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ORDER ALGEBRAS

Abstract. To every ordered set with a greatest element is assigned an order algebra, a Hilbert algebra occuring in intuicionistic logic. We prove some basic properties of order algebras and characterize their ideals and congruences. We introduce a concept of (relative) annihilator and show that it is a (relative) pseudocomplement in the lattice of all ideals.

Ordered sets as a basic algebraic structure were treated from various points of view. We introduce an approach mentioned by A. Diego [6] such that to every ordered set is assigned the so called order algebra and we will study its properties. This method is in fact that which was used for graphs where the so called graph algebras were introduced. The same appproach was used also for BCK-algebras inherited from posets by Tanaka in 1975 and by Kim in 1997, see also Romanowska and Traczyk [9], [10]. The aim of this paper is to investigate certain algebraic properties of order algebras and to show certain connections with algebras of mathematical logic.

DEFINITION 1. Let $(P, \leq)$ be an ordered set with a greatest element 1. Introduce a binary operation on P as follows:

$$\text{if } x \leq y \text{ then } x \cdot y = 1 \text{ and } x \cdot y = y \text{ otherwise.}$$

An algebra $\mathcal{P} = (P; \cdot, 1)$ of type $(2,0)$ is called an *order algebra (assigned to* $(P, \leq)$*)*.

REMARK 1. It is evident that an order algebra can be assigned to an arbitrary ordered set (not necessarily having a greatest element). If $(P, \leq)$ has not a greatest element, then an order algebra $\mathcal{P}$ has a base set $P_1 = P \cup \{1\}$, where $1 \notin P$ is a new element. Not distinguishing whether $(P, \leq)$ has or has not a greatest element and $\mathcal{P}$ is based on P or P_1, we will consider only ordered sets with 1. As mentioned, it is without loss of generality.

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The following three results can be easily proved directly by Definition 1 and they describe extremally simple structure of order algebras.

PROPOSITION 1. *Let $(P, \leq)$ be an ordered set with a greatest element 1 and $\mathcal{P} = (P; \cdot, 1)$ the assigned order algebra. For every subset $A \subseteq P$, the set $A \cup \{1\}$ is a universum of a subalgebra of $\mathcal{P}$ generated by A. Hence, the lattice Sub $\mathcal{P}$ of subalgebras of $\mathcal{P}$ is Boolean and if $|P| = n$ then Sub $\mathcal{P} \simeq \mathbf{2}^{n-1}$.*

Order algebras are connected with algebras modelling a logic connective implication in intuicionistic logic. For to show it, let us recall the concept of Hilbert algebra.

DEFINITION 2. A *Hilbert algebra* is an algebra $\mathcal{H} = (H; \cdot, 1)$ of type (2,0) satisfying the following three axioms:

(i) $x \cdot (y \cdot x) = 1$
(ii) $(x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1$
(iii) $x \cdot y = 1$ and $y \cdot x = 1$ imply $x = y$.

Let us note that Hilbert algebras were introduced in early 50-ties by L. Henkin and T. Skolem for some investigations in intuicionistic and other non-classical logics. In 60-ties, these algebras were studied by A. Horn and A. Diego from an algebraic point of view. A. Diego [6] proved that Hilbert algebras form a variety which is locally finite. Hilbert algebras were recently treated also by D. Busenag [3], W.A. Dudek [7] and the authors [5].

PROPOSITION 2. *For every ordered set $(P, \leq)$ with a greatest element* 1, *the assigned order algebra $\mathcal{P} = (P; \cdot, 1)$ is a Hilbert algebra.*

Proof. See e.g. [6]. ∎

Applying results of Hilbert algebras in [6] (collected also in [5] or [7]), we can state immediately:

COROLLARY. *For every order algebra the following assertions hold:*

(1) $x \cdot x = 1$
(2) $1 \cdot x = x$
(3) $x \cdot 1 = 1$
(4) $x \cdot (y \cdot z) = y \cdot (x \cdot z)$
(5) $x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$
(6) $x \leq y \Rightarrow z \cdot x \leq z \cdot y$ *and* $y \cdot z \leq x \cdot z$
(7) $x \leq (x \cdot y) \cdot y$
(8) $x \cdot y \leq (y \cdot z) \cdot (x \cdot z)$
(9) $y \cdot z \leq (x \cdot y) \cdot (x \cdot z)$.

By a "language of order algebras" we mean the first order language with added symbols $=, \leq, \cdot, 1$.

Denote by $\mathcal{W}$ the class of all order algebras. It is easy to see that $\mathcal{W}$ is closed under subalgebras and homomorphic images but not under direct products, i.e. $\mathcal{W}$ is not a variety or quasivariety. On the other hand, $\mathcal{W}$ can be easily described by formulas of our language:

PROPOSITION 3. *An algebra* $\mathcal{P} = (P; \cdot, 1)$ *of type* $(2,0)$ *is an order algebra if and only if it satisfies the following axioms*:

(i) $x \cdot (y \cdot x) = 1$
(ii) $(x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1$
(iii) $x \cdot y = 1$ *and* $y \cdot x = 1$ *imply* $x = y$
(iv) $x \cdot y \neq 1$ *imply* $x \cdot y = y$.

If $\mathcal{P}$ is a Hilbert algebra then (see [3], [6], [7]) the relation $\leq$ defined by

$$x \leq y \text{ if and only if } x \cdot y = 1 \tag{$*$}$$

is an order relation on P.

REMARK 2. The axiom (i) occuring in Proposition 3 can be replaced by two more simple ones:

$$x \cdot x = 1 \quad \text{and} \quad 1 \cdot x = x. \tag{$**$}$$

In fact, (i) ensures reflexivity and (ii) ensures transitivity of the relation introduced by $(*)$. Of course, (iii) ensures antisymmetry. Moreover, (i) ensures that 1 is a greatest element. Evidently, $(**)$ says that $\leq$ defined by $(*)$ is reflexive and 1 is a greatest element.

REMARK 3. Let $(P, \leq)$ be an ordered set with a greatest element and $\mathcal{P} = (P; \cdot, 1)$ its assigned order algebra. We can express in the language of order algebras whether $x, y \in P$ have an infimum or supremum:

(A) if $\exists z \in P$ such that $x \cdot z = 1 = y \cdot z$ and $x \cdot t = 1 = y \cdot t \Rightarrow z \cdot t = 1$ then $z = \sup(x, y)$,

(B) if $\exists w \in P$ such that $w \cdot x = 1 = w \cdot y$ and $t \cdot x = 1 = t \cdot y \Rightarrow t \cdot w = 1$ then $w = inf(x, y)$.

Hence, we can express the fact whether $(P, \leq)$ is a semilattice or a lattice in terms of an assigned order algebra.

Another algebra describing a more restrictive connective implication was introduced by J.C. Abbott [1] under the name implication algebra:

DEFINITION 3. An *implication algebra* is a grupoid $(A, \cdot)$ satisfying the following identities:

(a) $(x \cdot y) \cdot x = x$
(b) $(x \cdot y) \cdot y = (y \cdot x) \cdot x$
(c) $x \cdot (y \cdot z) = y \cdot (x \cdot z)$.

It was shown by J. C. Abbott [1] that every implication algebra satisfies the identity

$$x \cdot x = y \cdot y$$

thus $x \cdot x$ is an algebraic constant of $(A, \cdot)$; it is denoted by 1. As it was proved by A. Diego [6], every implication algebra is a Hilbert algebra. Hence, there is a natural question which order algebra is an implication algebra.

THEOREM 1. *Let $(P, \leq)$ be an ordered set with a greatest element. The assigned order algebra $\mathcal{P} = (P; \cdot, 1)$ is an implication algebra if and only if $(P, \leq)$ does not contain a chain of length $n > 2$.*

Proof. Suppose that $(P, \leq)$ contains at least three element chain, i.e. there are $a, b \in P$ with $a < b < 1$. Then

$$(b \cdot a) \cdot b = a \cdot b = 1 \neq b,$$

i.e. $\mathcal{P}$ does not satisfy (a) of Definition 3.

Conversely, suppose that $(P, \leq)$ does not contain at least three element chain, i.e. $a \| b$ for any distinct elements $a, b \in P$ with $a \neq 1 \neq b$. We prove the axioms of Definition 3.

ad(a):

if $x \| y$ then $(x \cdot y) \cdot x = y \cdot x = x$
if $y = 1$ then $(x \cdot 1) \cdot x = 1 \cdot x = x$
if $x = 1$ then $(1 \cdot y) \cdot 1 = y \cdot 1 = 1 = x$.

ad (b):

if $x \| y$ then $(x \cdot y) \cdot y = 1 = (y \cdot x) \cdot x$
if $y = 1$ then $(x \cdot 1) \cdot 1 = 1 = x \cdot x = (1 \cdot x) \cdot x$
if $x = 1$ then $(1 \cdot y) \cdot y = y \cdot y = 1 = (y \cdot 1) \cdot 1$.

ad (c): Suppose $y \not\leq z$.

If $x \not\leq z$ then $x \cdot (y \cdot z) = x \cdot z = z$ and

$$y \cdot (x \cdot z) = y \cdot z = z;$$

if $x < z$ then $z = 1$ and $x \cdot (y \cdot z) = x \cdot 1 = 1$

$$y \cdot (x \cdot z) = y \cdot 1 = 1;$$

if $x = z$ then $x \cdot (y \cdot z) = x \cdot z = x \cdot x = 1$

$$y \cdot (x \cdot z) = y \cdot (x \cdot x) = y \cdot 1 = 1.$$

Suppose $y < z$. Then $z = 1$ and

$$x \cdot (y \cdot z) = x \cdot 1 = 1, \quad y \cdot (x \cdot z) = y \cdot 1 = 1.$$

Suppose $y = z$. Then $x \cdot (y \cdot z) = x \cdot (y \cdot y) = x \cdot 1 = 1$;
if $x \leq y$ then $x \leq z$ and $y \cdot (x \cdot z) = y \cdot (x \cdot y) = y \cdot y = 1$
if $x \not\leq y$ then $y \cdot (x \cdot z) = y \cdot (x \cdot y) = y \cdot y = 1$.

In all the cases we infer $x \cdot (y \cdot z) = y \cdot (x \cdot z)$. Altogether, $(P, \cdot)$ is an implication algebra. ∎

REMARK 4. We have shown that an order algebra $\mathcal{P}$ assigned to $(P, \leq)$ is an implication algebra if and only if the diagram of $(P, \leq)$ looks as in Fig.1.

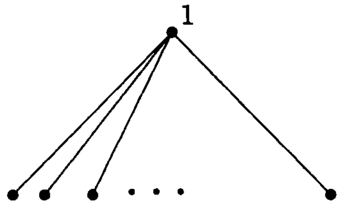

Fig. 1.

For an ordered set $(A, \leq)$, a non-void subset $F \subseteq A$ is called an *order-filter* if $x \in F$ and $x \leq y$ entail $y \in F$. If $B \subseteq A$, we denote by $F(B)$ an order-filter generated by B, i.e.

$$F(B) = \{x \in A; b \leq x \text{ for some } b \in B\}.$$

If $B = \{b\}$, we will write briefly $F(b)$ instead of $F(\{b\})$.

The following concept was introduced for Hilbert algebras in [5]:

DEFINITION 4. A non-void subset I of a Hilbert algebra $\mathcal{H} = (H; \cdot, 1)$ is called an *ideal of* $\mathcal{H}$ if

(I1) $x \in H, y \in I$ imply $x \cdot y \in I$
(I2) $x \in H, y_1, y_2 \in I$ imply $(y_2 \cdot (y_1 \cdot x)) \cdot x \in I$.

It is immediately clear that $1 \in I$ for every ideal I of $\mathcal{H}$ because $I \neq \emptyset$ and when $a \in I$ then, by (I1), also $1 = a \cdot a \in I$. We can describe connections of ideals of order algebras and filters of their ordered sets:

THEOREM 2. *Let $(P, \leq)$ be an ordered set with a greatest element and $\mathcal{P} = (P; \cdot, 1)$ be its assigned order algebra. A non-void subset $I \subseteq P$ is an ideal of $\mathcal{P}$ if and only if I is an order-filter of $(P, \leq)$.*

Proof. Let I be an ideal of $\mathcal{P} = (P; \cdot, 1)$. Immediately by (I1), I is an order-filter of $(P, \leq)$.

Conversely, suppose that I is an order-filter of $(P, \leq)$. If $x \in P$ and $y \in I$ then $x \leq 1$ and, by (6) of the Corollary, $y = 1 \cdot y \leq x \cdot y$, i.e. also $x \cdot y \in I$. Hence I satisfies (I1). If $a \in I$ and $a \cdot b \in I$ for $a, b \in P$ then:

if $a \not\leq b$ then $b = a \cdot b \in I$

if $a \leq b$ then $b \in I$, because I is an order-filter; we have shown

$$a \in I \text{ and } a \cdot b \in I \text{ imply } b \in I. \qquad (***)$$

Now, by (4) and (1) of the Corollary,

$$y \cdot ((y \cdot z) \cdot z) = (y \cdot z) \cdot (y \cdot z) = 1$$

thus $y \leq (y \cdot z) \cdot z$. Hence, if $y \in I$ thus also $(y \cdot z) \cdot z \in I$.

Consider $y_1, y_2 \in I$. In account of the Corollary, we infer for any $x \in P$

$$y_2 \cdot [(y_1 \cdot (y_2 \cdot x)) \cdot x] = (y_1 \cdot (y_2 \cdot x)) \cdot (y_2 \cdot x) = (y_1 \cdot z) \cdot z \in I$$

(where $z = y_2 \cdot x$). Applying $(***)$, we obtain $(y_1 \cdot (y_2 \cdot x)) \cdot x \in I$ proving (I2), i.e. I is an ideal of $\mathcal{P}$. ∎

The following concepts were introduced in [8] and [4].

Let $\mathcal{A} = (A, F)$ be an algebra. A congruence $\Theta \in Con\,\mathcal{A}$ is called a *Rees congruence* if it has at most one class which is not a singleton.

Let $\mathcal{A} = (A, F)$ be an algebra with a constant (e.g. denoted by 1) where the concept of ideal is defined. $\mathcal{A}$ is called a *Rees ideal algebra* if for each $\Theta \in Con\,\mathcal{A}$ the class $[1]_\Theta$ can be the only nonsingleton class which is an ideal of $\mathcal{A}$ and, moreover, for every ideal I of $\mathcal{A}$

$$\omega \cup (I \times I) \in Con\,\mathcal{A}$$

(where ω denotes the identity relation on A).

THEOREM 3. *Let $\mathcal{P} = (P; \cdot, 1)$ be an order algebra (assigned to $(P, \leq)$). For every ideal I of $\mathcal{P}$ there exists a unique congruence $\Theta \in Con\,\mathcal{P}$ such that $[1]_\Theta = I$. Moreover, this Θ is a Rees congruence and $\mathcal{P}$ is a Rees ideal algebra.*

Proof. Let I be an ideal of $\mathcal{P}$. Define a relation Θ_I on P as follows

$$\langle x, y\rangle \in \Theta_I \text{ if and only if } x = y \text{ or } x, y \in I.$$

Evidently, Θ_I is an equivalence on P and, since I is an order-filter of $(P, \leq)$, one can easily check that $\Theta_I \in Con\,\mathcal{P}$. Of course, $[1]_{\Theta_I} = I$. To prove that Θ_I is a unique congruence on $\mathcal{P}$ having the class I, we need only to show that every $\Theta \in Con\mathcal{P}$ is a Rees congruence.

Suppose $\Theta \in Con\,\mathcal{P}$. It is almost evident that $[1]_\Theta$ is an order-filter of $(P, \leq)$ and, by Theorem 2, it is an ideal of $\mathcal{P}$.
Suppose $\langle x, y\rangle \in \Theta$ for $x \neq y$. Then

$$\langle 1, x \cdot y\rangle = \langle x \cdot x, x \cdot y\rangle \in \Theta \text{ and}$$

$$\langle 1, y \cdot x\rangle = \langle x \cdot x, y \cdot x\rangle \in \Theta.$$

Since $x \neq y$, we have either $x \cdot y = y$ or $y \cdot x = x$, thus either $\langle 1, x\rangle \in \Theta$ or $\langle 1, y\rangle \in \Theta$. Due to transitivity of Θ, we have $x, y \in [1]_\Theta$ in the both possible cases. Thus Θ is a Rees congruence. Altogether, we have also shown that $\mathcal{P}$ is a Rees ideal algebra. ∎

REMARK 5. Let $\mathcal{P} = (P; \cdot, 1)$ be an order algebra and I its ideal. By Theorem 3, there is a unique $\Theta_I \in Con\,\mathcal{P}$ such that $[1]_{\Theta_I} = I$. Applying the method of the previous proof, Θ_I can be described in the language of order algebras as follows:

$$\langle x, y\rangle \in \Theta_I \text{ if and only if } x \cdot y \in I \text{ and } y \cdot x \in I.$$

We can study the congruence properties of order algebras. Let us recall the basic concepts. We say that an algebra $\mathcal{A}$ is

— *permutable* if $\Theta \circ \Phi = \Phi \circ \Theta$ for each $\Theta, \Phi \in Con\,\mathcal{A}$

— *congruence distributive* if $\Theta \cap (\Phi \vee \Psi) = (\Theta \cap \Phi) \vee (\Theta \cap \Psi)$ for each $\Theta, \Phi, \Psi \in Con\,\mathcal{A}$

— *congruence regular* if $[a]_\Theta = [a]_\Phi$ for some a of $\mathcal{A}$ implies $\Theta = \Phi$ for each $\Theta, \Phi \in Con\,\mathcal{A}$.

— *weakly regular* if $\mathcal{A}$ has a constant 1 and $[1]_\Theta = [1]_\Phi$ implies $\Theta = \Phi$ for each $\Theta, \Phi \in Con\,\mathcal{A}$.

THEOREM 4. *Every order algebra is permutable, congruence distributive and weakly regular. It is regular if and only if it has at most two elements.*

Proof. Let $(P, \leq)$ be an ordered set with 1 and $\mathcal{P} = (P; \cdot, 1)$ its assigned order algebra. Let I, J be ideals of $\mathcal{P}$. Since I, J are order-filters of $(P, \leq)$, also $I \cup J$ is an order-filter and hence ideal of $\mathcal{P}$. It is an easy exercise to prove that

$$\Theta_I \circ \Theta_J = \Theta_{I \cup J}$$

where these congruences are those having its 1-classes equal to I, J and $I \cup J$. Since $\mathcal{P}$ is a Rees ideal algebra, this yields immediately that $\mathcal{P}$ is permutable. In other words, we have $\Theta_I \vee \Theta_J = \Theta_{I \cup J}$ in $Con\,\mathcal{P}$. However, it is clear that $\Theta_I \cap \Theta_J = \Theta_{I \cap J}$ which immediately implies that $Con\,\mathcal{P}$ is distributive. By Theorem 3 we know that $\mathcal{P}$ is weakly regular.

Of course, if $|P| \leq 2$ then $\mathcal{P}$ is simple and hence congruence regular. On the contrary, if $|P| > 2$ then there exist $x, y \in P$, $y \neq 1$, with $x \notin F(y)$. By Theorem 3, $[x]_\Theta = \{x\}$ for $\Theta = \omega \cup (F(y) \times F(y)) \in Con\,\mathcal{P}$, i.e. $[x]_\Theta = [x]_\omega$ but $\Theta \neq \omega$, i.e. $\mathcal{P}$ is not congruence regular. ∎

The last concepts treated here will be annihilators of order algebras. The concept of an annihilator was already introduced and studied for commutative BCK-algebras in [2].

DEFINITION 5. Let $(P, \leq)$ be an ordered set and $\mathcal{P} = (P; \cdot, 1)$ its assigned order algebra. Let B, C be non-void subsets of P. The set

$$\langle C\rangle = \{x \in P; x \cdot c = c \text{ for each } c \in C\}$$

is called an *annihilator of* C. The set

$$\langle C, B\rangle = \{x \in P; (x \cdot c) \cdot c \in B \text{ for each } c \in C\}$$

is called a *relative annihilator of* C *with respect to* B.

If $C = \{c\}$ then $\langle C\rangle$ will be denoted briefly by $\langle c\rangle$.

For an ordered set $(P, \leq)$ and $\emptyset \neq M \subseteq P$ we denote by $U(M) = \{x \in P; p \leq x \text{ for each } p \in M\}$.

LEMMA. *Let* $(P, \leq)$ *be an ordered set with a greatest element and* $\mathcal{P} = (P; \cdot, 1)$ *its order algebra. For each non-void* $C \subseteq P$ *and every* $b \in P$ *we have*

$$\langle C\rangle = \{x \in P; x \not\leq c \textit{ for each } c \in C\backslash\{1\}\}$$

and $\langle F(B)\rangle = \{x \in P; U(x, b) = \{1\}\}$.

Proof. The first assertion follows directly from the definition of annihilator and the fact that $x \cdot c = c$ for $c \neq 1$ if and only if $x \not\leq c$. For the second assertion, it is clear that $F(b)$ contains all elements $y \in P$ such that $b \leq y$ and, by using of the first statement, $\langle F(b)\rangle$ is composed of all $x \in P$ which are non-comparable with every $y \neq 1$ such that $b \leq y$, i.e. $U(x, b) = \{1\}$. ∎

REMARK 6. The previous Lemma justifies the name "annihilator". Namely, for $b \in P$ the annihilator $\langle F(b)\rangle$ is the set of those $x \in P$ which "annihilate" every $y \in F(b)$ with respect to 1.

THEOREM 5. *Let* $\mathcal{P} = (P; \cdot, 1)$ *be an order algebra assigned to* $(P, \leq)$ *and* $b \in P$. *Then* $\langle b\rangle$ *is an ideal of* $\mathcal{P}$.

Proof. If $x \in \langle b\rangle$ and $x \leq y$ then, by the Lemma, also $y \in \langle b\rangle$. Thus $\langle b\rangle$ is an order-filter in $(P, \leq)$ and, by Theorem 2, an ideal of $\mathcal{P}$. ∎

It is clear that if $\{I_\gamma; \gamma \in \Gamma\}$ is a set of ideals of an order algebra $\mathcal{P} = (P; \cdot, 1)$ then also $I = \cap\{I_\gamma; \gamma \in \Gamma\}$ is an ideal of $\mathcal{P}$ (of course, $I \neq \emptyset$ since $1 \in I_\gamma$ for each $\gamma \in \Gamma$). Hence, the set $Id\mathcal{P}$ of all ideals of $\mathcal{P}$ forms a complete lattice with respect to set inclusion with the least element $\{1\}$ and the greatest element P.

In what follows, we can describe pseudocomplements of $Id\mathcal{P}$.

THEOREM 6. *Let* I *be an ideal of an order algebra* $\mathcal{P} = (P; \cdot, 1)$. *Then* $\langle I\rangle$ *is also an ideal of* $\mathcal{P}$ *and it is a pseudocomplement of* I *in the lattice* $Id\mathcal{P}$.

Proof. Let I be an ideal of $\mathcal{P}$. By Theorem 5, $\langle b\rangle$ is an ideal of $\mathcal{P}$ for each $b \in I$ and hence $\langle I\rangle = \cap\{\langle b\rangle; b \in I\}$, i.e. $\langle I\rangle \in Id\mathcal{P}$. Moreover, if $x \in I \cap \langle I\rangle$ then either $x = 1$ or $x \not\leq b$ for each $b \in I\backslash\{1\}$; taking $b = x$ we have $x \leq b$, a contradiction. Thus $I \cap \langle I\rangle = \{1\}$. Finally, let G be an ideal of $\mathcal{P}$ such that $G \cap I = \{1\}$ and suppose $g \in G, g \neq 1$. If $g \leq b$ for any $b \in I\backslash\{1\}$ then

$b \in I \cap G$, a contradiction. Thus $g \not\leq b$ whence $G \subseteq \langle I \rangle$. Together, $\langle I \rangle$ is the pseudocomplement of I in $Id\mathcal{P}$. ■

THEOREM 7. *Let B, C be ideals of an order algebra $\mathcal{P} = (P; \cdot, 1)$. Then $\langle B, C \rangle$ is a relative pseudocomplement of B with respect to C in the lattice $Id\mathcal{P}$.*

Proof. At first we show that

$$\langle B, C \rangle = \{x \in P; x \not\leq c \text{ for each } c \in C \backslash B\}.$$

Suppose $x \not\leq c$ for each $c \in C\backslash B$. Then $x \cdot c = c$ and hence $(x \cdot c) \cdot c = c \cdot c = 1 \in B$. Moreover, if $c \in B$ then $(x \cdot c) \cdot c \in \{c, 1\} \subseteq B$. Thus $(x \cdot c) \cdot c \in B$ for each $c \in C$, i.e. $x \in \langle B, C \rangle$.

Conversely, let $x \in \langle B, C \rangle$. Consider an arbitrary $c \in C\backslash B$. If $x \leq c$ then $(x \cdot c) \cdot c = 1 \cdot c = c \in B$, a contradiction. Hence $x \not\leq c$, i.e. the converse inclusion holds.

Now we show that $\langle B, C \rangle$ is an ideal of $\mathcal{P}$. Let $x \in \langle C, B \rangle$ and $x \leq y$. Then $x \not\leq c$ for each $c \in C\backslash B$ and hence also $y \not\leq c$. Thus $\langle B, C \rangle$ is an order-filter of $(P, \leq)$ and hence an ideal of $\mathcal{P}$.

Suppose that F is an ideal of $\mathcal{P}$ such that $C \cap F \subseteq B$. Let $f \in F$ and $c \in C\backslash B$. If $f \leq c$ then $c \in F \cap C \subseteq B$, a contradiction. Thus $f \not\leq c$ for each $c \in C\backslash B$ giving $F \subseteq \langle B, C \rangle$.

Finally, if $x \in C \cap \langle B, C \rangle$ then $x \not\leq c$ for each $c \in C\backslash B$ thus $x \in B$ proving $C \cap \langle B, C \rangle \subseteq B$. We have shown that $\langle B, C \rangle$ is the relative pseudocomplement of B with respect to C. ■

Our last result solves the problem whether an annihilator of every subset M of $\mathcal{P}$ is equal to the annihilator of an ideal generated by M.

THEOREM 8. *Let $(P, \leq)$ be an ordered set with a greatest element and $\mathcal{P}$ its assigned order algebra. The following conditions are equivalent:*

(A) *for each $M \subseteq P, M \neq \emptyset$ it holds $\langle M \rangle = \langle F(M) \rangle$;*

(B) *$\mathcal{P}$ is an implication algebra.*

Proof. Let $\mathcal{P} = (P; \cdot, 1)$ be an order algebra and $c \in P$. Take $M = \{c\}$ for $c \neq 1$. Then $\langle c \rangle = \{x \in P; x \cdot c = c\} = \{x \in P; x \not\leq c\}$. By the Lemma, $\langle F(c) \rangle = \{x \in P; U(x, c) = \{1\}\}$. Consider $b > c$. Then $b \in \langle c \rangle = \langle F(c) \rangle$ and hence $U(b, c) = \{1\}$, i.e. $b = 1$. We have shown that $(P, \leq)$ is a poset as visualized in Fig.1, i.e. $\mathcal{P}$ is an implication algebra.

Conversely, let $(P, \cdot)$ be an implication algebra. By Theorem 1, $(P, \leq)$ has only chains of length less or equal to 2, i.e. for $c \neq 1$ we have

$$\langle c \rangle = P \backslash \{c\} = \langle F(c) \rangle. \ ■$$

References

[1] J. C. Abbott, *Semiboolean algebras*, Matem. Vestnik 4 (1967), 177–198.
[2] H. A. S. Abujabal, M. A. Obaid, M. Aslam, A. B. Thaheem, *On annihilators of BCK-algebras*, Czech. Math. Journal 45 (1995), 727–735.
[3] D. Busenag, *A note on deductive systems of Hilbert algebra*, Kobe J. Math. 2 (1985), 29–35.
[4] I. Chajda, *Rees ideal algebras*, Math. Bohem. 122 (1997), 125–130.
[5] I. Chajda, R. Halaš, *Congruences and ideals in Hilbert algebras*, Kyungpook Math. J. 39 (1999), 429–432.
[6] A. Diego, *Sur les algèbres de Hilbert*, Collection de Logique Mathematique, Series A, (Ed. Hermann, Paris), 21 (1966), 1–52.
[7] W. A. Dudek, *On ideals in Hilbert algebras*, Acta Univ. Palack. Olomouc, Mathematica 38 (1999), 31–34.
[8] W. A. Dudek, R. Rousseau, *Set-theoretic relations and BCH-algebras with trivial structure* , Rev. Res. Fac. Sci. Novi Sad 25 (1995), 75–82.
[9] A. Romanowska, T. Traczyk, *On commutative BCK-algebras*, Math. Japonica 25 (1980), 567–583.
[10] A. Romanowska, T. Traczyk, *Commutative BCK-algebras. Subdirectly irreducible algebras and varieties*, Math. Japonica 27 (1981), 35–48.
[11] R. T. Tichy, *The Rees congruences in universal algebras*, Publ. Inst. Math. (Beograd) 29 (1981), 229–239.

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