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A NOTE ON OSTROWSKI TYPE INEQUALITIES

Abstract. In the present note we establish two new integral inequalities of the Ostrowski type involving a function of one independent variable. The discrete analogues of the main results are also given.

1. Introduction

In the year 1938, A. Ostrowski [1, p. 297], established the following interesting inequality.

Let f be differentiable function on (a, b) and on (a, b) $|f'(x)| \leq M$. Then for every $x \in (a, b)$

$$(1) \qquad \left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)M.$$

Over the years, this result has evoked considerable interest and many variants, generalizations and extensions has been appeared in the literature (see [2, pp. 468–484]). The aim of the present note is to establish two integral inequalities of the Ostrowski type involving a function of one independent variable. The discrete analogues of the main results are also given. The analysis used in the proofs is elementary and our results provide new estimates on inequalities of this type.

2. Main results

In what follows R denotes the set of real numbers and N the set of natural numbers. For a function $f : N \to R$ we define the difference operator Δ by $\Delta f(n) = f(n+1) - f(n)$.

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Our main results are given in the following theorems.

THEOREM 1. *Let $f : [a,b] \to R$ is C^2. Then we have the following inequalities:*

$$\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)[f(a)+f(b)]\right| \le \frac{1}{2}(b-a)\int_a^b |f'(x)|\,dx, \tag{2}$$

$$\left|\int_a^b f^2(x)\,dx - \frac{1}{2}(b-a)[f^2(a)+f^2(b)]\right| \le (b-a)\int_a^b |f(x)f'(x)|\,dx. \tag{3}$$

Proof. It is easy to observe that the following identities hold [3]

$$f(x) = f(a) + \int_a^x f'(t)\,dt, \tag{4}$$

$$f(x) = f(b) - \int_x^b f'(t)\,dt, \tag{5}$$

$$f^2(x) = f^2(a) + 2\int_a^x f(t)f'(t)\,dt, \tag{6}$$

$$f^2(x) = f^2(b) - 2\int_x^b f(t)f'(t)\,dt. \tag{7}$$

From (4), (5) and (6), (7) we have

$$f(x) = \frac{f(a)+f(b)}{2} + \frac{1}{2}\int_a^x f'(t)\,dt - \frac{1}{2}\int_x^b f'(t)\,dt, \tag{8}$$

$$f^2(x) = \frac{f^2(a)+f^2(b)}{2} + \int_a^x f(t)f'(t)\,dt - \int_x^b f(t)f'(t)\,dt. \tag{9}$$

Integrating both sides of (8) and (9) on (a,b) and by making elementary calculations we get respectively the required inequalities in (2) and (3). The proof is complete.

REMARK 1. From (8) and (9) it is easy to observe that the following inequalities also hold

$$\left|f(x) - \frac{f(a)+f(b)}{2}\right| \le \frac{1}{2}\int_a^b |f'(x)|\,dx, \tag{10}$$

$$\left|f^2(x) - \frac{f^2(a)+f^2(b)}{2}\right| \le \int_a^b |f(x)f'(x)|\,dx, \tag{11}$$

for $x \in (a,b)$.

The discrete versions of inequalities (2) and (3) are embodied in the following theorem.

THEOREM 2. *Let* $\{x_i\}_{i=1}^{n}$ *be a sequence of real numbers. Then the following inequalities hold*

$$\left|\sum_{i=0}^{n-1} x_i - n\left(\frac{x_0 + x_n}{2}\right)\right| \le \frac{1}{2} n \sum_{i=0}^{n-1} |\Delta x_i|, \tag{12}$$

$$\left|\sum_{i=0}^{n-1} x_i^2 - n\left(\frac{x_0^2 + x_n^2}{2}\right)\right| \le n \sum_{i=0}^{n-1} |(x_{i+1} + x_i)\Delta x_i|. \tag{13}$$

Proof. It is easy to observe that the following identities hold [4]

$$x_i = x_0 + \sum_{j=0}^{i-1} \Delta x_j, \tag{14}$$

$$x_i = x_n - \sum_{j=i}^{n-1} \Delta x_j, \tag{15}$$

$$x_i^2 = x_0^2 + \sum_{j=0}^{i-1} (x_{j+1} + x_j)(\Delta x_j), \tag{16}$$

$$x_i^2 = x_n^2 - \sum_{j=i}^{n-1} (x_{j+1} + x_j)(\Delta x_j). \tag{17}$$

From (14), (15) and (16), (17) we have

$$x_i = \frac{x_0 + x_n}{2} + \frac{1}{2}\sum_{j=0}^{i-1} \Delta x_j - \frac{1}{2}\sum_{j=i}^{n-1} \Delta x_j, \tag{18}$$

$$x_i^2 = \frac{x_0^2 + x_n^2}{2} + \frac{1}{2}\sum_{j=0}^{i-1} (x_{j+1} + x_j)(\Delta x_j) - \frac{1}{2}\sum_{j=i}^{n-1} (x_{j+1} + x_j)(\Delta x_j) \tag{19}$$

Summing both sides of (18) and (19) from 0 to $n - 1$ and by making elementary calculations we get the required inequalities in (12) and (13). The proof is complete.

REMARK 2. From (18) and (19) it is also easy to obtain the following inequalities

$$\left| x_i - \frac{x_0 + x_n}{2} \right| \le \frac{1}{2} \sum_{j=0}^{n-1} |\Delta x_j|, \tag{20}$$

$$\left| x_i^2 - \frac{x_0^2 + x_n^2}{2} \right| \le \frac{1}{2} \sum_{j=0}^{n-1} |(x_{j+1} + x_j)(\Delta x_j)|, \tag{21}$$

for $i = 1, 2, \ldots, n-1$.

References

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