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GENERATING RELATIONS INVOLVING HYPERGEOMETRIC FUNCTIONS BY MEANS OF INTEGRAL OPERATORS

Abstract. In this paper, the focus is on the results which involve exponential functions. The results of Pathan and Yasmeen [6] and Exton [3] are used with a view to obtaining generating functions which are partly unilateral and partly bilateral.

1. Introduction

Pathan and Yasmeen ([6]; p. 241 (1.2)) modified the result of Exton ([3]; p. 147(3)) in the form

$$\exp(s+t-\frac{xt}{s}) = \sum_{M=-\infty}^{\infty} \sum_{N=M^*}^{\infty} s^M t^N F_N^M(x), \tag{1.1}$$

where $M^* = \max\{0, -M\}$ and

$$F_N^M(x) = \frac{L_N^{(M)}(x)}{(M+N)!} = \begin{cases} \frac{1}{N!}\sum_{r=M^*}^{N} \frac{(-N)_r x^r}{(M+r)!r!} & \text{if } N \geq M^* \\ 0 & \text{if } 0 \leq N < M^*. \end{cases}$$

No factorials of negative integers occur in this definition, so all the terms have meaning.

Further, we know that if a three variables function $H(x,s,t)$ can be expanded in powers of t in the form

$$H(x,s,t) = \sum_{n=0}^{\infty} h_n f_n(x) g_n(s) t^n,$$

where h_n is independent of x, s and t, and $f_n(x)$ and $g_n(s)$ are different functions then $H(x,s,t)$ is called a bilateral generating function. We re-

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mark that the right hand side of result (1.1) is termed as partly unilateral and partly bilateral because one of the series from $-\infty$ to ∞ is bilateral and the second one is unilateral but the double series is neither bilateral nor unilateral. See, for example Exton [3] and Pathan and Yasmeen [6]. This result has been attracted a great deal of interest by several authors, for example, see Pathan and Yasmeen [6]–[8], Goyal and Gupta [1], Srivastava *et al.* [11]. An increasing number of such problems and properties are now capable of being elegantly represented by their use. A number of such generating functions are obtained in this paper. In Section 2, a theorem on Laplace transform is given. Further by invoking this theorem in Section 3, we derive generating relations involving hypergeometric functions and polynomials of Jacobi, Bessel and Schultz-Piszachich which are partly unilateral and partly bilateral.

2. Theorem on Laplace transform

THEOREM. *If* $Re(p) > 0$, $Re(p-s-t+\frac{yt}{s}) > 0$ *and* $L[f(u):p] = \phi(p)$, *then*

$$\phi\left(p-s-t+\frac{yt}{s}\right) = \sum_{M=-\infty}^{\infty} \sum_{N=M^*}^{\infty} \frac{s^M t^N}{(M+N)!} L[u^{M+N} L_N^{(M)}(yu) f(u);p], \tag{2.1}$$

$$M^* = \max\{0, -M\},\ N \geq M^*,$$

provided that $|f(u)|$ *and* $\left|u^{M+N} L_N^{(M)}(yu) f(u)\right|$ *exist and series involved in* (2.1) *are absolutely convergent.*

Proof. Since $\phi(p) = L[f(u);p]$ then on using ([2]; p. 129(5)), we have

$$\phi(p+a) = L[e^{-au} f(u);p].$$

Therefore for $a = -s-t+\frac{yt}{s}$, we get

$$\phi\left(p-s-t+\frac{yt}{s}\right) = L[e^{(s+t-\frac{yt}{s})u} f(u);p].$$

Now using the result (1.1), we obtain (2.1).

3. Applications

We shall now apply theorem of Section 2, to obtain the generating relations which are partly unilateral and partly bilateral.

Let

$$f(u) = u^{n+\mu-2}(1-xu)^n.$$

Now consider the polynomial $P_n^{(\mu)}(\frac{1}{p})$ of degree n ([5]; p. 118) defined as

$$(3.1)\quad P_n^{(\mu)}\left(\frac{1}{p}\right) = (-1)^n e^{-p} p^{n+\mu-1} \frac{d^n}{dp^n}(e^p p^{-n-\mu+1})$$
$$= \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{n-k}(n+\mu-1)\dots(n+\mu+k-2)}{p^k}.$$

The generating function for the polynomial $P_n^{(\mu)}(x)$ is given by

$$(3.2)\qquad \frac{2^{\mu-2}\exp((\frac{1}{2x})(\sqrt{(1-4tx)}-1))}{(\sqrt{(1-4tx)}+1)^{\mu-2}\sqrt{(1-4tx)}} = \sum_{n=0}^{\infty} P_n^{(\mu)}(x)\frac{t^n}{n!},$$

and we note that the polynomial $P_n^{(\mu)}(x)$ has the following integral representation ([5]; p. 125 (5.2.10))

$$(3.3)\qquad P_n^{(\mu)}(x) = \frac{(-1)^n}{\Gamma(n+\mu-1)} \int_0^{\infty} u^{n+\mu-2}(1-xu)^n e^{-u}\, du.$$

Using this integral representation with u replaced by pu and x replaced by $\frac{x}{p}$, expanding $(1-xu)^n$, and using the results ([2]; P.174(29)) and (2.1),we obtain (after making suitable adjustment in parameters and taking $p=1$)

$$(3.4)\quad (1-s-t+\frac{yt}{s})^{1-n-\mu} P_n^{(\mu)}\left(\frac{x}{(1-s-t+\frac{yt}{s})}\right)$$
$$= \sum_{M=-\infty}^{\infty} \sum_{N=M^*}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{n+k}\binom{n}{k}(n+\mu-1)_{M+2N+k}}{N!(M+N)!}[(1-y)t]^N x^k s^M$$
$$\times\, {}_2F_1\left[\begin{matrix}-N, 2-N-n-\mu-k : \frac{1}{1-y} \\ 2-M-2N-n-\mu-k;\end{matrix}\right],$$
$$Re\left(1-s-t+\frac{yt}{s}\right) > 0,\ Re(\mu) > 1,\ y > 0.$$

Now using the relation between hypergeometric function ${}_2F_1$ and Jacobi polynomial $P_n^{(m,c)}(x)$ ([9]; p. 255 (9)), equation (3.4) can alternatively be written as

$$(3.5)\quad (1-s-t+\frac{yt}{s})^{1-n-\mu} P_n^{(\mu)}\left(\frac{x}{(1-s-t+\frac{yt}{s})}\right)$$
$$= \sum_{M=-\infty}^{\infty} \sum_{N=M^*}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{n+k}\binom{n}{k}(n+\mu-1)_{M+N+k}}{(M+N)!}$$
$$\times\, x^k s^M t^N P_N^{(M,n+\mu+k-2)}(1-2y),$$
$$Re(\mu) > 1, y > 0.$$

The following special cases of (3.5) are worthy of note.

I. On taking $s = t = \frac{y}{2}$, equation (3.5) gives us

$$(3.6)\quad P_n^{(\mu)}(x) = \sum_{M=-\infty}^{\infty} \sum_{N=M^*}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{n+k}\binom{n}{k}(n+\mu-1)_{M+N+k}x^k(\frac{y}{2})^{M+N}}{(M+N)!} \times P_N^{(M,n+\mu+k-2)}(1-2y), \quad Re(\mu) > 1,\ y > 0.$$

Since $P_0^{(2)}(x) = 1$, so for $n = 0$ and $\mu = 2$, equations (3.5) and (3.6) reduce to special cases of ([6]; p.242(2.2)) and ([6]; p.242(2.3)) respectively, (for $a = c = 1$ and $x = y$).

II. Taking $s = t = \frac{y}{2}$, $\mu = 2$ and replacing x by $\frac{-1}{2x}$ in (3.5), we get

$$(3.7)\quad P_n^{(2)}\left(\frac{-1}{2x}\right) = \sum_{M=-\infty}^{\infty} \sum_{N=M^*}^{\infty} \sum_{k=0}^{n} \frac{(-1)^n\binom{n}{k}(n+1)_{M+N+k}\left(\frac{y}{2}\right)^{M+N}}{(M+N)!2^k x^k} \times P_N^{(M,n+k)}(1-2y), \quad y > 0.$$

An important consequence of (3.7) concerns

$$(3.8)\quad P_n^{(2)}\left(\frac{-1}{2x}\right) = \frac{S_n(x)}{(-x)^n},$$

and

$$(3.9)\quad P_n^{(2)}\left(\frac{-x}{2}\right) = (-1)^n y_n(x),$$

where $S_n(x)$ are the polynomials introduced by Schultz-Piszachich [10] and their series representaton is given by Werner and Pietzch ([12]; p. 167 (9)), and $y_n(x)$ are the familiar Bessel polynomials ([4]; p. 101(3)). By use of relation (3.8), equation (3.7) becomes

$$S_n(x) = \sum_{M=-\infty}^{\infty} \sum_{N=M^*}^{\infty} \sum_{k=0}^{n} \frac{\binom{n}{k}(n+1)_{M+N+k}x^{n-k}2^{-k}\left(\frac{y}{2}\right)^{M+N}}{(M+N)!} \times P_N^{(M,n+k)}(1-2y), \quad y > 0,$$

which after replacing x by $\frac{1}{x}$ and using the relation (3.9) yields

$$y_n(x) = \sum_{M=-\infty}^{\infty} \sum_{N=M^*}^{\infty} \sum_{k=0}^{n} \frac{\binom{n}{k}(n+1)_{M+N+k}(\frac{x}{2})^k\left(\frac{y}{2}\right)^{M+N}}{(M+N)!} \times P_N^{(M,n+k)}(1-2y), \quad y > 0.$$

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