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THE DELTA FUNCTION AND THE COMPOSITION OF DISTRIBUTIONS

Abstract. Let F be a distribution and let f be a locally summable function. The distribution $F(f)$ is defined as the neutrix limit of the sequence $\{F_n(f)\}$, where $F_n(x) = F(x) * \delta_n(x)$ and $\{\delta_n(x)\}$ is a certain sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. The distribution $\delta^{(s)}(x_+^\lambda)$ is evaluated for $\lambda > 0$ and $s = 0, 1, 2, \ldots$.

In the following we let N be the neutrix, see [3], having domain N' the positive integers and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^\lambda \ln^{r-1} n,\ \ln^r n: \qquad \lambda > 0, \quad r = 1, 2, \ldots$$

and all functions which converge to zero in the usual sense as n tends to infinity.

It follows that the neutrix limit of a function is unique if it exists and if the usual limit of a function exists, it exists as a neutrix limit and the two limits are equal.

To see how neutrices can be used to define distributions, see [6].

Now let $\rho(x)$ be an infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

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Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t)\rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

The following definition was given in [4].

DEFINITION 1. Let F be a distribution and let f be a locally summable function. We say that the distribution $F(f(x))$ exists and is equal to h on the open interval (a, b) if

$$\operatorname*{N-lim}_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all test functions φ with compact support contained in (a, b).

The following theorems were proved in [4] and [5] respectively:

THEOREM 1. *The distributions* $(x_-^\mu)_-^\lambda$ *and* $(x_+^\mu)_-^\lambda$ *exists and*

$$(x_-^\mu)_-^\lambda = (x_+^\mu)_-^\lambda = 0$$

for $\mu > 0$ *and* $\lambda\mu \neq -1, -2, \ldots$ *and*

$$(x_-^\mu)_-^\lambda = (-1)^{\lambda\mu}(x_+^\mu)_-^\lambda = \frac{\pi\operatorname{cosec}(\pi\lambda)}{2\mu(-\lambda\mu-1)!}\delta^{(-\lambda\mu-1)}(x)$$

for $\mu > 0$, $\lambda \neq -1, -2, \ldots$ *and* $\lambda\mu = -1, -2, \ldots$.

THEOREM 2. *The distribution* $(x_+^r)_-^{-s}$ *exists and*

$$(x_+^r)_-^{-s} = \frac{(-1)^{rs+s}c(\rho)}{r(rs-1)!}\delta^{(rs-1)}(x)$$

for $r, s = 1, 2, \ldots,$ *where*

$$c(\rho) = \int_0^1 \ln t\, \rho(t)\, dt.$$

Note that in the Theorem 2, the distribution x_-^{-s} is defined by

$$x_-^{-s} = -\frac{(\ln x_-)^{(s)}}{(s-1)!}$$

for $s = 1, 2, \ldots$ and not as in Gel'fand and Shilov [7].

We need the following easily proved lemma:

LEMMA.

$$\int_0^1 x^s \rho^{(s)}(x)\,dx = \tfrac{1}{2}(-1)^s s!.$$

We now prove the following theorem:

THEOREM 3. *The distribution* $\delta^{(s)}(x_+^\lambda)$ *exists and*

$$\delta^{(s)}(x_+^\lambda) = 0 \tag{1}$$

for $\lambda > 0$, $s = 0, 1, 2, \ldots$ *and* $(s+1)\lambda \neq 1, 2, \ldots$ *and*

$$\delta^{(s)}(x_+^\lambda) = \frac{(-1)^{(s+1)(\lambda+1)} s!}{2\lambda[(s+1)\lambda - 1]!}\delta^{((s+1)\lambda-1)}(x) \tag{2}$$

for $s = 0, 1, 2, \ldots$ *and* $(s+1)\lambda = 1, 2, \ldots$.

Proof. We have to evaluate

$$\operatorname*{N-lim}_{n\to\infty}\langle \delta_n^{(s)}(x_+^\lambda), \varphi(x)\rangle, \tag{3}$$

where $\varphi(x)$ is an arbitrary function in $\mathcal{D}$ with support contained in the interval $[a, b]$. For convenience, we may assume that $a < 0 < b$. By Taylor's Theorem we have

$$\varphi(x) = \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{k!}x^k + \frac{x^r}{r!}\varphi^{(r)}(\xi x),$$

where $0 < \xi < 1$ and r is an integer chosen so that $r > (s+1)\lambda$. In order to evaluate (3), we have to evaluate

$$\operatorname*{N-lim}_{n\to\infty}\langle \delta_n^{(s)}(x_+^\lambda), \varphi(x)\rangle = \operatorname*{N-lim}_{n\to\infty}\sum_{k=0}^{r-1}\frac{\varphi^{(k)}(0)}{k!}\int_a^b x^k \delta_n^{(s)}(x_+^\lambda)\,dx +$$

$$+ \operatorname*{N-lim}_{n\to\infty}\frac{1}{r!}\int_a^b x^r \delta_n^{(s)}(x_+^\lambda)\varphi^{(r)}(\xi x)\,dx. \tag{4}$$

Making the substitution $nx^\lambda = u$, we have for $n^{-1/\lambda} < b$

$$\int_a^b x^k \delta_n^{(s)}(x_+^\lambda)\,dx = \int_0^{n^{-1/\lambda}} x^k \delta_n^{(s)}(x_+^\lambda)\,dx + \int_a^0 x^k \delta_n^{(s)}(x_+^\lambda)\,dx$$

$$= \frac{n^{s-(k+1)/\lambda+1}}{\lambda}\int_0^1 u^{(k+1)/\lambda-1}\rho^{(s)}(u)\,du + n^{s+1}\rho^{(s)}(0)\int_a^0 x^k\,dx,$$

where $n^{s-(k+1)/\lambda+1}$ is a negligible function if $k \neq (s+1)\lambda - 1$ and it follows that

$$\operatorname*{N-lim}_{n\to\infty}\int_a^b x^k \delta_n^{(s)}(x_+^\lambda)\,dx = \begin{cases} \tfrac{1}{2}(-1)^s s!/\lambda, & k = (s+1)\lambda - 1, \\ 0, & k \neq (s+1)\lambda - 1, \end{cases} \tag{5}$$

on using the lemma.

Next, we have

$$\int_0^b |x^r \delta_n^{(s)}(x_+^\lambda)|\,dx = \int_0^{n^{-1/\lambda}} |x^r \delta_n^{(s)}(x_+^\lambda)|\,dx$$
$$= \frac{n^{s-(r+1)/\lambda+1}}{\lambda} \int_0^1 |u^{(k+1)/\lambda-1}\rho^{(s)}(u)|\,du$$
$$= O(n^{s-(r+1)/\lambda+1})$$

and it follows that

$$\lim_{n\to\infty} \int_0^b x^r \delta_n^{(s)}(x_+^\lambda)\varphi^{(r)}(\xi x)\,dx = 0. \tag{6}$$

Further

$$\int_a^0 x^r \delta_n^{(s)}(x_+^\lambda)\varphi^{(r)}(\xi x)\,dx = n^{s+1}\rho^{(s)}(0)\int_a^0 x^k \varphi^{(r)}(\xi x)\,dx$$

and it follows that

$$\mathrm{N}\!-\!\lim_{n\to\infty} \int_a^0 x^r \delta_n^{(s)}(x_+^\lambda)\varphi^{(r)}(\xi x)\,dx = 0. \tag{7}$$

It now follows from equations (4) to (7) that

$$\mathrm{N}\!-\!\lim_{n\to\infty}\langle \delta_n^{(s)}(x_+^\lambda), \varphi(x)\rangle = 0$$

for $(s+1)\lambda \neq 0, 1, 2, \ldots$, proving equation (1) and

$$\mathrm{N}\!-\!\lim_{n\to\infty}\langle \delta_n^{(s)}(x_+^\lambda), \varphi(x)\rangle = \frac{(-1)^s s!\varphi^{((s+1)\lambda-1)}(0)}{2\lambda[(s+1)\lambda-1]!}$$

for $(s+1)\lambda = 1, 2, \ldots$, proving equation (2).

COROLLARY 3.1 *The distribution* $\delta^{(s)}(x_-^\lambda)$ *exists and*

$$\delta^{(s)}(x_-^\lambda) = 0 \tag{8}$$

for $\lambda > 0$, $s = 0, 1, 2, \ldots$ *and* $(s+1)\lambda \neq 1, 2, \ldots$ *and*

$$\delta^{(s)}(x_-^\lambda) = \frac{(-1)^s s!}{2\lambda[(s+1)\lambda-1]!}\delta^{((s+1)\lambda-1)}(x) \tag{9}$$

for $s = 0, 1, 2, \ldots$ *and* $(s+1)\lambda = 1, 2, \ldots$.

Proof. Equations (8) and (9) follow on replacing x by $-x$ in equations (1) and (2) respectively.

THEOREM 4. *The distribution* $\delta^{(s)}(|x|^\lambda)$ *exists and*

$$\delta^{(s)}(|x|^\lambda) = 0 \tag{10}$$

for $\lambda > 0,\ s = 0, 1, 2, \ldots$ *and* $(s+1)\lambda = 2, 4, \ldots$ *and*

$$\delta^{(s)}(|x|^\lambda) = \frac{(-1)^{(s+1)(\lambda+1)} s!}{\lambda[(s+1)\lambda - 1]!} \delta^{((s+1)\lambda-1)}(x) \tag{11}$$

for $s = 0, 1, 2, \ldots$ *and* $(s+1)\lambda = 1, 3, \ldots$.

Proof. This time, with $n^{-1/\lambda} < \min\{-a, b\}$, we have

$$\begin{aligned}\int_a^b x^k \delta_n^{(s)}(|x|^\lambda)\, dx &= \int_{-n^{-1/\lambda}}^{n^{-1/\lambda}} x^k \delta_n^{(s)}(|x|^\lambda)\, dx \\ &= \frac{n^{s-(k+1)/\lambda+1}}{\lambda} \int_{-1}^{1} u^{(k+1)/\lambda - 1} \rho^{(s)}(|u|)\, du\end{aligned}$$

and it follows that

$$\begin{aligned}&\operatorname*{N-lim}_{n\to\infty} \int_a^b x^k \delta_n^{(s)}(|x|^\lambda)\, dx \\ &\qquad = \begin{cases} (-1)^s s!/\lambda, & \text{even } k = (s+1)\lambda - 1, \\ 0, & \text{odd } k = (s+1)\lambda - 1. \end{cases}\end{aligned} \tag{12}$$

Next, we have

$$\begin{aligned}\int_a^b |x^r \delta_n^{(s)}(|x|^\lambda)|\, dx &= \int_{-n^{-1/\lambda}}^{n^{-1/\lambda}} |x^r \delta_n^{(s)}(|x|^\lambda)|\, dx \\ &= \frac{n^{s-(r+1)/\lambda+1}}{\lambda} \int_{-1}^{1} |u^{(k+1)/\lambda-1} \rho^{(s)}(|u|)|\, du \\ &= O(n^{s-(r+1)/\lambda+1})\end{aligned}$$

and it follows that

$$\lim_{n\to\infty} \int_a^b x^r \delta_n^{(s)}(|x|^\lambda) \varphi^{(r)}(\xi x)\, dx = 0. \tag{13}$$

Equations (10) and (11) now follow as above from equations (12) and (13).

THEOREM 5. *The distribution* $\delta^{(s)}(\operatorname{sgn} x|x|^\lambda)$ *exists and*

$$\delta^{(s)}(\operatorname{sgn} x|x|^\lambda) = 0 \tag{14}$$

for $\lambda > 0$, $s = 0, 1, 2, \dots$ *and* $(s+1)\lambda = 1, 3, \dots$ *and*

$$\delta^{(s)}(\operatorname{sgn} x|x|^\lambda) = \frac{(-1)^{(s+1)(\lambda+1)} s!}{\lambda[(s+1)\lambda - 1]!} \delta^{((s+1)\lambda-1)}(x) \tag{15}$$

for $s = 0, 1, 2, \dots$ *and* $(s+1)\lambda = 2, 4, \dots$.

Proof. Again with $n^{-1/\lambda} < \min\{-a, b\}$, we have

$$\begin{aligned}\int_a^b x^k \delta_n^{(s)}(\operatorname{sgn} x|x|^\lambda)\, dx &= \int_{-n^{-1/\lambda}}^{n^{-1/\lambda}} x^k \delta_n^{(s)}(\operatorname{sgn} x|x|^\lambda)\, dx \\ &= \frac{n^{s-(k+1)/\lambda+1}}{\lambda} \int_{-1}^{1} u^{(k+1)/\lambda - 1} \rho^{(s)}(\operatorname{sgn} u|u|)\, du\end{aligned}$$

and it follows that

$$\operatorname*{N-lim}_{n\to\infty} \int_a^b x^k \delta_n^{(s)}(\operatorname{sgn} x|x|^\lambda)\, dx = \begin{cases} (-1)^s s!/\lambda, & \text{odd } k = (s+1)\lambda - 1, \\ 0, & \text{even } k = (s+1)\lambda - 1. \end{cases} \tag{16}$$

Next, we have

$$\begin{aligned}\int_a^b |x^r \delta_n^{(s)}(\operatorname{sgn} x|x|^\lambda)|\, dx &= \int_{-n^{-1/\lambda}}^{n^{-1/\lambda}} |x^r \delta_n^{(s)}(\operatorname{sgn} x|x|^\lambda)|\, dx \\ &= \frac{n^{s-(r+1)/\lambda+1}}{\lambda} \int_{-1}^{1} |u^{(k+1)/\lambda - 1} \rho^{(s)}(\operatorname{sgn} u|u|)|\, du \\ &= O(n^{s-(r+1)/\lambda+1})\end{aligned}$$

and it follows that

$$\lim_{n\to\infty} \int_a^b x^r \delta_n^{(s)}(\operatorname{sgn} x|x|^\lambda) \varphi^{(r)}(\xi x)\, dx = 0. \tag{17}$$

Equations (14) and (15) now follow as above from equations (16) and (17).

For further related results, see [1], [2] [8] and [9].

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