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LATTICES RESPECTING CONVEX DECOMPOSITIONS, I

Abstract. - We prove the following results: Let (L_1, L_2) be a convex decomposition of a lattice L. If L_1 and L_2 are 0-modular, then L is 0-modular. If L_1 is 0-distributive (or L_1 is distributive with 0) and L_2 is distributive (or L_2 is 0-distributive), then L is 0-distributive.

1. Introduction

In this paper we explore the idea of a lattice convexly decomposed into two sublattices. The construction is based on a general approach [1] partly explained in [2]. In [3] we used it to describe an algorithm how to decompose a finite distributive lattice into its Boolean blocks by convex decompositions. To avoid confusion, we finally remark that our convex decompositions are more general than the well known construction of Hall and Dilworth [6, 7]. For some other aspects concerning amalgams of lattices or their pasting see also [4].

The ordered couple (L_1, L_2) is said to be a *convex decomposition* of a lattice L (written $L = \overrightarrow{cd}(L_1, L_2)$) if L_1 and L_2 are sublattices of L such that $L_1 \neq L \neq L_2$, $L_1 \cap L_2 \neq \emptyset$, $L_1 \cup L_2 = L$, $L_1 = (L_1 \cap L_2]$ and $L_2 = [L_1 \cap L_2)$. Here $(L_1 \cap L_2] = \{a \in L; \exists b \in L_1 \cap L_2 \ a \leq b\}$ and the set $[L_1 \cap L_2)$ is defined dually.

Note that $L_1 \cap L_2$ is a convex subset of L.

Figure 1 illustrates a convex decomposition of a seven-element lattice L_7.

We now recapitulate some results on convex decompositions. The reader may find the corresponding details in [3].

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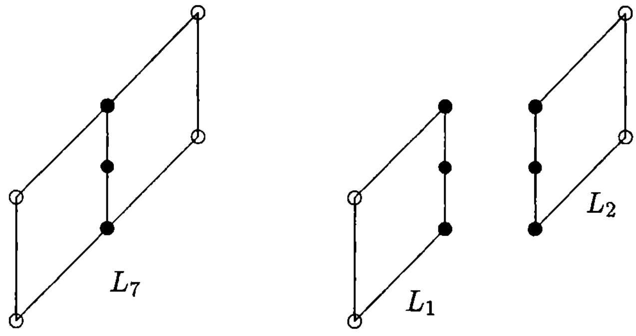

Figure 1

For the sake of brevity, let us first introduce the following notation: We will write $x \in \bullet$ or $\bullet \ni x$ for $x \in L_1 \cap L_2$ and $L_1 \cap L_2 \ni x$, respectively.

If $\vee_i$, $\wedge_i$ denote the lattice operations in the lattice L_i $(i = 1, 2)$ and if $\vee$, $\wedge$ are the operations in L, then

(1*) for any $a, b \in L_i$

$$a \wedge b = a \wedge_i b \ \& \ a \vee b = a \vee_i b;$$

(2*) for any $a \in L_1$ and $b \in L_2$

$$a \wedge b = a \wedge_1 (a^* \wedge_2 b) \ \& \ a \vee b = (a \vee_1 b_+) \vee_2 b$$

where b_+ and a^* are any elements of L such that $\bullet \ni b_+ \leq b$ and $a \leq a^* \in \bullet$;

(3*) for any $a \in L_1$ and every $b \in \bullet$, $a \vee b \in \bullet$; for any $c \in L_2$ and every $d \in \bullet$, $c \wedge d \in \bullet$.

Note that if (L_1, L_2) is a convex decomposition of L, then L possesses a smallest element if and only if L_1 possesses a smallest element and in this case the both elements coincide.

2. 0-modular lattices

Recall that a lattice L with 0 is called *0-modular* [13], [5] if, for all $a, b, c \in L$, $a \wedge b = 0$ and $c \leq b$ imply $c = (a \vee c) \wedge b$. (See also [10], [11], [8] and [12].)

THEOREM 2.1. *Let* $L = \overrightarrow{cd}(L_1, L_2)$ *and let* L_1 *and* L_2 *be 0-modular lattices. Then* L *is 0-modular.*

Proof. Suppose that $a, b, c \in L$, $a \wedge b = 0$ and $c \leq b$. Let ω denote the zero element of L_2. If $h \in L_1 \cap L_2$, then $\omega \leq h$. From $\omega \in L_2 = [L_1 \cap L_2)$ we can see that there exists d such that $\bullet \ni d \leq \omega \leq h \in \bullet$. Since $L_1 \cap L_2$ is convex, $\omega \in \bullet$.

We will consider several cases:

Case I: $a \in L_1$ *and* $b \in L_1$. Then $c \in L_1$ and, by the 0-modularity od L_1, $b \wedge (a \vee c) = b \wedge_1 (a \vee_1 c) = c$.

Case II: $a \in L_1$, $b \in L_2 \setminus L_1$ *and* $c \in L_2$. By (3*), we have $\alpha := a \vee \omega = = a \vee_1 \omega \in \bullet$. To show that

$$\alpha \wedge b = \omega, \tag{2.1}$$

we proceed as follows. By applying (2*), we see that $a \wedge_1 (\alpha \wedge_2 b) = = a \wedge b = 0$. Let $a' := a$, $c' := \omega$ and $b' := \alpha \wedge_2 b = \alpha \wedge b$. Since $c' \leq b'$ and $a' \wedge_1 b' = 0$ it follows from the 0-modularity of L_1 that

$$\omega = (a \vee_1 \omega) \wedge_1 (\alpha \wedge_2 b) = \alpha \wedge_1 (\alpha \wedge_2 b) = \alpha \wedge b$$

according to (2*).

Let $a'' := \alpha$, $b'' := b$ and $c'' := c$. Now $c'' \leq b''$ and $a'' \wedge_2 b'' = \omega$, by (2.1). Thus, by the 0-modularity of L_2, $c = b \wedge_2 (\alpha \vee_2 c)$. Since $\bullet \ni \omega \leq c$, it follows from (2*) that $\alpha \vee_2 c = (a \vee_1 \omega) \vee_2 c = a \vee c$ and we finally get $c = b \wedge (a \vee c)$.

Case III: $a \in L_1$, $b \in L_2 \setminus L_1$ *and* $c \in L_1 \setminus L_2$. Let $\gamma := c \vee_1 \omega$. By (3*) and (1*) it is easy to see that $\bullet \ni \gamma = c \vee \omega \leq b$. Obviously,

$$\omega \leq b \wedge (a \vee \omega). \tag{2.2}$$

From $a \leq a \vee_1 \omega \in \bullet$ and (2*) we obtain

$$a \wedge_1 [b \wedge_2 (a \vee_1 \omega)] = a \wedge b = 0. \tag{2.3}$$

Let $\overline{a} := a$, $\overline{b} := b \wedge (a \vee \omega)$ and $\overline{c} := \omega$. Using (2.2), (2.3) and the 0-modularity of L_1 applied to the triplet $\overline{a}, \overline{b}, \overline{c}$, we have

$$\overline{c} = (\overline{a} \vee_1 \overline{c}) \wedge_1 \overline{b} = (a \vee_1 \omega) \wedge_1 [b \wedge (a \vee \omega)].$$

By (1*) and (3*), $a \vee \omega = a \vee_1 \omega \in \bullet$ and $b \wedge (a \vee \omega) = b \wedge_2 (a \vee_1 \omega) \in \bullet$. From (1*) we now find that

$$\overline{c} = (a \vee_1 \omega) \wedge_2 [b \wedge_2 (a \vee_1 \omega)] = (a \vee_1 \omega) \wedge_2 b$$

and so

$$\omega = (a \vee_1 \omega) \wedge_2 b. \tag{2.4}$$

Let $\tilde{a} := a \vee_1 \omega$, $\tilde{b} := b$ and $\tilde{c} := \gamma$. Using (2.4) and the 0-modularity of L_2 applied to the triplet $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, we can see that

$$\gamma = \tilde{c} = (\tilde{a} \vee_2 \tilde{c}) \wedge_2 \tilde{b} = [(a \vee_1 \omega) \vee_2 \gamma] \wedge_2 b.$$

On the other hand, from (2*) and $\bullet \ni \omega \leq \gamma$ we get $(a \vee_1 \omega) \vee_2 \gamma = = a \vee \gamma$. Thus

$$\gamma = (a \vee \gamma) \wedge_2 b. \tag{2.5}$$

Since $\gamma \leq b$, it follows that $a \wedge \gamma \leq a \wedge b = 0$ and so $a \wedge \gamma = 0$.

Let $a_0 := a$, $b_0 := \gamma$ and $c_0 = c$. By the 0-modularity of L_1 applied to the triplet a_0, b_0, c_0,

$$c = c_0 = (a_0 \vee_1 c_0) \wedge_1 b_0 = (a \vee_1 c) \wedge_1 \gamma.$$

From (2.5), $a \vee c \leq a \vee \gamma \in \bullet$, (1*) and (2*), it follows that

$$\begin{aligned} c &= (a \vee_1 c) \wedge_1 [(a \vee \gamma) \wedge_2 b] = (a \vee_1 c) \wedge b = \\ &= (a \vee c) \wedge b. \end{aligned}$$

Case IV: $a \in L_2 \setminus L_1$ *and* $b \in L_1$. Then $c \in L_1$. Let $\gamma := c \vee_1 \omega$ and $\beta := b \vee_1 \omega$. Now, $b \wedge_1 \omega = b \wedge \omega \leq b \wedge a = 0$.

Let $\hat{a} := \omega$, $\hat{b} := b$ and $\hat{c} := c$. Applying the 0-modularity of L_1 to the triplet $\hat{a}, \hat{b}, \hat{c}$, we obtain

$$c = \hat{c} = (\hat{a} \vee_1 \hat{c}) \wedge_1 \hat{b} = (\omega \vee_1 c) \wedge_1 b = b \wedge_1 \gamma. \tag{2.6}$$

By $b \leq \beta \in \bullet$ and (2*), $0 = b \wedge a = b \wedge_1 (\beta \wedge_2 a)$.

Let $a_1 := b$, $b_1 := a \wedge_2 \beta$ and $c_1 := \omega$. Applying the 0-modularity of L_1 to the triplet a_1, b_1, c_1, it is clear that

$$\omega = c_1 = (a_1 \vee_1 c_1) \wedge_1 b_1 = (b \vee_1 \omega) \wedge_1 (a \wedge_2 \beta) = a \wedge \beta.$$

If we set $a_2 := a$, $b_2 := \beta$ and $c_2 := \gamma$, then by virtue of the 0-modularity of L_2 applied to the triplet a_2, b_2, c_2 we get

$$\gamma = c_2 = (a_2 \vee_2 c_2) \wedge_2 b_2 = (a \vee_2 \gamma) \wedge_2 \beta.$$

From (2.6), $b \leq \beta \in \bullet$ and (2*) we then obtain

$$c = b \wedge_1 \gamma = b \wedge_1 [\beta \wedge_2 (a \vee_2 \gamma)] = b \wedge (a \vee_2 \gamma). \tag{2.7}$$

Since $\bullet \ni \omega \leq a$, it follows from (2*) that $a \vee_2 \gamma = a \vee_2 (\omega \vee_1 c) = a \vee c$. According to (2.7), we have $c = b \wedge (a \vee c)$.

Case V: $a \in L_2 \setminus L_1$, $b \in L_2 \setminus L_1$. Then $0 = a \wedge b \in L_2$, which is a contradiction to our assumption that $L_2 \neq L$. ∎

3. 0-distributive lattices

A lattice L with 0 is called *0-distributive* [13] if it satisfies the implication

$$[(a \wedge b = 0 \ \& \ a \wedge c = 0] \Rightarrow \ a \wedge (b \vee c) = 0$$

for every a, b, c of L. Y. Rav [9] calls such a lattice semiprime. See also [10].

The seven–element lattice $L_7 = \overrightarrow{cd}(L_1, L_2)$ from Figure 1 is not 0-distributive but its sublattices L_1 and L_2 are 0-distributive. In general, we can salvage the 0-distributivity of L with a strenghthening of the assumptions on L_i, for example at the expense of requiring L_1 or L_2 to be distributive.

THEOREM 3.1. *Let $L = \overrightarrow{cd}(L_1, L_2)$, let L_1 be 0-distributive and let L_2 be distributive. Then L is 0-distributive.*

Proof. Suppose that $a, b, c \in L$, $a \wedge b = 0$ and $a \wedge c = 0$. Since $L_2 \neq L$, the situation where a and c belong to L_2 is not possible. For the same reason we can exclude the case where $a \in L_2$ and $b \in L_2$. Moreover, if a, b and c belong to L_1, then the assertion of the theorem is true by (1*).

By symmetry in b and c, it suffices to consider the following three cases.

Case I: $a \in L_1$, $b \in L_1$ *and* $c \in L_2$. Let $W := a \wedge (b \vee c)$. We intend to show that $W = 0$.

It is obvious by (2*) that

$$W = a \wedge_1 [a^* \wedge_2 (b \vee c)] = a \wedge_1 \{a^* \wedge_2 [(b \vee_1 c_+) \vee_2 c]\}$$

where $a \leq a^* \in \bullet$ and $\bullet \ni c_+ \leq c$. Let $V := a^* \wedge_2 [(b \vee_1 c_+) \vee_2 c]$. By virtue of (1*), (3*) and the distributivity of L_2 we have

$$V = [a^* \wedge_2 (b \vee_1 c_+)] \vee_2 (a^* \wedge_2 c) = [a^* \wedge_2 (b \vee_1 c_+)] \vee_1 (a^* \wedge_2 c).$$

Let $H := a^* \wedge_2 (b \vee_1 c_+)$ and $K := a^* \wedge_2 c$. Then from (1*) and (3*) it follows that $H = a^* \wedge_1 (b \vee_1 c_+)$ and it is simple to verify that $H, K \in L_1$ and that $W = a \wedge_1 (H \vee_1 K)$. At the same time

$$a \wedge_1 H = a \wedge_1 [a^* \wedge_1 (b \vee_1 c_+)] = (a \wedge_1 a^*) \wedge_1 (b \vee_1 c_+) = a \wedge_1 (b \vee_1 c_+).$$

By assumption and (1*), $a \wedge_1 b = a \wedge b = 0$. Similarly, $a \wedge_1 c_+ = a \wedge c_+ \leq a \wedge c = 0$. Since L_1 is 0-distributive, we have $a \wedge_1 H = 0$. By (2*), $a \wedge_1 K = a \wedge_1 (a^* \wedge_2 c) = a \wedge c = 0$. From the 0-distributivity of L_1 we see that $W = a \wedge_1 V = a \wedge_1 (H \vee_1 K) = 0$.

Case II: $a \in L_1$, $b \in L_2$ *and* $c \in L_2$. Let $Q := a \wedge (b \vee c)$ and $P := a^* \wedge_2 (b \vee_2 c)$ where $a \leq a^* \in \bullet$. Then $Q = a \wedge_1 P$. Since L_2 is distributive, from (1*) and (3*) we obtain

$$P = (a^* \wedge_2 b) \vee_2 (a^* \wedge_2 c) = (a^* \wedge_2 b) \vee_1 (a^* \wedge_2 c)$$

and so $Q = a \wedge_1 [(a^* \wedge_2 b) \vee_1 (a^* \wedge_2 c)]$. From (2*) it is seen that $a \wedge_1 (a^* \wedge_2 b) = a \wedge b = 0$ and that $a \wedge_1 (a^* \wedge_2 c) = a \wedge c = 0$. Since L_1 is 0-distributive, we have $Q = 0$.

Case III: $a \in L_2$, $b \in L_1$ *and* $c \in L_1$. Let $B := a \wedge (b \vee c)$. By (2*), $B = (b \vee_1 c) \wedge_1 [(b \vee_1 c)^* \wedge_2 a]$ where $b \vee_1 c \leq (b \vee_1 c)^* \in \bullet$. Put $A := (b \vee_1 c)^* \wedge_2 a$. By (3*), $A \in \bullet$ and, clearly, $B = (b \vee_1 c) \wedge_1 A$. Here $b \wedge_1 A = b \wedge_1 [(b \vee_1 c)^* \wedge_2 a]$. But $b \leq (b \vee c)^*$. Therefore, by (2*), $b \wedge_1 A = b \wedge a = 0$. Similarly, $c \wedge_1 A = c \wedge a = 0$. It then follows from the 0-distributivity of L_1 that $B = 0$. ∎

We end this section with a symmetric result.

THEOREM 3.2. *Let* $L = \overrightarrow{cd}(L_1, L_2)$, *let* L_1 *be a distributive lattice with 0 and let* L_2 *be 0-distributive. Then* L *is 0-distributive.*

Proof. Let ω denote the zero element of L_2. Suppose that $a, b, c \in L$, $a \wedge b = 0$ and $a \wedge c = 0$.

In a similar manner as in the proof of Theorem 3.1 we can see that it is sufficient to consider the following three cases.

Case I: $a \in L_1$, $b \in L_1$ *and* $c \in L_2$. Let $V := a \wedge (b \vee c)$. By (2*),

$$V = a \wedge_1 [a^* \wedge_2 (b \vee c)] = a \wedge_1 \{a^* \wedge_2 [(b \vee_1 c_+) \vee_2 c]\}.$$

Now $\bullet \ni \omega \leq c$ and, referring to (3*), we see that $a \leq a \vee_1 \omega \in \bullet$. Thus we can choose $a^* = a \vee_1 \omega$ which gives

$$V = a \wedge_1 \{(a \vee_1 \omega) \wedge_2 [(b \vee_1 \omega) \vee_2 c]\}.$$

From (3*) and (1*),

$$(a \vee_1 \omega) \wedge_2 (b \vee_1 \omega) = (a \vee_1 \omega) \wedge_1 (b \vee_1 \omega).$$

Since L_1 is distributive, it follows that

$$(a \vee_1 \omega) \wedge_2 (b \vee_1 \omega) = \omega. \tag{3.1}$$

Notice that

$$0 \leq \omega \leq (a \vee_1 \omega) \wedge_2 c \leq a \vee_1 \omega.$$

Moreover, by (2*),

$$a \wedge_1 [(a \vee_1 \omega) \wedge_2 c] = a \wedge c = 0.$$

Let $A := \omega$, $B := a$ and $C := (a \vee_1 \omega) \wedge_2 c$. Now $A \leq C$, $0 = B \wedge_1 C = B \wedge_1 A$ and $a \vee_1 \omega = B \vee_1 A = C \vee_1 A$. By the distributivity of L_1 we therefore get $C = A$, i.e.,

$$(a \vee_1 \omega) \wedge_2 c = \omega. \tag{3.2}$$

Since L_2 is 0-distributive, (3.1) and (3.2) yield

$$(a \vee_1 \omega) \wedge_2 [(b \vee_1 \omega) \vee_2 c] = \omega.$$

Consequently, $V = a \wedge_1 \omega \leq a \wedge c = 0$.

Case II: $a \in L_1$, $b \in L_2$ *and* $c \in L_2$. Let $W := a \wedge (b \vee c)$. By (2*), $W = a \wedge_1 [a^* \wedge_2 (b \vee_2 c)]$. In view of (3*) it is clear that we can choose

$a^* = a \vee_1 \omega$. Now, by (2*),

$$a \wedge_1 [(a \vee_1 \omega) \wedge_2 b] = a \wedge b = 0.$$

Since

$$0 \leq \omega \leq (a \vee_1 \omega) \wedge_2 b \leq a \vee_1 \omega,$$

the distributivity of L_1 implies that $a^* \wedge_2 b = (a \vee_1 \omega) \wedge_2 b = \omega$. Replacing b by c and vice–versa, we similarly get $a^* \wedge_2 c = \omega$. Since L_2 is 0-distributive, $a^* \wedge_2 (b \vee_2 c) = \omega$. It follows that $W = a \wedge_1 \omega \leq a \wedge c = 0$ and so $W = 0$.

Case III: $a \in L_2$, $b \in L_1$ *and* $c \in L_1$. Let $Z := a \wedge (b \vee c)$. By (2*) and (3*),

$$Z = (b \vee_1 c) \wedge_1 [(b \vee_1 c \vee_1 \omega) \wedge_2 a].$$

Since L_1 is distributive, it follows from (2*) that

$$\begin{aligned} Z &= \{b \wedge_1 [(b \vee_1 c \vee_1 \omega) \wedge_2 a]\} \vee_1 \{c \wedge_1 [(b \vee_1 c \vee_1 \omega) \wedge_2 a]\} \\ &= (b \wedge a) \vee (c \wedge a) = 0. \end{aligned}$$

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