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# ASYMPTOTICS OF PSEUDODIFFERENTIAL PARABOLIC EQUATIONS

**Abstract.** The paper provides new type examples covered by the general theory of global attractors for abstract parabolic equations presented in the monograph [C-D 1]. Inside the class of sectorial equations of the form

$$\dot{u} + Au = F(u),\ t > 0,\ u(0) = u_0, \tag{1}$$

we cover pseudodifferential parabolic problems

$$u_t = -(-\Delta)^{\alpha} u + f(u),\ \alpha \in (0,1)\,, \tag{2}$$

studied with suitable initial-boundary conditions and also their generalizations to problems with the main part being a finite sum of the fractional powers.

## 1. Part I. Abstract tools

**Introductory notes.** A class of equations with the main part being a fractional power of a uniformly elliptic operator (or a finite sum of such powers) will be studied here within the theory introduced in [FR], [HE], [PA] and developed in our recent monograph [C-D 1]. There is no difference, in general, in studying local solvability of abstract parabolic problems of the form (1) and pseudodifferential equations (2) (or its generalizations), since fractional powers of *sectorial positive operators* are, for $\alpha \in (0,1)$, sectorial and positive. In this discussion it is convenient to use a notion, due to H. Komatsu, of an *operator of the type* $(\omega, M(\theta))$ *with* $\omega < \frac{\pi}{2}$ in a Banach space $X$.

DEFINITION 1. We say that $A$ is of type $(\omega, M(\theta))$, $0 \le \omega < \pi$, if the domain $D(A)$ is dense in $X$, the resolvent set of $-A$ contains the sector

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$|arg\,\lambda| < \pi - \omega$ and the condition $\|\lambda(\lambda + A)^{-1}\| \leq M(\theta)$ holds on each ray $\lambda = re^{i\theta}$, $r \in (0, +\infty)$, $|\theta| < \pi - \omega$.

One may easily show that $A$ is of the type $(\omega, M(\theta))$, $\omega < \frac{\pi}{2}$, if and only if $A$ is a sectorial positive operator in the sense of [HE]. The equivalence of these two notions will be discussed in the Appendix.

An interesting theorem by T. Kato (see [KO, p. 320]) ensures that:

PROPOSITION 1. *If $A$ is of type $(\omega, M(\theta))$ and if $0 < \alpha < \frac{\pi}{\omega}$, then $A^\alpha$ is of type $(\alpha\omega, M_\alpha(\theta))$ with certain positive constant $M_\alpha(\theta)$. Furthermore, the resolvent of $A^\alpha$ is analytic in $\alpha$ and $\lambda$ in the domain $0 < \alpha < \frac{\pi}{\omega}$, $|arg\lambda| < \pi - \alpha\omega$.*

As a consequence, any proper fractional power $A^\alpha$, $\alpha \in (0, 1)$, of a sectorial positive operator $A$ will be sectorial and positive itself. Furthermore,

OBSERVATION 1. *If $A$ is of type $(\omega, M(\theta))$ with $\omega < \frac{\pi}{2}$, then the sum $A + A^\beta$ is a sectorial operator for any $\beta \in (0, 1)$.*

The above observation follows directly from [HE, Theorem 1.4.4] and [HE, Example 6, p. 19] since

$$\forall_{\varepsilon>0}\ \forall_{\beta\in(0,1)}\ \forall_{x\in D(A)}\ \|A^\beta x\|_X \leq \varepsilon\|Ax\|_X + C'\varepsilon^{\frac{-\beta}{1-\beta}}\|x\|_X.$$

Observation 1 may be formulated even more generally. As a direct consequence of [HE, Corollary 1.4.5] one has (see also [G-G-S]):

OBSERVATION 2. *Let $A$ be of type $(\omega, M(\theta))$ and the operators $B_j$, $j = 1, \ldots, m$, be linear on a base space $X$. If $B_j A^{-\alpha_j} \in \mathcal{L}(X, X)$ for some numbers $\alpha_j \in [0, 1)$, then the operator $A + \sum_{j=1}^m B_j$ is sectorial in $X$.*

**Local solvability of abstract parabolic equations.** As follows from the above considerations pseudodifferential equations of the type (2) fall into a class of abstract parabolic equations

$$\dot{u} + Au = F(u), \qquad t > 0, \qquad u(0) = u_0, \tag{3}$$

where $A : D(A) \to X$ is a sectorial positive operator in a Banach space $X$ and, for some $\beta \in [0, 1)$, the nonlinear term $F : X^\beta \to X$, $X^\beta = D(A^\beta)$, is Lipschitz continuous on bounded sets. Recall that (see [HE, Chapter 3], [C-D 1, Sections 2.1, 9.4]):

PROPOSITION 2. *Under the above assumptions there is a unique $X^\beta$ solution $u$ of (3) with $u_0 \in X^\beta$, defined on a maximal interval of existence $[0, \tau_{u_0})$, and such that*

$$u \in C([0, \tau_{u_0}), X^\beta) \cap C^1((0, \tau_{u_0}), X^\gamma) \cap C((0, \tau_{u_0}), X^1),\ \gamma \in [0, 1).$$

If the Lipschitz condition for $F$ is violated, solvability of (3) is a more delicate problem. The following result comes from [C-D 2] and is based on the original considerations of [L-M].

PROPOSITION 3. *If $A$ is a sectorial positive operator in a Banach space $X$, the resolvent of $A$ is compact and, for some $\beta \in [0,1)$, $F: X^\beta \to X$ is continuous and takes bounded subsets of $X^\beta$ into bounded subsets of $X$, then (3) has a local mild solution $u \in C([0,\tau), X^\beta)$ fulfilling the Cauchy integral formula*

$$(4) \qquad u(t,u_0) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}F(u(s))ds, \ t \in [0,\tau).$$

**Extendibility of the local solutions and a global attractor.** If the solutions are unique and exist globally in time, the problem (3) defines a $C^0$ semigroup on a phase space $X^\beta$. To describe the stability of (3) we recall a notion of a *global attractor* (see [HA]) which is a compact, nonempty, and invariant set $\mathcal{A}$, such that

$$\sup_{w_1 \in B} \inf_{w_2 \in \mathcal{A}} \|T(t)w_1 - w_2\|_{X^\beta} \to 0 \ \text{ as } \ t \to +\infty,$$

whenever $B$ is bounded in $X^\beta$.

The following result of [C-D 1] provides the abstract conditions for the existence of the global attractor.

PROPOSITION 4. *Under the assumptions of Proposition 2, when the resolvent of $A$ is compact, the following two conditions are equivalent:*

(i) *relation $T(t)u_0 = u(t,u_0)$, $t \geq 0$, defines on $X^\beta$ a $C^0$ semigroup of global solutions which has a global attractor,*

(ii) *It is possible to choose: a Banach space $Y$, with $D(A) \subset Y$, a locally bounded function $c: R^+ \longrightarrow R^+$, a nondecreasing function $g: R^+ \longrightarrow R^+$, and a certain number $\theta \in [0,1)$, such that,*

$$(5) \qquad \|u(t,u_0)\|_Y \leq c(\|u_0\|_{X^\beta}), \ t \in (0,\tau_{u_0}), \ u_0 \in X^\beta,$$

*and, simultaneously,*

$$(6) \quad \|F(u(t,u_0))\|_X \leq g(\|u(t,u_0)\|_Y)(1 + \|u(t,u_0)\|^\theta_{X^\beta}), \qquad t \in (0,\tau_{u_0}), u_0 \in X^\beta,$$

*where (5) is also asymptotically independent of $u_0 \in X^\beta$.*

Proposition 4 may be proved similarily as [C-D 1, Corollary 4.2.2]. We remark that $\theta \in [0,\frac{1}{\beta})$ is admissible in (6) provided that $Y \subset X$ (see [C-D 1, Remark 3.1.3]). Also, under the assumptions of Proposition 3, (ii) is sufficient for (3) to generate a semigroup having a global attractor in $X^\beta$ (see [C-D 2, Corollary 1]) whenever mild solutions (4) are unique.

**Excerpts from the theory of interpolation.** In applications it is important to have the embeddings of fractional power spaces in Sobolev and Hölder type spaces. It is even more convenient to have the complete characterization of $X^\beta$ spaces ($X$ being a complex Banach space) which is known, provided that the purely imaginary powers of $A$ are bounded; i.e.

$$\|A^{it}\|_{\mathcal{L}(X,X)} \le c_\varepsilon,\ t \in [-\varepsilon,\varepsilon]. \tag{7}$$

If this is the case, the following interpolation formula holds

$$D(A^{(1-\theta)\alpha+\theta\beta}) = [D(A^\alpha), D(A^\beta)]_\theta,\ \alpha,\beta \ge 0, \theta \in (0,1), \tag{8}$$

where $[\cdot,\cdot]_\theta$ denotes the *complex interpolation functor* (see [TR]). The well known examples of operators with the $\mathcal{BIP}$ property (7) are the maximal accretive operators in a Hilbert space with 0 in the resolvent set or, to be more specific, the self-adjoint, positive definite operators. For the discussion concerning the $\mathcal{BIP}$ property of the elliptic operators we refer to [C-D 1, Section 1.3], which provides the overview of the recent results within this field.

The operator $-\Delta_D$ in $L^p(\Omega)$, $p \in (1,+\infty)$, $\partial\Omega \in C^2$, appearing in the examples below has been recently studied in [P-S] where it was shown that

$$\forall_{\theta\in(0,\pi)} \exists_{M_p(\theta)>0}\ \|(-\Delta_D)^{it}\|_{\mathcal{L}(L^p(\Omega),L^p(\Omega))} \le M_p(\theta)e^{\theta t},\ t \in R.$$

Therefore we have a characterization (8) of the domains $X^\alpha_{L^p} = D(-\Delta_D)^\alpha$ and, in particular,

$$X^\alpha_{L^p} = [L^p(\Omega), W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)]_\alpha \subset H^{2\alpha}_p(\Omega),\ \alpha \in (0,1). \tag{9}$$

Using (9) and the embeddings for the spaces $H^{2\alpha}_p(\Omega)$ of *Bessel potentials* (see [TR]), we obtain *strict* inclusions:

$$X^\alpha_{L^p} \subset \begin{cases} W^{s,q}(\Omega) \text{ if } 2\alpha - \frac{n}{p} \ge s - \frac{n}{q},\ 2 \le p \le q < +\infty, \\ C^{k+\mu}(\overline{\Omega}) \text{ if } 2\alpha - \frac{n}{p} \ge k + \mu,\ k \in N,\ \mu \in (0,1). \end{cases} \tag{10}$$

Note that e.g. for $p = 2$ with further assumption $\partial\Omega \in C^{2+\eta}$, $\eta > 0$, the inclusion in (9) turns to the equality

$$X^\alpha_{L^2} = H^{2\alpha}_{2,Id}(\Omega),\ \alpha \in [0,1],\ 2\alpha \ne \frac{1}{2}, \tag{11}$$

(see [GU 1], [C-D 1] for details).

## 2. Part II. Examples

We provide here two special examples of equations having fractional powers of elliptic operators as the main part. Another equations and their physical motivation may be found in [B-P-F-S] and [F-S-Z].

EXAMPLE 1. As a first example consider a variant of an *equation of anomalous diffusion* studied e.g. in [B-K-W 2], that is the equation of the form:

(12) $$u_t + (-\Delta_D)^\alpha u + \mathbf{b}(x)\cdot\nabla(f(u)) = 0, \quad t>0,\ x\in\Omega,$$

where $\alpha \in (\frac{1}{2},1)$, $\mathbf{b} : R^n \supset \Omega \to R^n$ is a differentiable, bounded vector function and $f(u)$ was originally equal to $u|u|^{r-1}$ with some $r > 1$. Here $f : R \to R$ is $C^1$ with $f'$ locally Lipschitz continuous function and we require that

(13) $$div\,\mathbf{b}(x) = 0, \quad x\in\Omega.$$

The equation (12) will be studied in a bounded domain with the boundary $\partial\Omega \in C^{2+\eta}$, $\eta > 0$, together with homogeneous Dirichlet initial-boundary conditions

(14) $$\begin{cases} u(0,x) = u_0(x),\ x\in\Omega,\\ u_{|\partial\Omega} = 0. \end{cases}$$

**Equation (12) in $L^p(\Omega)$.** The problem (12), (14) will be considered first as an abstract parabolic equation (3) in the base space $X = L^p(\Omega)$ with $p > n$ and $A = (-\Delta_D)^\alpha$, $\Delta_D$ being the Laplacian with homogeneous Dirichlet boundary condition, in which case (10) gives

$$D((-\Delta_D)^\beta) = X^\beta_{L^p} \subset W^{1,p}(\Omega) \subset C(\overline{\Omega}), \quad \beta \geq \frac{1}{2}.$$

Define

(15) $$F(u) = \mathbf{b}\cdot\nabla(f(u))$$

and take a bounded set $\mathcal{U} \subset W^{1,p}(\Omega)$. For $\phi,\psi \in \mathcal{U}$, we obtain

(16) $$\begin{aligned} &\|F(\phi) - F(\psi)\|_{L^p(\Omega)}\\ &\leq \|(f'(\phi) - f'(\psi))\mathbf{b}\cdot\nabla\phi\|_{L^p(\Omega)} + \|f'(\psi)\mathbf{b}\cdot\nabla(\phi-\psi)\|_{L^p(\Omega)}\\ &\leq \|\mathbf{b}\|_{[L^\infty(\Omega)]^n}(\|f'(\phi) - f'(\psi)\|_{L^\infty(\Omega)}\|\phi\|_{W^{1,p}(\Omega)}\\ &\qquad + \|f'(\psi)\|_{L^\infty(\Omega)}\|\phi-\psi\|_{W^{1,p}(\Omega)})\\ &\leq C_{\mathcal{U}}\|\phi-\psi\|_{W^{1,p}(\Omega)}, \end{aligned}$$

where local Lipschitz continuity of $f'$ has been used. As a consequence of Proposition 2, to any $u_0 \in X^\beta_{L^p}$, $\beta \in (\frac{1}{2},\alpha)$, corresponds a unique $X^\beta_{L^p}$ solution of (12), (14). According to the theory developed in [C-D 1], for global solvability we need an additional *a priori estimate* of the local solutions in an auxilliary Banach space $Y$. In this example we shall choose $Y = L^\infty(\Omega)$.

**$L^\infty(\Omega)$ estimate.** Multiplying (12) by $u^{2k-1}$, $k = 1,2,\dots$, and integrating over $\Omega$ we obtain

(17) $$\frac{1}{2k}\frac{d}{dt}\int_\Omega u^{2k}dx = -\int_\Omega(-\Delta_D)^\alpha u\, u^{2k-1}dx - \int_\Omega \mathbf{b}(x)\cdot\nabla(f(u))u^{2k-1}dx.$$

Note that we have the equality

$$\int_\Omega \mathbf{b}(x)\cdot\nabla(f(u))u^{2k-1}dx = \int_\Omega \mathbf{b}(x)\cdot\nabla(g_k(u))dx,$$

where $g_k(s)=\int_0^s f'(z)z^{2k-1}dz$. Now the properties $g_k(0)=0$ and (13) ensure that the last integral in (17) vanishes. The first right hand side term in (17) will be transformed based on the Kato-Beurling-Deny inequality (40) with $q=2k$:

$$\begin{aligned}(18)\qquad -\int_\Omega(-\Delta_D)^\alpha u\, u^{2k-1}\,dx &= -\int_\Omega(-\Delta_D)^\alpha u\,|u|^{2k-1}\operatorname{sgn} u\,dx\\ &\le -\frac{2k-1}{k^2}\int_\Omega |(-\Delta_D)^{\frac{\alpha}{2}}(|u|^k)|^2dx.\end{aligned}$$

Note, that for arbitrary $t>0$ the local $X^\beta_{L^p}$ solution $u$ to (12), (14), $\beta\in(\frac{1}{2},\alpha)$, belongs to $X^\alpha_{L^p}\subset W^{1,p}(\Omega)\subset C(\overline{\Omega})$, so that $|u|^k\in H^1_0(\Omega)$, $k\in N$. Now, continuity of the inclusion $D((-\Delta_D)^\alpha)\subset L^2(\Omega)$ allows us to complete the $L^{2k}(\Omega)$ estimate

$$\frac{1}{2k}\frac{d}{dt}\int_\Omega u^{2k}dx \le -const.\frac{2k-1}{k^2}\int_\Omega u^{2k}dx$$

and to get the inequality

$$(19)\qquad \|u(t,u_0)\|_{L^{2k}(\Omega)} \le \|u_0\|_{L^{2k}(\Omega)}e^{-const.\frac{2k-1}{k^2}t}.$$

Letting $k$ tend to infinity, one finds

$$(20)\qquad \|u(t,u_0)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)},$$

so that the solutions are estimated globally in time in $L^\infty(\Omega)$.

REMARK 1. The a priori estimates (19), (20) are formally valid also when $\beta\in(0,\frac{1}{2}]$, provided that solutions under consideration are sufficiently smooth. However, for such range of the parameter $\beta$, equation (12) changes its type since the nonlinearity *is not* subordinated to the main linear part. In that case *we do not have* local existence of solutions within the approach used in this paper.

The *subordination condition* for the nonlinear term (see [C-D 1, condition (3.1.4)]) may now be written in the form:

$$(21)\quad \begin{aligned} &\|F(u(t,u_0))\|_{L^p(\Omega)} \\ &= \|f'(u(t,u_0))\mathbf{b}\cdot\nabla u(t,u_0)\|_{L^p(\Omega)} \\ &\le \sup_{|s|\le\|u(t,u_0)\|_{C(\overline{\Omega})}} |f'(s)|\|\mathbf{b}\|_{[L^\infty(\Omega)]^n}\|u(t,u_0)\|_{W^{1,p}(\Omega)}, \end{aligned}$$

thus the local $X^\beta_{L^p}$ solution to (12), (14) is in fact global in time for any $\beta\in(\frac{1}{2},\alpha)$.

Having (20) and (21) we know that the problem (12), (14) generates on $X^\beta_{L^p}$ a compact $C^0$ semigroup $\{T(t)\}$ of global solutions (see Remark 2 (i)), which has bounded orbits of bounded sets. This and (19) guarantee next that for each $u_0\in X^\beta_{L^p}$ the $\omega$-limit set $\omega_{X^\beta_{L^p}}(u_0)$ consists of a single element 0, which is the unique equilibrium of $\{T(t)\}$ and, furthermore,

$$T(t)u_0\to 0 \ \text{ in } \ X^\beta_{L^p} \ \text{ as } \ t\to+\infty.$$

We thus conclude that:

THEOREM 1. *For each $\beta\in(\frac{1}{2},\alpha)$ the problem* (12), (14) *generates on $X^\beta_{L^p}$ a $C^0$ semigroup of global solutions. The stationary solution 0 is globally asymptotically stable.*

REMARK 2. For $f(s)=s|s|^{r-1}$ Example 1 reduces to the problem studied in [B-K-W 2]. However, for the validity of the above calculations we need to take $r\ge 2$ which is not the case considered in [B-K-W 2]. To cover the case when $f$ is only of class $C^1(R)$ (e.g. $f(s)=s|s|^{r-1}$ with $r\in(1,2)$) we shall refer to Proposition 3. For this we need to check that: (i) the resolvent of $(-\Delta_D)^\alpha$ is compact and that (ii) $F: X^\beta_{L^p}\to X$ defined in (15) is a continuous function which takes bounded subsets of $X^\beta_{L^p}$, $\beta\in(\frac{1}{2},\alpha)$, into bounded subsets of $L^p(\Omega)$.

Property (i) is immediate, since $((-\Delta_D)^\alpha)^{-1}=(-\Delta_D)^{-\alpha}$ is a bounded linear operator defined on the whole of $X$ and $(-\Delta_D)^{-\alpha}(X)=X^\alpha_{L^p}$ is compactly embedded in $X$.

For the proof of (ii) we rewrite (21) in a form

$$(22)\quad \|F(\phi)\|_{L^p(\Omega)}\le \sup_{|s|\le\|\phi\|_{C(\overline{\Omega})}}|f'(s)|\|\mathbf{b}\|_{[L^\infty(\Omega)]^n}\|\phi\|_{W^{1,p}(\Omega)},\ \phi\in W^{1,p}(\Omega).$$

Since $p>n$ and $f'$ is locally bounded it is clear from (22) that $F(B)$ is bounded in $L^p(\Omega)$ whenever $B$ is bounded in $W^{1,p}(\Omega)$. Also, as seen from the second inequality in (16), convergence of $\{\phi_n\}$ in $W^{1,p}(\Omega)\subset C(\overline{\Omega})$ ensures convergence of $\{F(\phi_n)\}$ in $L^p(\Omega)$, $p>n$, which shows continuity of $F: X^\beta_{L^p}\to X$. Consequently, recalling Proposition 3, the problem (12), (14) possesses a (not necessarily unique) mild solution $u(\cdot,u_0)$ for each $u_0\in X^\beta_{L^p}$.

**Equation (12) in $L^2(\Omega)$.** It may be interesting to consider Example 1 in a larger base space, like $L^2(\Omega)$. We shall outline below solvability of (12), (14) in this case. Unlike in the case $X = L^p(\Omega)$, $p > n$, we need here to restrict the growth of the nonlinear term $f$ according to the condition

$$|f'(s_1) - f'(s_2)| \le c|s_1 - s_2|(1 + |s_1|^{r-1} + |s_2|^{r-1}),\ s_1, s_2 \in R, \tag{23}$$

where

$$1 < r \begin{cases} \text{arbitrarily large if } 4\beta \ge n, \\ \le \frac{4\beta-2}{n-4\beta} \text{ if } 4\beta < n < 8\beta - 2. \end{cases} \tag{24}$$

Using Hölder inequality and the assumptions (23), (24), one can check Lipschitz continuity of the nonlinear term (15) acting from $X^{\beta}_{L^2}$ to $L^2(\Omega)$, $\beta \in (\frac{1}{2}, \alpha)$. This justifies local solvability of (12), (14) in $X^{\beta}_{L^2}$, $\beta \in (\frac{1}{2}, \alpha)$.

For global solvability and the existence of a global attractor it suffices to find time independent estimate of $\|F(u(t,u_0))\|_{L^2(\Omega)}$. Considerations similar to those justifying the Lipschitz continuity of $F$ in $L^2(\Omega)$ give for $n > 2$

$$\begin{aligned}\|F(\phi)\|_{L^2(\Omega)} &\le const.\|(1 + |\phi|^r)|\nabla\phi|\|_{L^2(\Omega)} \\ &\le const.\left(\||\nabla\phi|\|_{L^2(\Omega)} + \|\phi\|^r_{L^{2pr}(\Omega)}\||\nabla\phi|\|_{L^{2q}(\Omega)}\right),\end{aligned}$$

with $p = \frac{n+\frac{n}{r}}{n-2}$, $q = \frac{nr+n}{2r+n}$, which leads to the estimate

$$\|F(\phi)\|_{L^2(\Omega)} \le g(\|\phi\|_{L^{\frac{2n(r+1)}{n-2}}(\Omega)})\|\phi\|_{W^{1,\frac{2n(r+1)}{2r+n}}(\Omega)}, \tag{25}$$

with certain nondecreasing function $g : R^+ \to R^+$. When $r$ fulfills (24), the inclusion $X^{\beta}_{L^2} \subset W^{1,\frac{2n(r+1)}{2r+n}}(\Omega)$ allows us to extend (25) to a subordination condition of type (6). The uniform in time and asymptotically independent of $u_0 \in X^{\beta}_{L^2}$ estimate of $\|u(t,u_0)\|_{L^{\frac{2n(r+1)}{n-2}}(\Omega)}$ follows as in (17)-(19).

For $n = 1, 2$ one may use the estimate

$$\|F(\phi)\|_{L^2(\Omega)} \le const.(\||\nabla\phi|\|_{L^2(\Omega)} + \|\phi\|^r_{L^{\frac{2sr}{s-2}}(\Omega)}\||\nabla\phi|\|_{L^s(\Omega)}), \quad s > 2,$$

together with the embedding $X^{\beta}_{L^2} \subset W^{1,s}(\Omega)$, valid for $n = 1, 2$, $\beta > \frac{1}{2}$ and $s > 2$ sufficiently close to 2, to get the similar conclusion.

Therefore, Theorem 1 remains true also when $X = L^2(\Omega)$ is the base space.

REMARK 3. It is possible to extend further the base space $X$. As a result of [GU 2, Theorem 1.7], the *Cauchy problem* for (12) in half space (i.e. when $(t,x) \in R^+ \times R^n$) may be studied in $L^1(R^n)$. In that case our abstract approach meets the considerations of [B-K-W 1], [B-K-W 2]. Since the space average of the solution is preserved in time (see [B-K-W 1]), the asymptotic behavior in $L^1(R^n)$ is *no more trivial.*

EXAMPLE 2. As a second example consider the *fractional dissipative equation* (see [VA]):

$$\begin{cases} u_t = -(-\Delta_D)^\alpha u + g(u), & t > 0,\ x \in \Omega, \\ u(0,x) = u_0(x),\ x \in \Omega, & u_{|\partial\Omega} = 0, \end{cases} \tag{26}$$

where $\Omega$ is a bounded domain in $R^n$ with $\partial\Omega \in C^2$ and $\alpha \in (0,1)$. We assume that $g : R \to R$ is a locally Lipschitz function and choose $p \geq 2$ such that $2\alpha - \frac{n}{p} > 0$.

The equation (26) may be rewritten as an abstract problem in $X = L^p(\Omega)$ with corresponding to $g$ Nemytskiĭ operator $G : X^\beta_{L^p} \to X$ being Lipschitz continuous on bounded sets. Here $X^\beta_{L^p}$ denotes the domain of $(-\Delta_D)^\beta$ in $L^p(\Omega)$ and $\beta$ is such that

$$2\beta - \frac{n}{p} > 0, \quad \beta \in \left[\frac{\alpha}{2}, \alpha\right), \tag{27}$$

so the embedding $X^\beta_{L^p} \subset C(\overline{\Omega})$ is continuous. The above justifies existence of the local $X^\beta_{L^p}$ solutions to (26).

**Global solutions to** (26). For the global solvability and the existence of a global attractor we shall assume additionally that $g$ fulfills the *dissipativeness condition:*

$$\limsup_{|s|\to+\infty} \frac{g(s)}{s} \leq 0. \tag{28}$$

As in the case of a scalar parabolic equation (see [HA, p. 75]), for $\beta$ given in (27) the formula

$$\mathcal{L}(\phi) = \frac{1}{2}\int_\Omega [(-\Delta_D)^{\frac{\alpha}{2}}\phi]^2\,dx - \int_\Omega\int_0^\phi g(s)\,ds\,dx, \qquad \phi \in X^\beta_{L^p}, \tag{29}$$

defines a Lyapunov function on $X^\beta_{L^p}$. Since $\mathcal{L}(u(t,u_0))$ is decreasing in time we have an a priori estimate

$$\frac{1}{2}\int_\Omega [(-\Delta_D)^{\frac{\alpha}{2}} u(t,u_0)]^2 dx \leq \mathcal{L}(u_0) + \int_\Omega \int_0^{u(t,u_0)} g(s)\,ds\,dx. \tag{30}$$

Following [HA, p. 76], the dissipativeness condition implies that

$$\int_0^z g(s)\,ds \leq \varepsilon z^2 + C_\varepsilon, \qquad z \in R,$$

with arbitrary $\varepsilon > 0$ and a corresponding constant $C_\varepsilon > 0$. Therefore, (30) extends to the estimate

$$\int_\Omega [(-\Delta_D)^{\frac{\alpha}{2}} u(t,u_0)]^2\, dx \le 4(\mathcal{L}(u_0) + const.|\Omega|), \tag{31}$$

being the required a priori estimate of the solutions in $X^{\frac{\alpha}{2}}_{L^2}$.

When $|G(s)| \le c(1+|s|^r)$ with

$$\begin{cases} r < 1 + \frac{2\alpha}{\frac{n}{2}-\alpha} \text{ and } \beta \text{ in (27)}, \ n > 2\alpha, \\ r \text{ arbitrarily large}, \ n \le 2\alpha, \end{cases}$$

the $X^{\frac{\alpha}{2}}_{L^2}$ estimate (31) is sufficient for global $X^{\beta}_{L^p}$ solvability of (26) (see [C-D 1, Section 5.2]). To avoid the growth restriction on $g$ we need to strengthen the estimate (31). Since (31) implies the $L^2(\Omega)$ estimate, an $L^\infty(\Omega)$ estimate of the $X^{\beta}_{L^p}$ solutions to (26) will follow from Lemma 1 (see Appendix).

**Asymptotic behavior of solutions to** (26). As known, the existence of a Lyapunov function having all properties of [HA, pp. 49–50] determines the asymptotic behavior of solutions. Following closely the presentation of [HA, pp. 76–77] one can check that the problem (26) generates on $X^{\beta}_{L^p}$ ($\beta$ satisfying (27)) a *gradient system* whenever $g$ is a $C^2$ function.

All $\omega$-limit sets of points are contained in the *set $E$ of stationary solutions* to (26), that is in the set of $X^{\alpha}_{L^p}$ solutions to

$$\begin{cases} (-\Delta_D)^\alpha v = g(v), \ v \in X^{\alpha}_{L^p}, \\ v_{|\partial\Omega} = 0. \end{cases} \tag{32}$$

Whenever the set $E$ is bounded in $X^{\alpha}_{L^p}$, compactness of the resolvent of $(-\Delta_D)^\alpha$ (see Remark 2) ensures the existence of a global attractor $\mathcal{A}$ for the semigroup generated by (26) on $X^{\beta}_{L^p}$, $\beta$ as in (27) (see [HA, Theorem 3.8.5]).

**Boundedness of the set** $E$. First we show that the set $E$ of solutions $v$ to (32) is bounded in $L^2(\Omega)$. Multiplying (32) by $v$, integrating and using (43), we find that

$$\int_\Omega (-\Delta_D)^\alpha v\, v dx \le C \int_\Omega v^2 dx + D|\Omega|, \tag{33}$$

and since

$$const. \int_\Omega \phi^2 dx \le \int_\Omega \left[(-\Delta_D)^{\frac{\alpha}{2}}\phi\right]^2 dx, \ \phi \in X^{\frac{\alpha}{2}}_{L^2},$$

then (33) provides an $L^2(\Omega)$ estimate of $v$ for $C = \frac{1}{2} const.$.

Using the recurrence technique of Moser-Alikakos, the $L^2(\Omega)$ estimate of the set $E$ may next be sharpened to an $L^\infty(\Omega)$ estimate, analogously as in the proof of Lemma 1 (see [C-D 1, Lemma 6.2.2] for details). Finally from (32) we find

$$(34)\qquad \int_\Omega |(-\Delta_D)^\alpha v|^p dx = \int_\Omega |g(v)|^p dx \le const.(\|v\|_{L^\infty(\Omega)}) < +\infty,$$

which shows that $E$ is bounded in $X^\alpha_{L^p}$.

We shall summarize the above studies in the following:

THEOREM 2. *The problem* (26) *studied on* $L^p(\Omega)$, $p \ge 2$, $2\alpha - \frac{n}{p} > 0$ *under the assumption* (28) *generates on* $X^\beta_{L^p(\Omega)}$, $\beta$ *as in* (27), *a* $C^0$ *semigroup which has a global attractor. Whenever* $g \in C^2$, *this semigroups is a* $C^1$ *gradient system.*

REMARK 4. Within the above approach one can similiarily consider an abstract parabolic equation

$$u_t = \sum_{i=1}^n A_i^{\alpha_i} u + f(x, u, D^\beta u),$$

where the main part is a sum of fractional powers of elliptic operators (see Observation 2).

## 3. Part III. Appendix

**Various definitions of generators of analytic semigroups.** We first compare Definition 2 of a *sectorial positive operator* by [HE] and Definition 1 of an *operator of type* $(\omega, M(\theta))$.

For $a \in R$ and $\phi \in (0, \frac{\pi}{2})$, by $S_{a,\phi}$ denote a sector of the complex plain

$$S_{a,\phi} := \{\lambda \in C : \phi \le |arg(\lambda - a)| \le \pi, \lambda \ne a\}.$$

DEFINITION 2. A linear, closed, densely defined operator $A : X \supset D(A) \to X$ acting in a Banach space $X$, is a sectorial operator if and only if there exist $a \in R$, $\phi \in (0, \frac{\pi}{2})$ and $M > 0$ such that the resolvent set $\rho(A)$ contains the sector $S_{a,\phi}$ and

$$\|(\lambda I - A)^{-1}\| \le \frac{M}{|\lambda - a|} \quad \text{for each } \lambda \in S_{a,\phi}.$$

We shall show that for $a = 0$ and $\omega < \frac{\pi}{2}$ Definitions 1 and 2 are equivalent, i.e. an operator $A$ is of type $(\omega, M(\theta))$ if and only if $A$ is sectorial with sector $S_{0,\omega}$. This will be a simple consequence of the proposition below.

PROPOSITION 5. *Let* $A : X \to X$ *be a closed linear operator in a Banach space* $X$, $\phi \in [0, \pi]$ *and* $S_{0,\phi} \subset \rho(A)$. *The following two conditions are equivalent:*

$$(35)\qquad \|(\lambda I - A)^{-1}\| \le \frac{M(\theta)}{|\lambda|} \quad \textit{for each } \lambda \in S_{0,\phi} \textit{ such that } arg\,\lambda = \theta,$$

(36) $$\|(\lambda I - A)^{-1}\| \le \frac{M}{|\lambda|} \quad \textit{for each } \lambda \in S_{0,\phi}.$$

Proof. It suffices to show that (35) implies (36). Fix $\phi < \pi$ and $\theta \in [\phi, 2\pi - \phi]$ and let $\alpha = \arcsin \frac{1}{2M(\theta)}$. Take $\lambda$ such that $arg\,\lambda \in (\theta - \alpha, \theta + \alpha)$. Then there exists $\lambda_0$ such that $arg\,\lambda_0 = \theta$, $|\lambda_0| > |\lambda|$ and $\frac{|\lambda - \lambda_0|}{|\lambda_0|} \le \frac{1}{2M(\theta)}$. Therefore we have

(37) $$|\lambda - \lambda_0| \le \frac{1}{2\|(\lambda_0 I - A)^{-1}\|}.$$

Using the result of [YO, p. 211]

$$(\lambda I - A)^{-1} = \sum_{i \in N} (\lambda_0 - \lambda)^i (\lambda_0 I - A)^{-(i+1)}$$

and the estimate (37), we obtain that

$$\|(\lambda I - A)^{-1}\| \le \frac{2M(\theta)}{|\lambda_0|} \le \frac{2M(\theta)}{|\lambda|},$$

for each $\lambda$ with $arg\,\lambda \in (\theta - \alpha, \theta + \alpha)$. Since the interval $[\phi, 2\pi - \phi]$ is compact, the proof is complete. ■

**The Kato-Beurling-Deny inequality.** A version of the famous Kato-Beurling-Deny inequality will be proved below for completeness of the presentation. We shall focus here on $A = -\Delta_D$ in $L^2(\Omega)$, $\Omega \subset R^n$ being a bounded $C^2$ domain, which is a special case of the general theory in [DA].

It is well known that $(sI + A)^{-1}$, $s \ge 0$, has a positive symmetric kernel $K_s = K_s(x, y)$, $x, y \in \Omega$, satisfying the estimate in [TA, 5.168, p. 210]. Also, the integral formula for fractional powers of $A$ (see [TR, §1.15.1 (6)]) reads:

$$A^\alpha v = \frac{\sin \pi\alpha}{\pi} \int_0^{+\infty} s^{\alpha - 1} A (sI + A)^{-1} v ds, \ v \in D(A), \ \alpha \in (0, 1).$$

Writing below for simplicity of the notation $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$ for the $L^2(\Omega)$ product, we obtain

$$\begin{aligned} &\langle A^\alpha v, v^{q-1} \rangle_{L^2(\Omega)} \\ &= \left\langle \frac{\sin \pi\alpha}{\pi} \int_0^{+\infty} s^{\alpha-1} (sI + A - sI)(sI + A)^{-1} v \, ds, v^{q-1} \right\rangle_{L^2(\Omega)} \\ &= \frac{\sin \pi\alpha}{\pi} \int_0^{+\infty} s^{\alpha-1} (\|v^q\|_{L^1(\Omega)} - \langle s(sI + A)^{-1} v, v^{q-1} \rangle_{L^2(\Omega)}) \, ds, \quad v \in C_0^+, \end{aligned}$$

where

$$C_0^+ = \{\phi \in C^2(\overline{\Omega}) : \phi \ge 0, \ \phi_{|\partial\Omega} = 0\}.$$

The properties of $K_s$ and an elementary inequality of [DA, p. 68]

$$(s-t)(s^{q-1}-t^{q-1}) \geq \frac{4(q-1)}{q^2}|s^{\frac{q}{2}}-t^{\frac{q}{2}}|^2, \qquad s \geq 0,\ t \geq 0,\ q \geq 2,$$

ensure next that

$$\begin{aligned}
(38) \quad & \|v^q\|_{L^1(\Omega)} - \langle s(sI+A)^{-1}v, v^{q-1}\rangle_{L^2(\Omega)} \\
&= \|v^q\|_{L^1(\Omega)} - s \int_{\Omega\times\Omega} v^{q-1}(x)v(y)K_s(x,y)\,dx\,dy \\
&= \frac{s}{2}\int_{\Omega\times\Omega} [v(x)-v(y)][v^{q-1}(x)-v^{q-1}(y)]K_s(x,y)\,dx\,dy \\
&\quad + \|v^q\|_{L^1(\Omega)} - s\int_{\Omega\times\Omega} v^q(x)K_s(x,y)\,dx\,dy \\
&= \frac{s}{2}\int_{\Omega\times\Omega} [v(x)-v(y)][v^{q-1}(x)-v^{q-1}(y)]K_s(x,y)\,dx\,dy \\
&\quad + \|v^q\|_{L^1(\Omega)} - s\|(sI+A)^{-1}(v^q)\|_{L^1(\Omega)} \\
&\geq \frac{4(q-1)}{q^2}\Big(\frac{s}{2}\int_{\Omega\times\Omega} |v^{\frac{q}{2}}(x)-v^{\frac{q}{2}}(y)|^2K_s(x,y)\,dx\,dy \\
&\quad + \|v^q\|_{L^1(\Omega)} - s\|(sI+A)^{-1}(v^q)\|_{L^1(\Omega)}\Big), \quad s>0,\ q\geq 2,\ v\in \mathcal{C}_0^+,
\end{aligned}$$

where in the last line above we have used additionally the inequality $1 \geq \frac{4(q-1)}{q^2}$, $q \geq 2$, and the contraction property of $A$ in $L^1(\Omega)$ (see [DA, Theorem 1.3.5]), which guarantees that

$$\|v^q\|_{L^1(\Omega)} - s\|(sI+A)^{-1}(v^q)\|_{L^1(\Omega)} \geq 0, \qquad s>0,\ v\in\mathcal{C}_0^+.$$

Similar calculations show that

$$\begin{aligned}
& \langle A^\alpha(v^{\frac{q}{2}}), v^{\frac{q}{2}}\rangle_{L^2(\Omega)} \\
&= \Big\langle \frac{\sin\pi\alpha}{\pi}\int_0^{+\infty} s^{\alpha-1}(sI+A-sI)(sI+A)^{-1}(v^{\frac{q}{2}})\,ds, v^{\frac{q}{2}}\Big\rangle_{L^2(\Omega)} \\
&= \frac{\sin\pi\alpha}{\pi}\int_0^{+\infty} s^{\alpha-1}(\|v^q\|_{L^1(\Omega)} - \langle s(sI+A)^{-1}(v^{\frac{q}{2}}), v^{\frac{q}{2}}\rangle_{L^2(\Omega)})\,ds \\
&= \frac{\sin\pi\alpha}{\pi}\int_0^{+\infty} s^{\alpha-1}(\|v^q\|_{L^1(\Omega)} - s\int_{\Omega\times\Omega} v^{\frac{q}{2}}(x)v^{\frac{q}{2}}(y)K_s(x,y)\,dx\,dy)ds
\end{aligned}$$

$$= \frac{\sin \pi\alpha}{\pi} \int_0^{+\infty} s^{\alpha-1}\Big( \|v^q\|_{L^1(\Omega)} + \frac{s}{2} \int_{\Omega\times\Omega} |v^{\frac{q}{2}}(x) - v^{\frac{q}{2}}(y)|^2 K_s(x,y)\,dx\,dy$$

$$- s \int_{\Omega\times\Omega} v^q(x) K_s(x,y)\,dx\,dy\Big) ds$$

$$= \frac{\sin \pi\alpha}{\pi} \int_0^{+\infty} s^{\alpha-1}\Big( \frac{s}{2} \int_{\Omega\times\Omega} |v^{\frac{q}{2}}(x) - v^{\frac{q}{2}}(y)|^2 K_s(x,y)\,dx\,dy$$

$$+ \|v^q\|_{L^1(\Omega)} - s\|(sI + A)^{-1}(v^q)\|_{L^1(\Omega)}\Big) ds,$$

$s > 0$, $q \geq 2$, $v \in C_0^+$. As a consequence of the relation (38) and the obvious estimates when $\alpha = 0$ or $\alpha = 1$ we obtain the proposition below.

PROPOSITION 6. *Let $\Omega \subset R^n$ be a bounded domain, $\partial\Omega \in C^2$, and $A = -\Delta_D$ on $L^2(\Omega)$. Then, for $\alpha \in [0,1]$, $q \in [2,+\infty)$ and $\phi \in C_0^+$ the following inequality holds:*

$$(39)\quad \int_\Omega A^\alpha \phi\, \phi^{q-1} dx = \int_\Omega A^{\frac{\alpha}{2}} \phi\, A^{\frac{\alpha}{2}}(\phi^{q-1}) dx \geq \frac{4(q-1)}{q^2} \int_\Omega \left[A^{\frac{\alpha}{2}}(\phi^{\frac{q}{2}})\right]^2 dx.$$

**Extension of (39) to functions with arbitrary sign.** It is well known that the resolvent $(\lambda I - \Delta_D)^{-1}$, $\lambda > 0$, preserves positivity (see [DA, Theorem 1.3.5]). This property extends directly to the resolvent of $(-\Delta_D)^\alpha$, $\alpha \in (0,1)$, because of the formula ([KO, p. 319]):

$$(\lambda I + (-\Delta_D)^\alpha)^{-1}$$
$$= \frac{\sin \pi\alpha}{\pi} \int_0^{+\infty} \frac{\tau^\alpha}{\lambda^2 + 2\lambda\tau^\alpha \cos \pi\alpha + \tau^{2\alpha}} (\tau I - \Delta_D)^{-1}\, d\tau, \quad \lambda > 0,$$

since the denominator above is positive. Next, for $\phi \in X^\alpha_{L^2}$ with $|\phi|^{q-1} \in X^{\frac{\alpha}{2}}_{L^2}$, Theorem 1.3.2 of [DA] gives us that $|\phi| \in X^{\frac{\alpha}{2}}_{L^2}$ and

$$\int_\Omega (-\Delta_D)^\alpha \phi |\phi|^{q-1} \operatorname{sgn} \phi\, dx \geq \int_\Omega (-\Delta)^{\frac{\alpha}{2}}(|\phi|)(-\Delta_D)^{\frac{\alpha}{2}}(|\phi|^{q-1})\, dx.$$

Together with (39), the last estimate justifies that

COROLLARY 1. *For $\alpha \in [0,1]$, $q \in [2,+\infty)$, $\phi \in X^\alpha_{L^2}$ and $|\phi|^{q-1} \in X^{\frac{\alpha}{2}}_{L^2}$, the following estimate holds:*

$$(40)\qquad \int_\Omega (-\Delta_D)^\alpha \phi |\phi|^{q-1} \operatorname{sgn} \phi\, dx \geq \frac{4(q-1)}{q^2} \int_\Omega [(-\Delta_D)^{\frac{\alpha}{2}}(|\phi|^{\frac{q}{2}})]^2 dx.$$

REMARK 5. Formulas (39), (40) hold also, with the same proof, for $(-\Delta_D)$ replaced by arbitrary operator $A$ given by a symmetric differential operator

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right), \quad x \in \Omega \subset R^n,$$

considered with homogeneous Dirichlet condition in a bounded domain $\Omega$ with $\partial\Omega \in C^2$. We need to assume additionally that the coefficients $a_{ij}$ are real, $a_{ij} \in C^1(\overline{\Omega})$, $a_{ij} = a_{ji}$, $i,j = 1,\dots,n$, and the following *ellipticity condition* is satisfied:

$$\exists_{\eta>0}\forall_{x\in\Omega}\forall_{\xi\in R^n} \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \eta|\xi|^2.$$

For the weaker possible assumptions on the coefficients and the boundary of $\Omega$ one may refer to [DA].

**The Moser-Alikakos technique.** The lemma below has been used in the estimates of Example 2.

LEMMA 1. *For $X^{\beta}_{L^p}$ solutions to* (26), *$\beta$ as in* (27), *the following implication holds:*

$$(41) \qquad (\|u(t,u_0)\|_{L^1(\Omega)} \leq const., \ t \geq 0) \Longrightarrow (\|u(t,u_0)\|_{L^\infty(\Omega)} \leq const.', \ t \geq 0).$$

Proof. For an arbitrary domain $Q \subset R^n$ (bounded or not) and any $\mu > 0$ the interpolation inequality for $L^p$ spaces

$$\|\phi\|_{L^2(Q)} \leq \|\phi\|^{\frac{2+\mu}{2+2\mu}}_{L^{2+\mu}(Q)}\|\phi\|^{\frac{\mu}{2+2\mu}}_{L^1(Q)}, \quad \phi \in L^1(Q) \cap L^{2+\mu}(Q),$$

and the Young inequality lead to the estimate

$$\forall_{\varepsilon>0}\forall_{\mu>0}\exists_{C_{\varepsilon,\mu}>0} \ \|\phi\|^2_{L^2(Q)} \leq \varepsilon\|\phi\|^2_{L^{2+\mu}(Q)} + C_{\varepsilon,\mu}\|\phi\|^2_{L^1(Q)}.$$

Choosing $\mu > 0$ such that $X^{\beta}_{L^2} \subset L^{2+\mu}(\Omega)$ we obtain

$$(42) \qquad \forall_{\varepsilon>0}\exists_{C'_\varepsilon>0} \ \|\phi\|^2_{L^2(\Omega)} \leq \varepsilon\|\phi\|^2_{X^{\beta}_{L^2}} + C'_\varepsilon\|\phi\|^2_{L^1(\Omega)}, \quad \phi \in X^{\beta}_{L^2},$$

which is a counterpart of the formula [C-D 1, (9.3.8)]. Observe next that the dissipativeness condition (28) implies

$$(43) \qquad \forall_{C>0}\exists_{D>0}\forall_{s\in R} \ sg(s) \leq Cs^2 + D,$$

which in order corresponds to [C-D 1, (9.3.5)].

Finally we recall the estimate (40) with $q = 2^k$:

$$(44) \qquad \frac{(2^k-1)}{2^{2k-2}}\int_\Omega [(-\Delta_D)^{\frac{\alpha}{2}}(|\phi|^{2^{k-1}})]^2 dx \leq \int_\Omega [(-\Delta_D)^{\alpha}\phi]|\phi|^{2^k-1}\operatorname{sgn}\phi \, dx.$$

With the above conditions (42), (43), (44), we are able to repeat step by step the calculations of [C-D 1, Lemma 9.3.1] and get (41). The proof is complete. ■

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