Q;f;c,d),$$

where

$$(1.11)\qquad \mathcal{M}(P;f;a,b)=\frac{\int_a^b f(x)\,dP(x)}{P(b)}$$

will also be investigated where consequently $P(t)$ and $Q(t)$ are also of bounded variation on $[a,b]$ and $[c,d]$ respectively.

For $p(t)$ and $q(t)$ continuous on their respective intervals $[a,b]$ and $[c,d]$ then (1.8) – (1.9) results from (1.10) – (1.11). Only continuity is required since $p(t)$ and $q(t)$ are positive.

It is demonstrated that a limiting approach, under suitable continuity assumptions, produces an identity for the weighted Ostrowski functional from which bounds may be obtained.

2. Some analytic inequalities from identities

Prior to obtaining bounds on (1.8) it is useful to demonstrate the validity of an identity involving the kernel $K : [a,b] \to \mathbb{R}$ defined by

$$K(t) := \begin{cases} \dfrac{P(t)}{P(b)}, & a \le t \le c; \\ \dfrac{P(t)}{P(b)} - \dfrac{Q(t)}{Q(d)}, & c < t < d; \\ \dfrac{P(t)}{P(b)} - 1, & d \le t \le b, \end{cases} \tag{2.1}$$

where the weight functions $p : [a,b] \to \mathbb{R}_+$ and $q : [c,d] \to \mathbb{R}_+$ are such that $0 < P(b) = \int_a^b p(t)\,dt < \infty$ and $0 < Q(d) = \int_c^d q(t)\,dt < \infty$ with $[c,d] \subset [a,b]$.

LEMMA 1. *Let $f : [a,b] \to \mathbb{R}$ be of bounded variation on $[a,b]$. Further, let $p : [a,b] \to \mathbb{R}_+$ and $q : [c,d] \to \mathbb{R}_+$, $[c,d] \subset [a,b]$ be positive weight functions of bounded variation and $P(t) = \int_a^t p(x)\,dx$, $Q(t) = \int_c^t q(x)\,dx$, $t \in [a,b]$ and $t \in [c,d]$ respectively. Then,*

$$-\int_a^b K(t)\,df(t) = \Delta(P,Q;f;a,c,d,b), \tag{2.2}$$

where $K(t)$ is the kernel function as defined by (2.1) *and $\Delta(P,Q;f;a,c,d,b)$ by* (1.10).

Proof. We start with

$$\int_a^b K(t)\,df(t) = \int_a^c \frac{P(t)}{P(b)}df(t) + \int_c^d \left(\frac{P(t)}{P(b)} - \frac{Q(t)}{Q(d)}\right) df(t) + \int_d^b \left(\frac{P(t)}{P(b)} - 1\right) df(t)$$

and using integration by parts of the Riemann-Stieltjes integrals, we get

$$\begin{aligned} \int_a^b K(t)\,df(t) &= \frac{P(t)}{P(b)} f(t)\Big]_a^c - \frac{1}{P(b)} \int_a^c f(t)\,dP(t) \\ &+ \left(\frac{P(t)}{P(b)} - \frac{Q(t)}{Q(d)}\right) f(t)\Big]_c^d - \int_c^d f(t) \left(\frac{dP(t)}{P(b)} - \frac{dQ(t)}{Q(d)}\right) \\ &+ \left(\frac{P(t)}{P(b)} - 1\right) f(t)\Big]_d^b - \frac{1}{P(b)} \int_d^b f(t)\,dP(t). \end{aligned}$$

Now, using the fact that $P(a) = Q(c) = 0$ and some simple algebra, the identity (2.2) results. ■

REMARK 1. For $p(t)$ and $q(t)$ continuous then $dP(t) = p(t)\,dt$ and $dQ(t) = q(t)\,dt$ and so from (2.2) the identity

$$-\int_a^b K(t)\,df(t) = D(p,q;f;a,c,d,b), \tag{2.3}$$

holds, where $K(t)$ and $D(p,q;f;a,c,d,b)$ are as defined by (2.1) and (1.8) respectively.

Further, for $f(\cdot)$ absolutely continuous on $[a,b]$ then $df(t) = f'(t)\,dt$.

The following well known lemmas will prove useful and are stated here for lucidity.

LEMMA 2. *Let $g, v : [a,b] \to \mathbb{R}$ be such that g is continuous and v is of bounded variation on $[a,b]$. Then the Riemann-Stieltjes integral $\int_a^b g(t)\,dv(t)$ exists and is such that*

$$\left|\int_a^b g(t)\,dv(t)\right| \le \sup_{t\in[a,b]} |g(t)| \bigvee_a^b (v), \tag{2.4}$$

where $\bigvee_a^b (v)$ is the total variation of v on $[a,b]$.

LEMMA 3. *Let $g, v : [a,b] \to \mathbb{R}$ be such that g is Riemann integrable on $[a,b]$ and v is L-Lipschitzian on $[a,b]$. Then*

$$\left|\int_a^b g(t)\,dv(t)\right| \le L \int_a^b |g(t)|\,dt \tag{2.5}$$

with v is L-Lipschitzian if it satisfies

$$|v(x) - v(y)| \le L|x-y|$$

for all $x, y \in [a,b]$.

LEMMA 4. *Let $g, v : [a,b] \to \mathbb{R}$ be such that g is Riemann-integrable on $[a,b]$ and v is monotonic nondecreasing on $[a,b]$. Then*

$$\left|\int_a^b g(t)\,dv(t)\right| \le \int_a^b |g(t)|\,dv(t). \tag{2.6}$$

It should be noted that if v is nonincreasing then $-v$ is nondecreasing.

THEOREM 5. *Let $f : [a,b] \to \mathbb{R}$ be of bounded variation on $[a,b]$. Further, let the weight functions $p : [a,b] \to \mathbb{R}_+$ and $g : [c,d] \to \mathbb{R}_+$ also be of bounded variation with $[c,d] \subset [a,b]$. Then the following bounds hold*

$$(2.7) \qquad P(b)\left|\Delta(P,Q;f;a,c,d,b)\right|$$

$$\leq \begin{cases} \left[\int_a^c P(t)\,dt + \frac{1}{Q(d)}\int_c^d |\theta(t)|\,dt \right. \\ \qquad \left. + \int_d^b (P(b)-P(t))\,dt\right] \|f'\|_\infty, & f' \in L_\infty[a,b]; \\ \left[\int_a^c P^\beta(t)\,dt + \frac{1}{Q^\beta(d)}\int_c^d |\theta(t)|^\beta\,dt \right. \\ \quad \left. + \int_d^b (P(b)-P(t))^\beta dt\right]^{\frac{1}{\beta}} \|f'\|_\alpha, & f' \in L_\alpha[a,b],\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \max\left\{P(c), \frac{\gamma}{Q(d)}, P(b)-P(d)\right\}\|f'\|_1, & f' \in L_1[a,b]; \\ \max\left\{P(c), \frac{\gamma}{Q(d)}, P(b)-P(d)\right\}\bigvee_a^b(f'), & f' \textit{ of bounded variation} \\ \left[\int_a^c P(t)\,dt + \frac{1}{Q(d)}\int_c^d |\theta(t)|\,dt \right. \\ \qquad \left. + \int_d^b (P(b)-P(t))\,dt\right] L, & f' \textit{ is } L\textit{-Lipschitzian,} \\ \int_a^c P(t)\,df(t) + \frac{1}{Q(d)}\int_c^d |\theta(t)|\,df(t) \\ \qquad + \int_d^b (P(b)-P(t))\,df(t), & f' \textit{ is monotonic nondecreasing,} \end{cases}$$

where

$$(2.8) \qquad \theta(t) := Q(d)P(t) - P(b)Q(t), \quad t \in (c,d), \qquad \gamma = ess \sup_{t\in[c,d]} |\theta(t)|.$$

Further, the Lebesgue norms $\|\cdot\|$ *are defined in the usual way as*

$$\|h\|_\alpha := \left(\int_a^b |h(t)|^\alpha\,dt\right)^{\frac{1}{\alpha}} \quad for \quad h \in L_\alpha[a,b],\ \alpha \geq 1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1$$

and

$$\|h\|_\infty := ess \sup_{t\in[a,b]} |h(t)| \quad for \quad h \in L_\infty[a,b].$$

Proof. We start by assuming that f is absolutely continuous, then from Remark 1 and identity (2.2) we have

$$\Delta(P,Q;f;a,c,d,b) = -\int_a^b K(t)\,f'(t)\,dt,$$

and so

$$(2.9) \qquad |\Delta(P,Q;f;a,c,d,b)| = \left|\int_a^b K(t)\,f'(t)\,dt\right| \leq \int_a^b |K(t)|\,|f'(t)|\,dt.$$

Now, for $f' \in L_1[a,b]$ we have

$$\int_a^b |K(t)| |f'(t)| \, dt \le \int_a^b |K(t)| \, dt \cdot \|f'\|_\infty$$

so that

$$\int_a^b |K(t)| \, dt = \frac{1}{P(b)} \int_a^c P(t) \, dt + \frac{1}{P(b) Q(d)} \int_c^d |\theta(t)| \, dt + \frac{1}{P(b)} \int_d^b (P(b) - P(t)) \, dt, \tag{2.10}$$

where $\theta(t)$ is as given by (2.8).

For the second inequality in (2.7) we utilise Hölder's integral inequality from (2.9) to give

$$\left| \int_a^b K(t) f'(t) \, dt \right| \le \left(\int_a^b |K(t)|^\beta \, dt \right)^{\frac{1}{\beta}} \|f'\|_\alpha,$$

$$f' \in L_\alpha[a,b], \ \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

From (2.1) we have

$$\begin{aligned} P^\beta(b) \int_a^b |K(t)|^\beta \, dt &= \int_a^c P^\beta(t) \, dt + \frac{1}{Q^\beta(d)} \int_c^d |\theta(t)|^\beta \, dt + \int_d^b (P(b) - P(t))^\beta \, dt \end{aligned} \tag{2.11}$$

and so the second inequality in (2.7) results.

The third inequality in (2.7) is obtained from (2.9) for $f' \in L_1[a,b]$

$$\left| \int_a^b K(t) f'(t) \, dt \right| \le ess \sup_{t \in [a,b]} |K(t)| \cdot \|f'\|_1,$$

where $|K(t)|$ is nondecreasing over $[a,c]$ and nonincreasing over $[d,b]$ and so

$$ess \sup_{t \in [a,b]} |K(t)| = \max \left\{ P(c), \frac{\gamma}{Q(d)}, P(b) - P(d) \right\}, \tag{2.12}$$

where $\gamma = ess \sup_{t \in [c,d]} |\theta(t)|$ with $\theta(t)$ as defined in (2.8).

The fourth inequality follows directly from (2.4) by associating $K(t)$ with $g(t)$ and $f(t)$ with $v(t)$ while making use of (2.12). The fifth follows from (2.5) and (2.10) while making the same associations.

Finally, for the sixth inequality, utilising (2.6) gives

$$\left|\int_a^b K(t)\,df(t)\right| \le \int_a^b |K(t)|\,df(t), \tag{2.13}$$

where from (2.1)

$$P(b)\int_a^b |K(t)|\,df(t) = \int_a^c P(t)\,df(t) + \frac{1}{Q(d)}\int_c^d |\theta(t)|\,df(t) + \int_d^b (P(b)-P(t))\,df(t), \tag{2.14}$$

and hence the theorem is proved. ∎

REMARK 2. In (2.7) the integrals over $[a,c]$ and $[d,b]$ may be further developed, however, as for the integrals over $[c,d]$, more explicit knowledge regarding the weight functions $p(t)$ and $q(t)$ is required in order to determine the location of the zeros of $\theta(t)$. In particular,

$$\int_a^c P(t)\,dt = (t-c)P(t)\Big]_a^c - \int_a^c (t-c)\,df(t) = \int_a^c (c-t)\,dP(t) = cP(c) - \nu(P;a,c) \tag{2.15}$$

and

$$\int_d^b (P(b)-P(t))\,dt = (t-d)(P(b)-P(t))\Big]_d^b + \int_d^b (t-d)\,dP(t) = \int_d^b (t-d)\,dP(t) = \nu(P;d,b) - d(P(b)-P(d)), \tag{2.16}$$

with

$$\nu(P;d,b) = \int_a^b t\,dP(t). \tag{2.17}$$

COROLLARY 1. *Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$, then for $a\le c<d\le b$ and $p:[a,b]\to\mathbb{R}_+$ continuous, with $0<P(b)=\int_a^b p(t)\,dt<\infty$ the inequalities*

$$\mathfrak{A}(p;a,b)\,|D(p,p;f;a,c,d,b)| = \mathfrak{A}(p;a,b)\,|\mathfrak{M}(p;f;a,b) - \mathfrak{M}(p;f;c,d)| \tag{2.18}$$

$$\leq \begin{cases} B_1 \|f'\|_\infty, & f' \in L_\infty [a,b]; \\ B_\beta^{\frac{1}{\beta}} \|f'\|_\alpha, & f' \in L_\alpha [a,b], \ \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ B_\infty \|f'\|_1, & f' \in L_1 [a,b]; \\ B_\infty \bigvee_a^b (f'), & f' \text{ of bounded variation} \\ B_1 L, & f' \text{is } L\text{-Lipschitzian} \\ B_m < m(a,c) f(a) + m(d,b) f(b) & \\ \quad + \dfrac{m(a,c) + m(d,b)}{m(c,d)} f(c), & f' \text{ is monotonic nondecreasing} \end{cases}$$

hold where

$$(2.19) \quad \mathfrak{A}(p;a,b) = P(b) = \int_a^b p(t)\, dt = m(a,b), \quad M(a,b) = \int_a^b tp(t)\, dt,$$

$$(2.20) \quad \phi(t) = A(p;c,d)\, A(p;a,t) - A(p;a,b)\, A(p;c,t), \quad t \in [c,d];$$

$$(2.21) \quad B_\beta = \int_a^c P^\beta(t)\, dt + \frac{1}{m^\beta(c,d)} \int_c^d |\theta(t)|^\beta\, dt + \int_d^b [P(b) - P(t)]^\beta\, dt$$

$$\begin{aligned} (2.22) \qquad B_1 = {} & cm(a,c) - M(a,c) + \frac{m(a,c) + m(d,b)}{m(c,d)} \\ & \times [M(c,t^*) - M(t^*,d) + dm(t^*,d) - cm(c,t^*)] \\ & + M(d,b) - dm(d,b), \quad \text{with } \phi(t^*) = 0, \end{aligned}$$

$$(2.23) \quad B_\infty = \max \left\{ m(a,c), \frac{1}{2m(c,d)} [\phi(c) - \phi(d) + |\phi(c) + \phi(d)|], m(d,b) \right\},$$

$$\begin{aligned} (2.24) \qquad B_m = {} & -\int_a^c p(t)\, f(t)\, dt - \frac{m(a,c) + m(d,b)}{m(c,d)} \cdot \int_c^d p(t)\, f(t)\, dt \\ & + \int_d^b p(t)\, f(t)\, dt \end{aligned}$$

and $\bigvee_a^b (h)$ *is the total variation of* h *over* $[a,b]$.

Proof. From Lemma 1 and Theorem 5, let $q(t) \equiv p(t)$. Define a new kernel for this specific $q(t)$ then from (2.1) we obtain

$$
(2.25)\qquad \kappa(t) := \begin{cases} \dfrac{\mathfrak{A}(p;a,t)}{\mathfrak{A}(p;a,b)}, & t \in [a,c], \\ \dfrac{\mathfrak{A}(p;a,t)}{\mathfrak{A}(p;a,b)} - \dfrac{\mathfrak{A}(p;c,t)}{\mathfrak{A}(p;c,d)}, & t \in (c,d) \\ \dfrac{\mathfrak{A}(p;a,t) - \mathfrak{A}(p;a,b)}{\mathfrak{A}(p;a,b)}, & t \in [d,b]. \end{cases}
$$

We also note Remark 1 as exemplified by (2.3) so that $D(p,q;f;a,c,d,b)$ for p,q continuous satisfy the same identity as $\Delta(P,Q;f;a,c,d,b)$ for p,q of bounded variation.

Now, we shall investigate the behaviour of $\kappa(t)$.

We note that $\kappa(a) = \kappa(b) = 0$ and $\kappa(t)$ is continuous at c and d. Further, from (2.20),

$$
\begin{aligned}
\phi(c) &= \mathfrak{A}(p;c,d)\,\mathfrak{A}(p;a,c) > 0 \quad \text{and} \\
\phi(d) &= \mathfrak{A}(p;c,d)\,[\mathfrak{A}(p;a,d) - \mathfrak{A}(p;a,b)] < 0
\end{aligned}
$$

and so there is at least one point $t^* \in (c,d)$ such that $\phi(t^*) = 0$. For $p(t)$ continuous

$$
\phi'(t) = p(t)\,[\mathfrak{A}(p;s,d) - \mathfrak{A}(p;a,b)] < 0
$$

and so there is only one point t^* such that $\phi(t^*) = 0$ and $t^* \in (c,d)$.

It should be noted that $\phi(t)$ in (2.20) is equivalent to $\theta(t)$ of (2.8) with $q(t) = p(t)$ and $Q(d) = \int_c^d p(t)\,dt$. We thus need to determine the expressions in (2.7) in an explicit form.

We note from (2.15) and (2.16) that

$$
(2.26)\qquad \int_a^c P(t)\,dt = cP(c) - M(a,c) = cm(a,c) - M(a,c),
$$

$$
\begin{aligned}
(2.27)\qquad \int_d^b (P(b) - P(t))\,dt &= M(d,b) - d\cdot(P(b) - P(d)) \\
&= M(d,b) - dm(d,b),
\end{aligned}
$$

where we have used (2.19).

Further, from (2.20) and the above behaviour of $\phi(t)$

$$
\begin{aligned}
(2.28)\qquad \operatorname{ess\,sup}|\phi(\mathrm{t})| &= \max\{\phi(c), -\phi(d)\} \\
&= \frac{1}{2}[\phi(c) - \phi(d) + |\phi(c) + \phi(d)|]
\end{aligned}
$$

and

$$\int_c^d |\phi(t)|\,dt = \int_c^{t^*} \phi(t)\,dt - \int_{t^*}^d \phi(t)\,dt. \tag{2.29}$$

Now, integration by parts gives

$$\begin{aligned}\int_c^{t^*} \phi(t)\,dt &= -\int_c^{t^*} (t-c)\,\phi'(t)\,dt \\ &= \left[-\int_c^{t^*} tp(t)\,dt + c\int_c^{t^*} p(t)\,dt\right][\mathfrak{A}(p;c,d) - \mathfrak{A}(p;a,b)] \\ &= [cm(c,t^*) - M(c,t^*)][\mathfrak{A}(p;c,d) - \mathfrak{A}(p;a,b)]\end{aligned}$$

and

$$\begin{aligned}-\int_{t^*}^d \phi(t)\,dt &= -\int_{t^*}^d (t-d)\,\phi'(t)\,dt \\ &= \left[\int_{t^*}^d (t-d)\,p(t)\,dt\right][\mathfrak{A}(p;c,d) - \mathfrak{A}(p;a,b)] \\ &= [M(t^*,d) + dm(t^*,d)][\mathfrak{A}(p;c,d) - \mathfrak{A}(p;a,b)].\end{aligned}$$

Hence from (2.26) we obtain

$$\begin{aligned}\int_c^d |\phi(t)|\,dt &\\ &= [M(c,t^*) - M(t^*,d) + dm(t^*,d) - cm(c,t^*)][\mathfrak{A}(p;c,d) - \mathfrak{A}(p;a,b)] \\ &= [M(c,t^*) - M(t^*,d) + dm(t^*,d) - cm(c,t^*)][m(a,c) + m(d,b)].\end{aligned} \tag{2.30}$$

Combining (2.26), (2.27) and (2.30) gives from the first and fifth inequalities of (2.7) the respective inequalities in (2.18) with B_1 as given by (2.22). The second inequality in (2.18) is obtained from the corresponding inequality in (2.7) for specifically $q(t) = p(t)$.

Further, from (2.28) and the third and fourth inequalities in (2.7) gives the respective inequalities in (2.18) where B_∞ is as given in (2.23) and of course $q(t) \equiv p(t)$.

Now for the final inequality. From the last inequality in (2.7) with $q(t) = p(t)$ so that $\theta(t) = \phi(t)$ and $P(b) = m(a,b)$, $Q(d) = m(c,d)$ we have on integrating by parts

$$\begin{aligned}\int_a^c P(t)\,df(t) &= P(c)\,f(c) - \int_a^c p(t)\,f(t)\,dt \\ &= m(a,c)\,f(c) - \int_a^c p(t)\,f(t)\,dt,\end{aligned} \tag{2.31}$$

$$(2.32)\quad \frac{1}{m(c,d)}\int_c^d |\phi(t)|\,df(t) = \frac{1}{m(c,d)}\left\{\int_c^{t^*} \phi(t)\,df(t) - \int_{t^*}^d \phi(t)\,df(t)\right\}$$

and

$$(2.33)\quad \int_d^b [P(b)-P(t)]\,df(t) = (P(b)-P(d))\,f(d) + \int_d^b p(t)\,f(t)\,dt$$
$$= m(d,b)\,f(d) + \int_d^b p(t)\,f(t)\,dt.$$

Using the fact from (2.20) that

$$(2.34)\quad \phi'(t) = p(t)\,[m(c,d) - m(a,b)] = -p(t)\,[m(a,c) + m(d,b)]$$

then

$$\int_c^{t^*} \phi(t)\,df(t) = -\phi(c)\,f(c) - \int_c^{t^*} \phi'(t)\,f(t)\,dt$$
$$= -m(c,d)\,m(a,c)\,f(c)$$
$$- [m(a,c) + m(d,b)]\int_c^{t^*} p(t)\,f(t)\,dt$$

and

$$-\int_{t^*}^d \phi(t)\,df(t) = -\phi(d)\,f(d) - [m(a,c) + m(d,b)]\int_{t^*}^d p(t)\,f(t)\,dt$$
$$= -m(c,d)\,m(d,b)\,f(d)$$
$$- [m(a,c) + m(d,b)]\int_{t^*}^d p(t)\,f(t)\,dt.$$

Thus,

$$(2.35)\quad \frac{1}{m(c,d)}\left\{\int_c^{t^*} \phi(t)\,df(t) - \int_{t^*}^d \phi(t)\,df(t)\right\}$$
$$= -m(a,c)\,f(c) - m(d,b)\,f(d) - \frac{m(a,c) + m(d,b)}{m(c,d)}\int_c^d p(t)\,f(t)\,dt.$$

Combining (2.31), (2.32), (2.33) into the final inequality of (2.7) gives (2.24) upon using (2.35). The upper bound on the final inequality in (2.18) is obtained on noting that for $h(t) > 0$, $w(a,b) = \int_a^b h(t)\,dt$ and $f(t)$ monotonic nondecreasing then

$$-\int_a^b h(t)\,f(t)\,dt < w(a,b)\,f(a)$$

and

$$\int_a^b h(t) f(t)\, dt < w(a,b) f(b).$$

The corollary is now completely proven. ∎

REMARK 3. If we take the weight functions to be unity then earlier results may be recaptured as particular cases.

REMARK 4. If we assume that there is a point $x \in (a,b)$ for which the function is continuous then we may recapture bounds for the weighted Ostrowski functional

$$\Theta(p;f)(x) := \mathcal{M}(p;f) - f(x), \tag{2.36}$$

where

$$\mathcal{M}(p;f) = \frac{\int_a^b p(x) f(x)\, dx}{\int_a^b p(x)\, dx}.$$

Indeed, if we assume that $c = x \in (a,b)$, $d = x + \varepsilon \in (a,b)$, then from (1.8) – (1.9), on assuming also that $p(\cdot)$ is continuous at x,

$$|D(p,p;f;a,x,x+\varepsilon,b)| = \left|\mathcal{M}(p;f;a,b) - \frac{\int_x^{x+\varepsilon} p(t) f(t)\, dt}{\int_x^{x+\varepsilon} p(t)\, dt}\right|.$$

Taking the limit as $\varepsilon \to 0+$ gives

$$|\Theta(p;f)(x)| = \lim_{\varepsilon \to 0} |D(p,p;f;a,x,x+\varepsilon,b)|.$$

Moreover, from the identity (2.3) and the kernel $\kappa(t)$ as defined by (2.25) gives an identity for the weighted Ostrowski functional

$$\Theta(p;f)(x) = -\int_a^b \kappa_0(x)\, df(x), \tag{2.37}$$

where

$$\int_a^b p(t)\, dt \cdot \kappa_0(x) = \begin{cases} \int_a^t p(x)\, dx, & t \in [a,x]; \\ \\ \int_t^b p(x)\, dx, & t \in (x,b]. \end{cases}$$

See the work [12] for bounds on $\Theta(p;f)(x)$ obtained from the identity (2.37).

If the weight $p(x)$ is taken as unity and $f(t)$ is absolutely continuous, then (2.37) may be recognised as Montgomery's identity, [4]. See also [1], [5]–[14].

3. Concluding remarks

The current work has investigated differences between weighted integral means over the intervals $[c,d] \subset [a,b]$. One can envisage a process in which access is restricted to a subinterval $[c, d]$ so that the current work may prove useful in approximating the integral mean over a larger interval by having information over a subinterval. The work also allows for a different weighting envisaged as operation under changed conditions over the subinterval.

This sort of problem was described in Barnett et al. [2] where the problem of determining the mean quality of a continuous stream process was examined in which the sampling was done over a subinterval. The current article may be looked upon as a similar problem where an external influence exemplified by the weight function is accommodated within the formulation.

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