DEMONSTRATIO MATHEMATICA
Vol. XXXV No 2 2002

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THE HAUSDORFF METRIC AND ITS EXTENSIONS

Abstract. We consider a complete metric space (X, ρ) such that closed balls are compact. The paper is devoted to the Hausdorff distance d defined on $\Re$, the space of all nonempty compact subsets of X. We construct an embedding $(\Re, d) \hookrightarrow (\Im, \mathrm{dist}_f)$, where $\Im$ is the family of all closed subsets of X. We show that $(\Im, \mathrm{dist}_f)$ is compact.

0. Introduction and preliminaries

Let (X, ρ) be a metric space. It is well known that when (X, ρ) is separable and locally compact then the set $\hat{X} = X \cup \{\infty\}$ (Alexandroff compactification) is metrizable, where ∞ is the point that does not belong to X and $(\hat{X}, \hat{\rho})$ is compact (see [3], page 55). In the paper we use the Hausdorff type distances to define a metric on $\hat{X}$ which is more direct and efficient.

By d we denote the Hausdorff metric defined by

$$d(A, B) = \max\{\rho_s(A, B), \rho_s(B, A)\}$$

for all A, B in $\Re$, the space of all nonempty compact subsets of X, where $\rho_s(A, B)$ is the Hausdorff semidistance (see [5]). The Hausdorff metric d may be infinite when extended to $\Im$, the space of all nonempty closed subsets of X. In [5], a bounded metric on $\Im$ is defined. We use extensions of the Hausdorff metric to obtain a bounded metric on $\Im$ including an empty subset of X.

We extend d to a bounded metric h on $\Im$, the space of all closed subsets of X (including the empty set). We admit the following $\rho_s(\emptyset, A) = 0$, for all $A \subseteq X$ and $\rho_s(A, \emptyset) = \infty$ for all nonempty $A \subseteq X$. For a real valued continuous function $f(t)$, such that $f(t) > 0$ for all $t \in [0, \infty)$ and $\int_{[0,\infty)} f(t)dt < \infty$ we define a metric $dist_f$ on the space of all closed subsets of X (including the empty set) by

1991 *Mathematics Subject Classification*: 26A09, 28A25, 54E35, 54E45, 54E50.
Key words and phrases: metric space, Hausdorff metric, locally compact.

$$dist_f(A,B) = \int_{[0,\infty)} f(t)h(A \cap K(x_0,t), B \cap K(x_0,t))dt.$$

We discuss convergence and related properties of $dist_f$. In particular we show that $(\Im, dist_f)$ is compact whenever nonempty finite closed balls $K(x_0,r) = \{y \in X : \rho(x_0,y) \le r\}$ are compact in (X,ρ). These studies are important in the light of recent developments in the study of fractals and semifractals (see [5] and [6]).

1. The extensions of the Hausdorff metric

We start with the following commonly known fact:

PROPOSITION 1.1. *Let d be the Hausdorff metric on the space of closed and bounded subsets of a set X (including the empty set). Then the function h defined by*

$$h(A,B) = \begin{cases} \frac{d(A,B)}{1+d(A,B)} & \text{if } d(A,B) < \infty \\ 1 & \text{if } d(A,B) = \infty \end{cases}$$

is a metric on the space of closed and bounded subsets of X.

DEFINITION 1.2. Let $x_0 \in X$ be a fixed point. Given arbitrary closed subsets A, $B \subseteq X$ we define the function $L_{A,B} : [0,\infty) \to \mathbb{R}$ by $L_{A,B}(t) = d(A \cap K(x_0,t), B \cap K(x_0,t))$.

LEMMA 1.3. *The function $L_{A,B}$ is right continuous.*

Proof. We need to prove that if t_n is a decreasing sequence in $[0,\infty)$ such that $t_n \to t_0$ as $n \to \infty$, then $L_{A,B}(t_n) \to L_{A,B}(t_0)$. The case when $B \cap K(x_0,t_n) = \emptyset$ for some n is trivial because then we will have $B \cap K(x_0,t_0) = \emptyset$ which leads to $L_{A,B}(t_n) = \infty$ and $L_{A,B}(t_0) = \infty$ if $A \cap K(x_0,t_0) \neq \emptyset$ or $A \cap K(x_0,t_n) = \emptyset = B \cap K(x_0,t_n)$ for some n and then $L_{A,B}(t_j) = 0 = L_{A,B}(t_0)$ for all $j \ge n$. Therefore we may assume that $L_{A,B}(t_n) < \infty$ for all n.

Now let us suppose that $\lim_{n\to\infty} L_{A,B}(t_n)$ exists and it is finite but $\lim_{n\to\infty} L_{A,B}(t_n) \neq L_{A,B}(t_0)$ (we may assume the convergence of $L_{A,B}(t_n)$ because we may always choose a convergent subsequence). Then either we have $\lim_{n\to\infty} L_{A,B}(t_n) > L_{A,B}(t_0)$ or $\lim_{n\to\infty} L_{A,B}(t_n) < L_{A,B}(t_0)$. In the first case we put $\alpha = \lim_{n\to\infty} L_{A,B}(t_n)$ and choose $\epsilon > 0$ such that $\alpha > L_{A,B}(t_0)+\epsilon$. For sufficiently large n there exist either $a_n \in A \cap K(x_0,t_n)$ such that $\rho(a_n,b) > \alpha - \frac{\epsilon}{2}$ for all $b \in B \cap K(x_0,t_n)$, (i.e. $K(a_n, \alpha - \frac{\epsilon}{2}) \cap B \cap K(x_0,t_n) = \emptyset$) or $b_n \in B \cap K(x_0,t_n)$ such that $\rho(b_n,a) > \alpha - \frac{\epsilon}{2}$ for all $a \in A \cap K(x_0,t_n)$ (i.e. $K(b_n, \alpha - \frac{\epsilon}{2}) \cap A \cap K(x_0,t_n) = \emptyset$). From $a_n \in A \cap K(x_0,t_n)$ we choose a subsequence which converges to $a_0 \in A \cap K(x_0,t_0)$ (by compactness of $A \cap K(x_0,t_0)$). Then $K(x_0, \alpha - \frac{3\epsilon}{4}) \cap B \cap K(x_0,t_0) = \emptyset$

which gives the contradiction $L_{A,B}(t_0) \geq \alpha - \frac{3\epsilon}{4} > \alpha - \epsilon > L_{A,B}(t_0)$. The proof for $b_n \in B \cap K(x_0, t_n)$ is similar.

In the second case, for some $\epsilon > 0$, we have $\alpha = \lim_{n\to\infty} L_{A,B}(t_n) < L_{A,B}(t_0) - \epsilon$. Let us fix $a \in A \cap K(x_0, t_0)$ and $b \in B \cap K(x_0, t_0)$. Then for sufficiently large n, $K(a, \alpha + \frac{\epsilon}{2}) \cap B \cap K(x_0, t_n) \neq \emptyset$ and $K(b, \alpha + \frac{\epsilon}{2}) \cap A \cap K(x_0, t_n) \neq \emptyset$. We choose $b_n \in K(a, \alpha + \frac{\epsilon}{2}) \cap B \cap K(x_0, t_n)$ and by compactness of $K(a, \alpha + \frac{\epsilon}{2}) \cap B \cap K(x_0, t_n)$ we have $\lim_{j\to\infty} b_{n_j} = \widetilde{b} \in B \cap K(x_0, t_0)$ for some subsequence b_{n_j}. But $\rho(a, \widetilde{b}) = \lim_{j\to\infty} \rho(a, b_{n_j}) \leq \alpha + \frac{\epsilon}{2}$. Similarly we have $\rho(b, \widetilde{a}) \leq \alpha + \frac{\epsilon}{2}$ for some $\widetilde{a} \in A \cap K(x_0, t_0)$. These imply that $L_{A,B}(t_0) \leq \alpha + \frac{\epsilon}{2}$ which is a contradiction because $\alpha + \frac{\epsilon}{2} < L_{A,B}(t_0) - \frac{\epsilon}{2}$. Finally we have proved that $\lim_{n\to\infty} L_{A,B}(t_n) = L_{A,B}(t_0)$ and therefore $L_{A,B}(t)$ is right continuous. ∎

COROLLARY 1.4. *The function* $t \to h(A \cap K(x_0, t), B \cap K(x_0, t))$ *is Borel measurable.*

REMARK 1.5. Given a right continuous function g and $\alpha \in \mathbb{R}$, let $E = \{x \in [0, \infty) : g(x) > \alpha\}$. Then $[x, x + \delta) \subseteq E$ for some $\delta > 0$ depending on $x \in E$. In particular $E = \cup_{x\in E}[x, x + \delta)$, may be represented as a countable union of left closed intervals and therefore E is a Borel set.

EXAMPLE 1. In $X = \mathbb{R}$ with the Euclidean metric we construct compact subsets A and B by defining

$$A = \{\tfrac{1}{2m} : m = 1, 2, \dots\} \cup \{0\} \text{ and } B = \{\tfrac{1}{2m-1} : m = 1, 2, 3, \dots\} \cup \{0\}.$$

EXAMPLE 2. In $\mathbb{R}^2$ with the Euclidean metric we will construct compact subsets A and B in $\mathbb{R}^2$ such that $L_{A,B}(t)$ is discontinuous at infinitely many points. Let us take $\alpha_m > 0, \beta_m > 0$ such that $\frac{\beta_m}{\alpha_m} = m$ and $\sqrt{\alpha_m^2 + \beta_m^2} = \frac{1}{m}$, i.e. $\alpha_m = \frac{1}{m\sqrt{1+m^2}}$, $\beta_m = \frac{1}{\sqrt{1+m^2}}$, $m = 1, 2, 3, \dots$. Now define $x_m = (\alpha_m, \beta_m) \in \mathbb{R}^2$. The sets

$$A = \{x_m : m \in \mathbb{N}\} \cup \{(0,0)\} \text{ and}$$

$$B = \{t_m x_m : m \in \mathbb{N}\} \cup \{(0,0)\}, \text{ where } \mathrm{t_m} = \frac{1}{2}(\frac{1}{\mathrm{m}} + \frac{1}{\mathrm{m}-1}), \text{ and } \mathrm{t_1} = 2$$

are compact. We calculate a few points of A and B and see that

$$A = \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{2\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \left(\frac{1}{3\sqrt{10}}, \frac{1}{\sqrt{10}}\right), \dots\right\} \cup \{(0,0)\}$$

and

$$B = \left\{\left(\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right), \left(\frac{3}{4\sqrt{5}}, \frac{1}{2\sqrt{5}}\right), \left(\frac{5}{12\sqrt{10}}, \frac{5}{4\sqrt{10}}\right), \dots\right\} \cup \{(0,0)\}.$$

By plotting the points of A and B in each of the above Examples we can draw the graph of $L_{A,B}(t)$ (in fact the graph of $L_{A,B}(t)$ in the above

Examples resembles that of a step function) and finally observe that $L_{A,B}(t)$ is really discontinuous at infinitely (countably) many points.

REMARK 1.6. By considering $\mathbb{R}$ with the usual metric, by $\mathbb{R}_+$ we denote the set of nonnegative real numbers and let $A_n = [0, n] \subset \mathbb{R}_+$, where the n are natural. Clearly the A_n exhaust $\mathbb{R}_+$. But for the metric h we always have $h(A_n, \mathbb{R}_+) = 1$. Loosely speaking A_{n+1} is "closer" to $\mathbb{R}_+$ than A_n but the metric h fails to illustrate this property. Hence we are led to the following:

DEFINITION 1.7. Let f be a real valued continuous function, such that $f(t) > 0$ for all $t \in [0, \infty)$ and $\int_{[0,\infty)} f(t)dt < \infty$. We define a distance between closed subsets A and B of X by

$$dist_f(A, B) = \int\limits_{[0,\infty)} f(t)h(A \cap K(x_0, t), B \cap K(x_0, t))dt.$$

THEOREM 1.8. *$dist_f$ is metric on $\Im$, the space of all closed subsets of X.*

Proof. Certainly we have $0 \leq dist_f(A, B) < \infty$ as $0 \leq dist_f(A, B) \leq \int_{[0,\infty)} f(t)dt < \infty$. Suppose that $dist_f(A, B) = 0$. Since $f(t) > 0$ for all $t \in [0, \infty)$ it follows that $h(A \cap K(x_0, t), B \cap K(x_0, t)) = 0$ for almost all $t \in [0, \infty)$. This implies that $A \cap K(x_0, t) = B \cap K(x_0, t)$ for almost all $t \in [0, \infty)$, since h is a metric. Hence $A = B$.

Conversely suppose that $A = B$. Then $dist_f(A, B) = dist_f(A, A) = 0$ as $h(A \cap K(x_0, t), A \cap K(x_0, t)) = 0$ for all $t \geq 0$. By recalling that $h(A, B) = h(B, A)$ we obtain $dist_f(A, B) = dist_f(B, A)$.

Finally we recall that $h(A, B) \leq h(A, C) + h(C, B)$ for compact subsets A, B and C in X. As the $A \cap K(x_0, t)$ are compact then by using the properties of integrals we obtain $dist_f(A, B) \leq dist_f(A, C) + dist_f(C, B)$. So the proof is complete. ■

2. Compactness of $(\Im, dist_f)$

THEOREM 2.1. *Let A_n (where $n \in \mathbb{N}$) and A be compact subsets of (X, ρ). Consider the following conditions:*

(i) $\lim_{n\to\infty} d(A_n, A) = 0$,

(ii) $\lim_{n\to\infty} L_{A_n,A}(t) = 0$ *for all except countably (or finitely) many* $t \in [0, \infty)$,

(iii) $\lim_{n\to\infty} dist_f(A_n, A) = 0$.

Then we have (i) ⇒ (ii) ⇒ (iii).

Proof. (i)⇒(ii) Let $\psi(t) = \limsup_{n\to\infty} d(A_n \cap K(x_0, t), A \cap K(x_0, t))$ and $r_0 = inf\{r \geq 0 : A \cap K(x_0, r) \neq \emptyset\}$ and $R_0 = sup\{r \geq 0 : A \cap K(x_0, r)^c \neq \emptyset\}$. Then certainly $\psi(t) = 0$ for all $t < r_0$ or $t > R_0$. In particular for such t,

$\lim_{n\to\infty} d(A_n, A) = 0$ implies $\lim_{n\to\infty} L_{A_n,A}(t) = 0$. Now we show that there are at most countably many $t \in [r_0, R_0]$ such that $\psi(t) > 0$.

Given a natural number k, let $E_k = \{t \in [0,\infty) : \psi(t) > \frac{1}{k}\}$. We want to show that E_k is finite. We assume that it is infinite and finally arrive at a contradiction. We may choose an infinite convergent sequence $t_n \in E_k \subseteq [r_0, R_0]$ such that $\psi(t_n) > \frac{1}{2k} = \delta$ for $n \geq N$.

Since A is compact then we can define $N(A,\delta)$ to be the maximum number of points $x_1, x_2, \ldots, x_m$ in A such that $\rho(x_i, x_j) > \delta$ for $i \neq j$. Now $N(A,\delta)$ is well defined since by compactness of A we can choose a finite number of points in A that are δ-separated .

Suppose that E_k is infinite. Given $m = N(A,\delta)$ we choose $m+2$ elements $t_1 < t_2 < \ldots < t_{m+2}$ from E_k and define $r_1 = \frac{(t_2-t_1)\wedge\frac{1}{k}}{2}$, $r_l = \frac{(t_{l+1}-t_l)\wedge\frac{1}{k}\wedge r_{l-1}}{2}$ for $l = 1, 2, \ldots, m+1$.

Let n_1 be so large that $d(A_n, A) < \frac{(t_2-t_1)\wedge\frac{1}{k}}{2}$ for all $n \geq n_1$ and $L_{A_{n_1},A}(t_1) > \frac{1}{k}$.

(1) Either there exists $a \in A \cap K(x_0, t_1)$ such that $K(a, \frac{1}{k}) \cap A_{n_1} \cap K(x_0, t_1) = \emptyset$ and then we set $a_1 = a$ or

(2) there exists $\widetilde{a} \in A_{n_1} \cap K(x_0, t_1)$ such that $K(\widetilde{a}, \frac{1}{k}) \cap A \cap K(x_0, t_1) = \emptyset$, and then we may always find $a \in A \cap K(x_0, t_1 + \frac{t_2-t_1}{2}) \cap K(\widetilde{a}, r_1)$, where $r_1 = \frac{(t_2-t_1)\wedge\frac{1}{k}}{2}$. We again set $a_1 = a$.

Suppose we have chosen $a_1, a_2, \ldots, a_l \in A, n_1 < n_2 < \ldots < n_l$, where

$a_1 \in A \cap K(x_0, \frac{t_1+t_2}{2})$ and $\rho(a_1, A_n) < r_1$ for all $n \geq n_1$
$a_2 \in A \cap K(x_0, \frac{t_2+t_3}{2})$ and $\rho(a_2, A_n) < r_2$ for all $n \geq n_2$

$\vdots$

$a_l \in A \cap K(x_0, \frac{t_l+t_{l+1}}{2})$ and $\rho(a_l, A_n) < r_l$ for all $n \geq n_l$ and $\rho(a_j, a_i) > \frac{1}{2k}$ for all $1 \leq i \neq j \leq l$.

We find $n_{l+1} > n_l$ so large that $d(A_n, A) < r_{l+1}$ for all $n \geq n_{l+1}$ and $L_{A_{n_{l+1}},A}(t_{l+1}) > \frac{1}{k}$.

As in (1), if there exists $a \in A \cap K(x_0, t_{l+1})$ such that $K(a, \frac{1}{k}) \cap A_{n_{l+1}} \cap K(x_0, t_{l+1}) = \emptyset$ we set $a_{l+1} = a$. We notice that $\rho(a_j, a_{l+1}) > \frac{1}{2k}$ for $j = 1, 2, \ldots, l$.

On the other hand, if there exists $\widetilde{a} \in A_{n_{l+1}} \cap K(x_0, t_{l+1})$ such that $K(\widetilde{a}, \frac{1}{k}) \cap A \cap K(x_0, t_{l+1}) = \emptyset$ then there exists

$$a \in A \cap K(x_0, t_{l+1} + \frac{t_{l+2} - t_{l+1}}{2}) \cap K(\widetilde{a}, r_{l+1}).$$

We set $a_{l+1} = a$. Clearly $a \in A \cap K(x_0, \frac{t_{l+1}+t_{l+2}}{2})$, $\rho(a_{l+1}, A_n) < r_{l+1}$ for all $n \geq n_{l+1}$ and $\rho(a_{l+1}, a_j) > \frac{1}{2k}$ for all $j = 1, 2, \ldots, l$. It follows by induction that there exists a sequence $a_1, a_2, \ldots, a_{m+1} \in A$ such that $\rho(a_i, a_j) > \frac{1}{2k}$ for all $i \neq j$, which contradicts the maximality of m. Therefore E_k is finite.

Now $E = \{t \in [0, \infty) : \psi(t) > 0\} = \bigcup_{k=1}^{\infty} E_k$ is at most countable. So the Lebesgue measure of E is zero. Hence (i)⇒(ii).

(ii)⇒(iii): Let $\lim_{n\to\infty} L_{A_n,A}(t) = 0$ for all except countably or finitely many $t \in [0, \infty)$. Clearly this implies

$$\lim_{n\to\infty} \frac{L_{A_n,A}(t)}{1 + L_{A_n,A}(t)} = \lim_{n\to\infty} h(A_n \cap K(x_0, t), A \cap K(x_0, t)) = 0,$$

except for countably or finitely many $t \in [0, \infty)$. Hence

$$\lim_{n\to\infty} dist_f(A_n, A) = 0$$

as

$$\lim_{n\to\infty} \int_{[0,\infty)} f(t) h(A_n \cap K(x_0, t), A \cap K(x_0, t)) dt = 0,$$

by the Lebesgue dominated convergence theorem. Hence (ii)⇒(iii). ∎

COROLLARY 2.2. *Let A_n (where n is natural) and A be compact subsets of (X, ρ). Suppose that there exists a number $R > 0$ such that A_n and A are included in $K(x_0, R)$.*

Then

$$\lim_{n\to\infty} dist_f(A_n, A) = 0$$

implies that

$$\lim_{n\to\infty} d(A_n, A) = 0.$$

Proof. Let A_n and A be as stated above and assume that there exists $R > 0$ such that A_n, $A \subseteq K(x_0, R)$ for all n. Suppose that $\lim_{n\to\infty} dist_f(A_n, A) = 0$.

Then

$$dist_f(A_n, A) = \int_{[0,\infty)} f(t) h(A_n \cap K(x_0, t), A \cap K(x_0, t)) dt \geq \frac{d(A_n, A)}{1 + d(A_n, A)} \int_{[R,\infty)} f(t) dt.$$

Thus

$$\lim_{n\to\infty} \frac{d(A_n, A)}{1 + d(A_n, A)} \int_{[R,\infty)} f(t) dt \leq 0,$$

and since $f(t) > 0$ we must have $\lim_{n\to\infty} d(A_n, A) = 0$. Hence the required result. ∎

REMARK 2.3. From the above results we see that (i), (ii) and (iii) in Theorem 2.1 are equivalent once A_n and A are restricted to bounded regions of (X, ρ).

EXAMPLE 3. Consider $X = \mathbb{R}$ with the Euclidean metric, and let $f(t) = e^{-t}$, $A = \{0\}$, $A_n = \{0, n\}$, and $x_0 = 0$. Clearly $d(A_n, A) = n$, and

$$\lim_{n\to\infty} d(A_n, A) = \infty > 0.$$

But

$$\begin{aligned} dist_f(A_n, A) &= \int_{[0,\infty)} e^{-t} h(\{0,n\} \cap [-t,t], \{0\} \cap [-t,t])\, dt \\ &= \int_{[0,n]} e^{-t} h(\{0\},\{0\}) dt + \int_{[n,\infty)} e^{-t} h(\{0,n\},\{0\})\, dt \\ &= \frac{n}{1+n} \int_{[n,\infty)} e^{-t} dt = \frac{n}{e^n(1+n)}. \end{aligned}$$

In particular,

$$\lim_{n\to\infty} dist_f(A_n, A) = \lim_{n\to\infty} \frac{n}{e^n(1+n)} = 0.$$

Therefore (iii)⇒(i) in Theorem 2.1 does not hold in general.

It is well known (see [2] and [4]) that closed and bounded subsets are not necessarily compact in an arbitrary metric space and that separable Hilbert spaces are isometrically isomorphic (see [4]). In particular, closed finite balls of l^2 are not compact either. This will be used in the next example.

EXAMPLE 4. Let $X = l^2$ be a Hilbert space with the norm $\|.\|_2$, α_k be a dense sequence in the interval $[1, 2]$, and let e_k denote standard base vectors in X. We define

$$A = \{\alpha_k e_k : \text{for } \ k = 1, 2, \ldots\},$$

and let x_0 be the zero vector. Clearly A is a closed and bounded subset of X (as A is discrete). We define $A_n = \{(\alpha_k + \frac{1}{n})e_k : k = 1, 2, \ldots\}$. Obviously A_n is also closed and bounded in X and $d(A_n, A) \le \|\frac{1}{n}e_k\|_2 = \frac{1}{n}$. So that $\lim_{n\to\infty} d(A_n, A) = 0$. On the other hand if $t \in (1, 2)$, and since $\{\alpha_k\}$ is dense in $[1, 2]$, then for every n there exists k such that $\alpha_k e_k < t < (\alpha_k + \frac{1}{n})e_k$. In particular $\alpha_k e_k \in A \cap K(x_0, t)$ but $(\alpha_k + \frac{1}{n})e_k \notin A_n \cap K(x_0, t)$. Now $\inf_{y\in A_n\cap K(x_0,t)} \|\alpha_k e_k - y\|_2 \ge \sqrt{2}$. This means that $L_{A_n,A}(t) \ge \sqrt{2}$ for every $t \in (1, 2)$ and finally $\psi(t) \ge \sqrt{2}$ on the whole interval $t \in (1, 2)$. In particular, Theorem 2.1 may not be extended to noncompact A_n, A and $K(x_0, t)$.

THEOREM 2.4. *Let (X, ρ) be a metric space such that finite closed balls are compact. Then the space of all closed subsets including the empty set is compact for the metric $dist_f$.*

Proof. Let $\{A_n\}$ be a sequence of closed subsets in X. We want to find a convergent subsequence$\{A_{n_l}\}$ of $\{A_n\}$ in $dist_f$. So let $A_n^m = A_n \cap K(x_0, m) \subseteq$

$K(x_0, m)$. By compactness of $K(x_0, m)$ we may find a convergent subsequence of A_n^m in $(K(x_0, m), d)$ for each m. Using the diagonal method (Cantor method) we may choose a subsequence A_{n_s} such that for every $m \in [1, \infty)$ the sequence $A_{n_s}^m \to A^m$ in the Hausdorff metric d as $s \to \infty$. Since $A_{n_s}^m = A_{n_s}^{m'}$ on $K(x_0, m \wedge m')$ thus $A^m = A^{m'}$ on $K(x_0, m \wedge m')$. Let $A = \cup_{m=1}^{\infty} A^m$. Clearly the set A is closed in X. Then $L_{A_{n_s}, A}(t) = L_{A_{n_s}^m, A^m}(t) \to 0$ as $s \to \infty$ except at most countably many $t \in [0, m)$. By letting $m \to \infty$ we obtain $\lim_{s \to \infty} L_{A_{n_s}, A}(t) = 0$ except at most countably many $t \in [0, \infty)$. Hence by Theorem 2.1, $\lim_{s \to \infty} dist_f(A_{n_s}, A) = 0$, which completes the proof. ∎

Finally let (X, ρ) be a metric space such that finite closed balls are compact. We define the function

$$\hat{d}_f : \hat{X} \times \hat{X} \to [0, \infty)$$

by

$$\hat{d}_f(x, y) = dist_f(\{x\}, \{y\})$$

for all $x, y \in \hat{X}$. By identifying the point ∞ in $\hat{X}$ with the empty subset of X, the function $\hat{d}_f$ defines a metric on $\hat{X}$. Letting $f(t) = e^{-t}$, we get

$$\hat{d}_{e^{-t}}(x, y) = e^{-min\{\rho(x_0, x), \rho(x_0, y)\}} - \frac{e^{-max\{\rho(x_0, x), \rho(x_0, y)\}}}{1 + \rho(x, y)}.$$

In particular, when $X = \mathbb{R}$ equipped with the usual metric, and $x_0 = 0$, then

$$\hat{d}_{e^{-t}}(x, y) = e^{-(|x| \bigwedge |y|)} - \frac{e^{-(|x| \bigvee |y|)}}{1 + |x - y|}.$$

For some results on the extensions of X (one-point compactification) and the topology on $\Im$ see [1], [7] and [8].

Acknowledgements. The Author would like to express deep gratitude to Prof. W. K. Bartoszek and Prof. I. W. Alderton whose guidance and support were crucial for the successful completion of this project. The Author also wish to thank the referee of the paper for the comments that helped to improve the paper.

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Received September 22, 2000; revised version September 19, 2001.