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ON CERTAIN INTEGRAL INEQUALITIES AND THEIR DISCRETE ANALOGUES

Abstract. In the present paper we establish new integral inequalities in two variables and their discrete analogues which provide explicit bounds on unknown functions. The inequalities given here can be used as tools in the study of certain differential, integral and difference equations.

1. Introduction

The fundamental role played by the integral and finite difference inequalities in the development of the theory of differential, integral and finite difference equations is well known. During the past few years some new integral and finite difference inequalities have been developed, which provide a natural and effective means for further development of these equations, see [1, 3–6] and the references given therein. In the qualitative analysis of certain classes of differential, integral and finite difference equations the bounds provided by the existing literature are inadequate and it is necessary to seek some new inequalities in order to achieve a diversity of desired goals. In this paper we offer some basic integral inequalities involving functions of two independent variables and their discrete analogues which can be used more conveniently in certain new applications for which the inequalities given earlier do not apply directly.

2. Main results

In what follows, R denotes the set of real numbers and $R_+ = [0, \infty)$, $N_0 = \{0, 1, 2, \ldots\}$ be subsets of R. We use the usual conventions that empty sums and products are taken to be 0 and 1 respectively. We assume that all the functions which appear in the inequalities are real valued and all the

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integrals, sums and products involved exist on the respective domains of their definitions.

An interesting and useful integral inequality is established in the following theorem.

THEOREM 1. *Let $u(x,y)$, $a(x,y)$, $b(x,y)$, $c(x,y)$ be nonnegative continuous functions defined for $x,y \in R_+$ and $p>1$ be a real constant. If*

$$u^p(x,y) \le a(x,y) + b(x,y)\int_0^x\int_y^\infty c(s,t)u(s,t)dtds, \tag{2.1}$$

for $x,y \in R_+$, then

$$u(x,y) \le \left[a(x,y) + b(x,y)A(x,y)\exp\left(\int_0^x\int_y^\infty c(s,t)\frac{b(s,t)}{p}dtds\right)\right]^{1/p}, \tag{2.2}$$

for $x,y \in R_+$, where

$$A(x,y) = \int_0^x\int_y^\infty c(s,t)\left(\frac{p-1}{p} + \frac{a(s,t)}{p}\right)dtds, \tag{2.3}$$

for $x,y \in R_+$.

Proof. Define a function $z(x,y)$ by

$$z(x,y) = \int_0^x\int_y^\infty c(s,t)u(s,t)dtds, \tag{2.4}$$

then (2.1) can be restated as

$$u^p(x,y) \le a(x,y) + b(x,y)z(x,y). \tag{2.5}$$

From (2.5) and using the elementary inequality (see, [2, p. 30])

$$\alpha^{1/p}\beta^{1/q} \le \frac{\alpha}{p} + \frac{\beta}{q},$$

where $\alpha \ge 0$, $\beta \ge 0$ and $1/p + 1/q = 1$ with $p > 1$, we observe that

$$\begin{aligned} u(x,y) &\le [a(x,y) + b(x,y)z(x,y)]^{1/p}[1]^{p-1/p} \\ &\le \frac{p-1}{p} + \frac{a(x,y)}{p} + \frac{b(x,y)}{p}z(x,y). \end{aligned} \tag{2.6}$$

From (2.4) and (2.6) we have

$$\begin{aligned} z(x,y) &\le \int_0^x\int_y^\infty c(s,t)\left(\frac{p-1}{p} + \frac{a(s,t)}{p} + \frac{b(s,t)}{p}z(s,t)\right)dtds \\ &= A(x,y) + \int_0^x\int_y^\infty c(s,t)\frac{b(s,t)}{p}z(s,t)dtds, \end{aligned} \tag{2.7}$$

where $A(x,y)$ is defined by (2.3). Clearly $A(x,y)$ is nonnegative continuous, nondecreasing in x and nonincreasing in y for $x, y \in R_+$. First we assume that $A(x,y) > 0$ for $x, y \in R_+$. From (2.7) it is easy to observe that

$$\frac{z(x,y)}{A(x,y)} \le 1 + \int_0^x \int_y^\infty c(s,t) \frac{b(s,t)}{p} \frac{z(s,t)}{A(s,t)} dt ds. \tag{2.8}$$

Define a function $v(x,y)$ by the right hand side of (2.8). Then $v(x,y) > 0$, $\frac{z(x,y)}{A(x,y)} \le v(x,y)$, $v(x,y)$ is nonincreasing in y for $y \in R_+$ and

$$\begin{aligned} v_x(x,y) &= \int_y^\infty c(x,t) \frac{b(x,t)}{p} \frac{z(x,t)}{A(x,t)} dt \\ &\le \int_y^\infty c(x,t) \frac{b(x,t)}{p} v(x,t) dt \le v(x,y) \int_y^\infty c(x,t) \frac{b(x,t)}{p} dt. \end{aligned} \tag{2.9}$$

Treating y fixed in (2.9), dividing both sides of (2.9) by $v(x,y)$, setting $x = s$ and integrating the resulting inequality from 0 to x, $x \in R_+$, we get

$$v(x,y) \le \exp\left(\int_0^x \int_y^\infty c(s,t) \frac{b(s,t)}{p} dt ds \right). \tag{2.10}$$

Using (2.10) in $\frac{z(x,y)}{A(x,y)} \le v(x,y)$ we have

$$z(x,y) \le A(x,y) \exp\left(\int_0^x \int_y^\infty c(s,t) \frac{b(s,t)}{p} dt ds \right). \tag{2.11}$$

If $A(x,y)$ is nonnegative in (2.7), then we carry out the above procedure with $A(x,y) + \varepsilon$ instead of $A(x,y)$, where $\varepsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit with $\varepsilon \to 0$ to obtain (2.11). The desired inequality in (2.2) follows from (2.5) and (2.11).

We next establish the following integral inequality which can be used in certain situations.

THEOREM 2. *Let $u(x,y)$, $a(x,y)$, $b(x,y)$ be nonnegative functions defined and continuous for $x, y \in R_+$, $F : R_+^3 \to R_+$ be a continuous function which satisfies the condition*

$$0 \le F(x,y,u) - F(x,y,v) \le G(x,y,v)(u-v),$$

for $u \ge v \ge 0$, where $G : R_+^3 \to R_+$ is a continuous function and $p > 1$ be a real constant. If

$$u^p(x,y) \le a(x,y) + b(x,y) \int_0^x \int_y^\infty F(s,t,u(s,t)) dt ds, \tag{2.12}$$

for $x, y \in R_+$ *then*

$$(2.13)\quad u(x,y) \le \Bigg[a(x,y) + b(x,y)\overline{A}(x,y)\exp\Bigg(\int_0^x\int_y^\infty G(s,t,\frac{p+1}{p}+\frac{a(s,t)}{p})\times \times\frac{b(s,t)}{p}dtds\Bigg)\Bigg]^{1/p},$$

for $x, y \in R_+$, *where*

$$(2.14)\qquad \overline{A}(x,y) = \int_0^x\int_y^\infty F(s,t,\frac{p-1}{p}+\frac{a(s,t)}{p})dtds,$$

for $x, y \in R_+$.

Proof. Define a function $z(x,y)$ by

$$(2.15)\qquad z(x,y) = \int_0^x\int_y^\infty F(s,t,u(s,t))dtds.$$

Then as in the proof of Theorem 1, from (2.12) we see that the inequalities (2.5) and (2.6) hold. From (2.15), (2.6) and the conditions on F, it follows that

$$(2.16)\quad z(x,y) \le \int_0^x\int_y^\infty \Bigg[F\bigg(s,t,\frac{p-1}{p}+\frac{a(s,t)}{p}+\frac{b(s,t)}{p}z(s,t)\bigg) - F\bigg(s,t,\frac{p-1}{p}+\frac{a(s,t)}{p}\bigg) + F\bigg(s,t,\frac{p-1}{p}+\frac{a(s,t)}{p}\bigg)\Bigg]dtds$$
$$\le \overline{A}(x,y) + \int_0^x\int_y^\infty G(s,t,\frac{p-1}{p}+\frac{a(s,t)}{p})\frac{b(s,t)}{p}z(s,t)dtds,$$

where $\overline{A}(x,y)$ is defined by (2.14). The rest of the proof can be completed by closely looking at the proof of Theorem 1 given above. Here we omit the further details.

3. Discrete analogues

In this section we establish the discrete versions of Theorems 1 and 2 which can be used in the study of certain partial finite difference and sum-difference equations.

THEOREM 3. *Let* $u(m,n)$, $a(m,n)$, $b(m,n)$, $c(m,n)$ *be nonnegative functions defined for* $m, n \in N_0$ *and* $p > 1$ *be a real constant. If*

$$(3.1)\qquad u^p(m,n) \le a(m,n) + b(m,n)\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} c(s,t)u(s,t),$$

for $m, n \in N_0$, *then*

$$(3.2)\quad u(m,n) \le \left[a(m,n)+b(m,n)e(m,n)\prod_{s=0}^{m-1}\left[1+\sum_{t=n+1}^{\infty} c(s,t)\frac{b(s,t)}{p}\right]\right]^{1/p},$$

for $m, n \in N_0$, *where*

$$(3.3)\qquad e(m,n) = \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} c(s,t)\left(\frac{p-1}{p} + \frac{a(s,t)}{p}\right),$$

for $m, n \in N_0$.

Proof. Define a function $z(m,n)$ by

$$(3.4)\qquad z(m,n) = \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} c(s,t)u(s,t).$$

Then (3.1) can be written as

$$(3.5)\qquad u^p(m,n) \le a(m,n) + b(m,n)z(m,n).$$

From (3.5) as in the proof of theorem 1, we get

$$(3.6)\qquad u(m,n) \le \frac{p-1}{p} + \frac{a(m,n)}{p} + \frac{b(m,n)}{p}z(m,n).$$

From (3.4) and (3.6) we have

$$(3.7)\qquad z(m,n) \le \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} c(s,t)\left(\frac{p-1}{p} + \frac{a(s,t)}{p} + \frac{b(s,t)}{p}z\left(s,t\right)\right)$$
$$= e(m,n) + \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} c(s,t)\frac{b(s,t)}{p}z(s,t),$$

where $e(m,n)$ is defined by (3.3). Clearly $e(m,n)$ is nonnegative, nondecreasing in m and nonincreasing in n for $m, n \in N_0$. First we assume that $e(m,n) > 0$ for $m, n \in N_0$. From (3.7) we observe that

$$\frac{z(m,n)}{e(m,n)} \le 1 + \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} c(s,t)\frac{b(s,t)}{p}\frac{z(s,t)}{e(s,t)}.$$

Define a function $v(m,n)$ by

$$(3.8)\qquad v(m,n) = 1 + \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} c(s,t)\frac{b(s,t)}{p}\frac{z(s,t)}{e(s,t)}.$$

Then $\frac{z(m,n)}{e(m,n)} \le v(m,n)$ and

$$(3.9)\qquad \begin{aligned}[v(m+1,n)-v(m,n)]-[v(m+1,n+1)-v(m,n+1)]&\\ =c(m,n+1)\frac{b(m,n+1)}{p}\frac{z(m,n+1)}{e(m,n+1)}&\\ \le c(m,n+1)\frac{b(m,n+1)}{p}v(m,n+1).&\end{aligned}$$

From (3.9) and using the facts that $v(m,n)>0$, $v(m,n+1)\le v(m,n)$ for $m,n\in N_0$, we observe that

$$(3.10)\qquad \frac{[v(m+1,n)-v(m,n)]}{v(m,n)}-\frac{[v(m+1,n+1)-v(m,n+1)]}{v(m,n+1)} \le c(m,n+1)\frac{b(m,n+1)}{p}.$$

Keeping m fixed in (3.10), set $n=t$ and sum over $t=n,n+1,\ldots,r-1$ ($r\ge n+1$ is arbitrary in N_0) to obtain

$$(3.11)\qquad \frac{[v(m+1,n)-v(m,n)]}{v(m,n)}-\frac{[v(m+1,r)-v(m,r)]}{v(m,r)} \le \sum_{t=n+1}^{r} c(m,t)\frac{b(m,t)}{p}.$$

Noting that $\lim_{r\to\infty} v(m+1,r)=\lim_{r\to\infty} v(m,r)=1$ and by letting $r\to\infty$ in (3.11) we get

$$\frac{[v(m+1,n)-v(m,n)]}{v(m,n)}\le \sum_{t=n+1}^{\infty} c(m,t)\frac{b(m,t)}{p},$$

i.e.

$$(3.12)\qquad v(m+1,n)\le [1+\sum_{t=n+1}^{\infty} c(m,t)\frac{b(m,t)}{p}]v(m,n).$$

Now by keeping n fixed in (3.12), setting $m=s$ and substituting $s=0,1,2,\ldots,m-1$, successively and using the fact that $v(o,n)=1$, we get

$$(3.13)\qquad v(m,n)\le \prod_{s=0}^{m-1}\left[1+\sum_{t=n+1}^{\infty} c(s,t)\frac{b(s,t)}{p}\right].$$

Using (3.13) in $\frac{z(m,n)}{e(m,n)}\le v(m,n)$ we have

$$(3.14)\qquad z(m,n)\le e(m,n)\prod_{s=0}^{m-1}\left[1+\sum_{t=n+1}^{\infty} c(s,t)\frac{b(s,t)}{p}\right].$$

The required inequality in (3.2) follows from (3.5) and (3.14). If $e(m,n)$ is nonnegative, then we carry out the above procedure with $e(m,n)+\varepsilon$ instead of $e(m,n)$, where $\varepsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit with $\varepsilon \to 0$ to obtain (3.2).

THEOREM 4. *Let $u(m,n)$, $a(m,n)$, $b(m,n)$ be a nonnegative functions defined for $m,n \in N_0$, $L : N_0^2 \times R_+ \to R_+$ be a function which satisfies the condition*

$$0 \leq L(m,n,u) - L(m,n,v) \leq K(m,n,v)(u-v),$$

for $u \geq v \geq 0$, where $K(m,n,v)$ is a nonnegative function defined for $m,n \in N_0$, $v \in R_+$ and let $p > 1$ be a real constant. If

$$u^p(m,n) \leq a(m,n) + b(m,n) \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} L(s,t,u(s,t)), \tag{3.15}$$

for $m,n \in N_0$, then

$$u(m,n) \leq \Big[a(m,n) + b(m,n)\bar{e}(m,n) \prod_{s=0}^{m-1} \Big[1 + \sum_{t=n+1}^{\infty} K\Big(s,t,\frac{p-1}{p} + \frac{a(s,t)}{p}\Big)\frac{b(s,t)}{p}\Big]\Big]^{1/p}, \tag{3.16}$$

for $m,n \in N_0$, where

$$\bar{e}(m,n) = \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} L\Big(s,t,\frac{p-1}{p} + \frac{a(s,t)}{p}\Big), \tag{3.17}$$

for $m,n \in N_0$.

The proof of this theorem follows by closely looking at the proofs of the above theorems. Here we omit the details.

4. An application

In this section, we present an immediate application of Theorem 1 to obtain the bound on the solution of a nonlinear integral equation of the form

$$u^p(x,y) = f(x,y) + \int_0^x \int_y^{\infty} g(x,y,s,t,u(s,t))dtds, \tag{4.1}$$

where $p > 1$ is a real constant, $u, f : R_+^2 \to R$, $g : R_+^5 \to R$ are continuous functions such that

$$|f(x,y)| \leq a(x,y), \tag{4.2}$$

$$|g(x,y,s,t,u(s,t))| \leq b(x,y)c(s,t)|u(s,t)|, \tag{4.3}$$

for $0 \leq s \leq x$, $0 \leq t \leq y$, $x,y \in R_+$, where a, b, c are as defined in Theorem 1. Let $u(x,y)$ be a solution of (4.1) for $x,y \in R_+$. From (4.1)–(4.3) it follows that

$$|u(x,y)|^p \le a(x,y) + b(x,y)\int_0^x \int_y^\infty c(s,t)|u(s,t)|dtds. \tag{4.4}$$

Now a suitable application of Theorem 1 to (4.4) yields

$$|u(x,y)| \le \left[a(x,y) + b(x,y)A(x,y)\exp\left(\int_0^x \int_y^\infty c(s,t)\frac{b(s,t)}{p}dtds\right)\right]^{1/p}, \tag{4.5}$$

where $A(x,y)$ is defined by (2.3). The right-hand side of (4.5) gives the bound on the solution of (4.1) in terms of the known functions.

Finally, we note that the bounds obtained in Theorems 1-4 are independent of the unknown functions and will have many possible applications to boundedness, uniqueness, continuous dependence and other properties of the solutions of certain classes of partial differential and finite difference equations. However, various applications of these inequalities will be reported elsewhere.

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