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ON THE ZEROS OF SOLUTIONS OF THE DIFFERENTIAL EQUATION $\omega^{(2m)} + p(z)\omega = 0$

1. We consider a linear differential equation of order n:

$$\omega^{(n)} + p_1(z)\omega^{(n-1)} + \ldots + p_n(z)\omega = 0, \tag{1}$$

where the complex-valued functions $p_k(z)$, $k = 1, 2, \ldots, n$ are analytic functions which are regular in a region D of the complex plane.

The differential equation (1) is said to be disconjugate in D if no nontrivial solution of (1) has more than $n-1$ zeros (where the zeros are counted with their multiplicities) in D. The equation (1) is said to be (m, m)-disconjugate in D if $n = 2m$ and if no nontrivial solution of (1) has two zeros of order m in D.

In [1] the following result for differential equations of arbitrary even order was obtained:

THEOREM A. *The differential equation*

$$\omega^{(2m)} + p(z)\omega = 0, \tag{2}$$

where the function $p(z)$ is analytic in $|z| < 1$, is (m, m)-disconjugate, if

$$|p(z)| \le \frac{B(2m)}{(1 - |z|^2)^{2m}}, \qquad |z| < 1,$$

where $B(2) = 1$, $B(4) = 9$ and

$$B(2m) = 9 \prod_{k=3}^{m} (4k - 3), \qquad m = 3, 4, \ldots$$

In [2] the following result was obtained:

THEOREM B. *The differential equation* (2), *where the function* $p(z)$ *is analytic in* $|z| < 1$, *is* (m,m)*-disconjugate, if*

$$|p(z)| \leq \frac{\prod_{k=1}^{m}(2k-1)^2}{(1-|z|^2)^{2m}}, \qquad |z| < 1.$$

In this paper using integral inequalities we prove the theorem:

THEOREM 1. *The differential equation* (2), *where the function* $p(z)$ *is analytic in* $|z| < 1$, *is* (m,m)*-disconjugate, if*

$$|p(z)| \leq \frac{\Gamma(m)}{(1-|z|^2)^m}, \qquad |z| < 1, \tag{3}$$

where

$$\Gamma(m) = \begin{cases} 2^n \prod_{k=1}^{m}(4k-1)(4k), & \text{if } m = 2n, \\ 2^{n+1}(2n+1)\prod_{k=1}^{m}(4k-1)(4k), & \text{if } m = 2n+1. \end{cases} \tag{4}$$

2. In a recent paper [3] there was established an integral inequality involving a function and its second derivative of the form

$$\int_I sh^2\,dt \leq \int_I rh''^2\,dt, \qquad h \in H, \tag{5}$$

where $I = (a,b)$, $-\infty \leq a < b \leq \infty$, r and s are real functions of the variable t, H is a class of functions absolutely continuous on I. We denote by $AC(I)$ the class of real functions defined and absolutely continuous on the interval I, and by $AC^1(I)$ the class of functions $f \in AC(I)$ such that $f' \in AC(I)$.

Let us take $I = (-1,1)$ and the function $r = (1-t^2)^{-\alpha}$ $(\alpha \geq 0)$. From Theorem [3] we obtain that the inequality of the form (5) holds:

LEMMA 1. *If* $\alpha \geq 0$ *and the function* $h \in AC^1((-1,1))$ *satisfies the integral condition*

$$\int_{-1}^{1} \frac{h''^2}{(1-t^2)^\alpha}\,dt < \infty$$

and the limit conditions

$$h(-1) = h'(-1) = h(1) = h'(1) = 0$$

then the inequality

$$\int_{-1}^{1} \frac{h''^2}{(1-t^2)^\alpha}\,dt \geq 2(2\alpha+3)(2\alpha+4)\int_{-1}^{1} \frac{h^2}{(1-t^2)^{\alpha+2}}\,dt \tag{6}$$

holds. The inequality (6) *becomes an equality if and only if* $h = c(1-t^2)^{\alpha+2}$, *where* $c = \text{const}$.

LEMMA 2. *If a nontrivial real function h of $C^m[-1,1]$ has two zeros of order m at $t=-1$ and $t=1$, then*

$$\int_{-1}^{1} [h^{(m)}]^2\,dt > \Gamma(m) \int_{-1}^{1} \frac{h^2}{(1-t^2)^m}\,dt, \tag{7}$$

where $\Gamma(m)$ is defined by (4).

Proof. By the inequality (6), we have the sequence of inequalities for $m=2n$:

$$\int_{-1}^{1} [h^{(2n)}]^2 dt \geq 2\cdot(2\cdot 0+3)(2\cdot 0+4) \int_{-1}^{1} \frac{[h^{(2n-2)}]^2}{(1-t^2)^2}\,dt,$$

$$\int_{-1}^{1} \frac{[h^{(2n-2)}]^2}{(1-t^2)^2}\,dt \geq 2\cdot(2\cdot 2+3)(2\cdot 2+4) \int_{-1}^{1} \frac{[h^{(2n-4)}]^2}{(1-t^2)^4}\,dt,$$

$$\cdots\cdots\cdots\cdots$$

$$\int_{-1}^{1} \frac{h''^2}{(1-t^2)^{2(n-1)}}\,dt \geq 2\cdot[2\cdot(2n-1)+3][2\cdot(2n-1)+4] \int_{-1}^{1} \frac{h^2}{(1-t^2)^{2n}}\,dt.$$

Multiplying these inequalities, we get

$$\int_{-1}^{1} [h^{(2n)}]^2\,dt \geq 2^n \prod_{k=1}^{n} (4k-1)(4k)\cdot \int_{-1}^{1} \frac{h^2}{(1-t^2)^{2n}}\,dt. \tag{8}$$

If $m=2n+1$, then by the inequality (6) we have

$$\int_{-1}^{1} [h^{(2n+1)}]^2\,dt \geq 2^n \prod_{k=1}^{n} (4k-1)(4k) \int_{-1}^{1} \frac{h'^2}{(1-t^2)^{2n}}\,dt. \tag{9}$$

But

$$\int_{-1}^{1} \frac{h'^2}{(1-t^2)^{2n}}\,dt \geq 2(2n+1) \int_{-1}^{1} \frac{h^2}{(1-t^2)^{2n+1}}\,dt. \tag{10}$$

Indeed, we have

$$0 \leq \int_{-1}^{1} \frac{1}{(1-t^2)^{2n}} \left(h' + \frac{2(2n+1)t}{1-t^2} h\right)^2 dt =$$

$$= \int_{-1}^{1} \frac{h'^2}{1-t^2)^{2n}}\,dt - 2(2n+1) \int_{-1}^{1} \frac{h^2}{(1-t^2)^{2n+1}}\,dt,$$

the last step following from an integration by parts. From the inequalities (8), (9) and (10) we have the inequality (7). Lemma is proved.

3. Now we prove Theorem 1 using Lemma 2.

Proof of Theorem 1. Suppose the theorem is false and there exists a solution $\omega(z)$ of (2) with two zeros, $z = z_1$, $z = z_2$ of order m in the unit disk. Then there exists a unique circle which passes through this two points and is orthogonal to the circle $|z| = 1$. The circle passing through z_1 and z_2 orthogonal to $|z| = 1$ is divided by $|z| = 1$ into two arcs. We denote the arc inside $|z| < 1$ by C. Without loss of generality, we may assume that C is on the upper half plane $\operatorname{Re} z \geq 0$, and is symmetric with respect to the imaginary axis $\operatorname{Im} z$ of the complex plane. In the opposite case by a rotation $\zeta = \alpha z$, $|\alpha| = 1$, the points z_1, z_2 can be brought into a position on the upper half plane symmetric with respect to the imaginary axis. Hence we will assume that the arc C is this position.

The linear transformation

$$\zeta = \frac{z - i\beta}{1 + i\beta z}, \quad 0 < \beta < 1 \tag{11}$$

maps $|z| < 1$ on $|\zeta| < 1$ and C on the linear segment $-1 < \zeta < 1$, and the equation (2) is transformed into the equation

$$y^{(2m)}(\zeta) + q(\zeta)y(\zeta) = 0, \tag{12}$$

where $y(\zeta) = (1 - i\beta\zeta)\omega(z(\zeta))$, and

$$q(\zeta) = p(z(\zeta))\left(\frac{dz}{d\zeta}\right)^{2m}. \tag{13}$$

Moreover, for a linear mapping the unit circle into itself, the relation

$$\left|\frac{dz}{d\zeta}\right| = \frac{1 - |z|^2}{1 - |\zeta|^2}$$

holds.

We now show that $|z| \geq |\zeta|$. Set $\zeta = \xi + i\eta$, then we have

$$|z|^2 - |\zeta|^2 = \left|\frac{\zeta + i\beta}{1 - i\beta\zeta}\right|^2 - |\zeta|^2 = \frac{|\zeta + i\beta|^2 - |\zeta|^2 \cdot |1 - i\beta\zeta|^2}{|1 - i\beta\zeta|^2} =$$
$$= \frac{\zeta^2 + \beta^2 - \zeta^2(1 + \beta^2\zeta^2)}{|1 - i\beta\zeta|^2} = \frac{\beta^2(1 - \zeta^4)}{|1 - i\beta\zeta|^2} \geq 0.$$

Since $|z|^2 \geq |\zeta|^2$, we have

$$\left|\frac{dz}{d\zeta}\right| = \frac{1 - |z|^2}{1 - |\zeta|^2} \leq 1. \tag{14}$$

Now using (13), (14) and (3) we estimate $q(\zeta)$

$$
\begin{aligned}
|q(\zeta)| &= |p(z(\zeta))| \cdot \left|\frac{dz}{d\zeta}\right|^{2m} \le \\
&\le \frac{\Gamma(m)}{(1-|\frac{\zeta+i\beta}{1-i\beta\zeta}|^2)^m} \cdot \left|\frac{1-\beta^2}{(1-i\beta\zeta)^2}\right|^m \cdot \left|\frac{dz}{d\zeta}\right|^m \le \\
&\le \frac{\Gamma(m)(1-\beta^2)^m}{[|1-i\beta\zeta|^2 - |\zeta+i\beta|^2]^m} = \\
&= \frac{\Gamma(m)(1-\beta^2)^m}{[(1-\beta^2)(1-\zeta^2)]^m} = \frac{\Gamma(m)}{(1-\zeta^2)^m}.
\end{aligned}
$$

This follows from the fact that the restriction (3) is translated to the restriction

$$
|p(\zeta)| \le \frac{\Gamma(m)}{(1-\zeta^2)^m}. \tag{15}
$$

The linear transformation (11) maps zeros z_1 and z_2 of the solution $\omega(z)$ equation (2) on the zeros $\zeta = -\rho$ and $\zeta = \rho$ of the solution $y(\zeta)$ of the equation (12), where $0 < \rho < 1$.

Multiplying the equation (12) by $\overline{y}(\zeta)d\zeta$ and integrating from $\zeta = -\rho$ to $\zeta = \rho$ along the real axis, we obtain

$$
(-1)^m \int_{-\rho}^{\rho} |y^{(m)}(\zeta)|^2 d\zeta = \int_{-\rho}^{\rho} q(\zeta)|y(\zeta)|^2 d\zeta. \tag{16}
$$

We set $y(\zeta) = u(\zeta) + iv(\zeta)$. Then

$$
|y(\zeta)|^2 = u^2(\zeta) + v^2(\zeta), \quad |y^m(\zeta)|^2 = [u^{(m)}(\zeta)]^2 + [v^{(m)}(\zeta)]^2,
$$

and (16) becomes

$$
(-1)^m \int_{-\rho}^{\rho} \left[\left(\frac{\partial^m u}{\partial \zeta^m}\right)^2 + \left(\frac{\partial^m v}{\partial \zeta^m}\right)^2\right] d\zeta = \int_{-\rho}^{\rho} q(\zeta)[u^2(\zeta) + v^2(\zeta)]d\zeta. \tag{17}
$$

Using (15), from (17) we get

$$
\begin{aligned}
\int_{-\rho}^{\rho} \left[\left(\frac{\partial^m u}{\partial \zeta^m}\right)^2 + \left(\frac{\partial^m v}{\partial \zeta^m}\right)^2\right] d\zeta &\le \int_{-\rho}^{\rho} |q(\zeta)|[u^2(\zeta) + v^2(\zeta)]d\zeta \le \\
&\le \Gamma(m) \int_{-\rho}^{\rho} \frac{u^2(\zeta) + v^2(\zeta)}{(1-\zeta^2)^m} d\zeta < \\
&< \Gamma(m)\rho^{2m} \int_{-\rho}^{\rho} \frac{u^2(\zeta) + v^2(\zeta)}{(\rho^2-\zeta^2)^m} d\zeta.
\end{aligned}
$$

However, this contradicts with the inequality (7), according to which

$$\Gamma(m)\int_{-\rho}^{\rho}\frac{u^2(\zeta)+v^2(\zeta)}{(1-\zeta^2)^m}d\zeta \le \int_{-\rho}^{\rho}\left[\left(\frac{\partial^m u}{\partial\zeta^m}\right)^2+\left(\frac{\partial^m v}{\partial\zeta^m}\right)^2\right]d\zeta,$$

of $u, v \in C^m[-\rho,\rho]$ and u and v have zeros of order m at $\zeta = -\rho$ and $\zeta = +\rho$. This contradiction proves the theorem.

Sufficient conditions of a different type can be obtained by means of the following results [4], [5]: If the function $p(z)$ is analytic in the unit disk $|z| < 1$, $z = x + iy$ and $|z'| < 1$, then

$$(18)\qquad |p(z')| \le \frac{\iint\limits_{|z|<1} |p(z)|dxdy}{\pi(1-|z|^2)^2},$$

and

$$(19)\qquad |p(z')| \le \frac{\int_0^{2\pi} |p(e^{i\theta})|d\theta}{2\pi(1-|z'|^2)}.$$

THEOREM 2. *If the function $p(z)$ is analytic in the unit disk $|z| < 1$, it is (m, m)-disconjugate if*

$$(20)\qquad \iint\limits_{|z|<1} |p(z)|^{2/m}dxdy \le \pi\sqrt[m]{\Gamma^2(m)},\qquad |z|<1$$

or

$$(21)\qquad \int_0^{2\pi} |p(e^{i\theta})|^{1/m}d\theta \le 2\pi\sqrt[m]{\Gamma(m)}.$$

Proof. From (18) and (3), we see that

$$|p(z)|^{2/m} \le \frac{\iint\limits_{|z|<1} |p(z)|^{2/m}dxdy}{\pi(1-|z|^2)^2} \le \frac{\Gamma^2(m)}{(1-|z|^2)^2}.$$

Therefore, we obtain (20). In an analogous way from (19) and (3) we obtain (21). This completes the proof.

4. Let us denote the non-Euclidean distance of any two points z_1 and z_2 in the unit disk $|z| < 1$ by $\Lambda(z_1, z_2)$. This distance is defined by

$$\Lambda(z_2, z_2) = \int_{[z_1 z_2]} \frac{|dz|}{1-|z|^2},$$

where the integration is along the orthogonal arc between z_1 and z_2 which we denote by $[z_1 z_2]$.

THEOREM 3. *Let $p(z)$ be a regular function $|z| < 1$ and assume that*

$$|p(z)| \le \frac{a}{(1-|z|^2)^m}, \qquad a > 1, \quad |z| < 1. \tag{22}$$

Let the nontrivial solution $\omega(z)$ of equation (2) have two zeros z_1 and z_2 of order m in unit disk $|z| < 1$.

Then

$$\Lambda(z_1, z_2) > \ln \frac{\sqrt[m]{a/\Gamma(m)} + 1}{\sqrt[m]{a/\Gamma(m)} - 1}. \tag{23}$$

Proof. We choose again the transformation (11) so that z_1 and z_2 go into $\zeta = \pm\rho$, $0 < \rho < 1$.

By the invariance of the non-Euclidean distance, we have

$$\Lambda(z_1, z_2) = \Lambda(-\rho, \rho) = \int_{-\rho}^{\rho} \frac{dx}{1-x^2} = \ln \frac{1+\rho}{1-\rho}.$$

Therefore, (23) will be established if we can show that

$$\log \frac{1+\rho}{1-\rho} > \ln \frac{\sqrt[m]{a/\Gamma(m)} + 1}{\sqrt[m]{a/\Gamma(m)} - 1} = \ln \frac{1 + \sqrt[m]{a/\Gamma(m)}}{1 - \sqrt[m]{a/\Gamma(m)}},$$

i.e., that

$$\rho > \sqrt[m]{\Gamma(m)/a}.$$

Assume, conversely, that

$$a\rho^m \le \Gamma(m). \tag{24}$$

This implies from $0 < \rho < 1$ that

$$(\rho^2 - x^2)^m = \rho^{2m}\left(1 - \left(\frac{x}{\rho}\right)^2\right)^m \le \rho^{2m}(1-x^2)^m. \tag{25}$$

Multiplying the last two inequalities (24) and (25) we obtain

$$a(\rho^2 - x^2)^m \le \rho^m \Gamma(m)(1-x^2)^m; \qquad -\rho \le x \le \rho$$

with equality possible only at $x = 0$. Therefore

$$\frac{a}{(1-x^2)^m} \le \frac{\rho^m \Gamma(m)}{(\rho^2 - x^2)^m}. \tag{26}$$

By the transformation (11), the equation (2) is transformed into (12) with a solution $y(\zeta) \not\equiv 0$, $y(\zeta) = u(\zeta) + iv(\zeta)$ such that the solution $y(\zeta)$ has two zeros $\zeta = -\rho$ and $\zeta = \rho$ of order m. In fact the condition (22) is again invariant with respect to the transformation (11), i.e.

$$|q(\zeta)| \le \frac{a}{(1-\zeta^2)^m}. \tag{27}$$

Multiplying the equation (12) by $\overline{y}(\zeta)d\zeta$ and integrating from $\zeta = -\rho$ to $\zeta = \rho$, we obtain

$$(28) \qquad (-1)^m \int_{-\rho}^{\rho} |y^m(\zeta)|^2 d\zeta = \int_{-\rho}^{\rho} q(\zeta)|y(\zeta)|^2 d\zeta.$$

Now using the inequalities (27) and (26) we estimate (28). Then we have

$$\int_{-\rho}^{\rho} ([u^{(m)}(\zeta)]^2 + [v^{(m)}(\zeta)]^2) d\zeta \leq$$

$$\leq \int_{-\rho}^{\rho} |q(\zeta)|[u^2(\zeta) + v^2(\zeta)] d\zeta \leq a \int_{-\rho}^{\rho} \frac{u^2(\zeta) + v^2(\zeta)}{(1-\zeta^2)^m} d\zeta <$$

$$< \rho^m \Gamma(m) \int_{-\rho}^{\rho} \frac{u^2(\zeta) + v^2(\zeta)}{(\rho^2-\zeta^2)^m} d\zeta < \Gamma(m) \int_{-\rho}^{\rho} \frac{u^2(\zeta) + v^2(\zeta)}{(\rho^2-\zeta^2)^m} d\zeta.$$

This gives a contradiction with the inequality (7). By (7) we have

$$\int_{-\rho}^{\rho} ([u^{(m)}(\zeta)]^2 + [v^{(m)}(\zeta)]^2) d\zeta \geq \Gamma(m) \int_{-\rho}^{\rho} \frac{u^2(\zeta) + v^2(\zeta)}{(\rho^2-\zeta^2)^m} d\zeta.$$

This contradiction proves the theorem.

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