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SOME RESULTS RELATED TO CARISTI'S FIXED POINT THEOREM AND EKELAND'S VARIATIONAL PRINCIPLE

1. Introduction and preliminaries

Caristi's fixed point theorem [3, Theorem (2.1)′] and its equivalent Ekeland's variational principle [6, Theorem 1.1], which was not basically formulated as a fixed point theorem, and is an abstraction of a lemma of Bishop and Phelps [1] (see also [2]), have been of continuing interest in fixed point theory because of their numerous applications (see [4], [5], [8]–[10], [14]–[17]). Recently Jung et. al [11], [12] have obtained some minimization theorems and coincidence theorems for mappings in fuzzy metric spaces. Further, they utilized their results to obtain analogues of Caristi's fixed point theorem, the well-known Downing and Kirk theorem [5, Theorem 2.1], and a more general type of Ekeland's variational principle in fuzzy metric spaces. The purpose of this paper is to generalize the above results of Jung et. al [11], [12] in the same direction. The results obtained herein improve and include many known results.

For the sake of completeness, we shall recollect some definitions and results from [13]. We denote the set of all upper semi-continuous normal convex fuzzy numbers by E and the set of all non-negative fuzzy members in E by G respectively. The additive and multiplicative identities of fuzzy numbers are denoted by $\overline{0}$ and $\overline{1}$, respectively.

The α-level set $[x]_\alpha$ of a fuzzy number $x \in E$ is a closed interval $[a^\alpha, b^\alpha]$, where the values $a^\alpha = -\infty$ and $b^\alpha = \infty$, are admissible. When $a^\alpha = -\infty$, for instance, then $[a^\alpha, b^\alpha]$ means the interval $[-\infty,, b^\alpha]$.

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DEFINITION 1.1. A partial ordering $\leq$ in E is defined by $x \leq y$ if and only if $a_1^\alpha \leq a_2^\alpha$ and $b_1^\alpha \leq b_2^\alpha$ for all $\alpha \in (0,1]$, where $x, y \in E$, $[x]_\alpha = [a_1^\alpha, b_1^\alpha]$ and $[y]_\alpha = [a_2^\alpha, b_2^\alpha]$.

DEFINITION 1.2. A sequence $\{x_n\}$ in E is called α-level convergence to $x \in E$, if $\lim_n a_n^\alpha = a^\alpha$ and $\lim_n b_n^\alpha = b^\alpha$ for all $\alpha \in (0,1]$, where $[x_n]_\alpha = [a_n^\alpha, b_n^\alpha]$ and $[x]_\alpha = [a^\alpha, b^\alpha]$.

Throughout. the set E will be endowed with the above partial ordering and the α-level convergence.

DEFINITION 1.3. Let X be a non-empty set, d be a mapping from $X \times X$ into G and the mappings $L, R : [0,1] \times [0,1] \to [0,1]$ be symmetric, non-decreasing in both arguments and satisfy $L(0,0) = 0, R(1,1) = 1$. Denote by $[d(x,y)]_\alpha = [\lambda_\alpha(x,y), \rho_\alpha(x,y)]$ for all $\alpha \in (0,1]$ and $x, y \in X$. Then the quadruple (X, d, L, R) is called a fuzzy metric space and d a fuzzy metric, if

(1) $d(x,y) = \overline{0}$ if and only if $x = y$,
(2) $d(x,y) = d(y,x)$ for all $x, y \in X$,
(3) for all $x, y, z \in X$;
 (i) $d(x,y)(s+t) \geq L(d(x,z)(s), d(z,y)(t))$.
 whenever $s \leq \lambda_1(x,z), t \leq \lambda_1(z,y)$ and $s + t \leq \lambda_1(x,y)$,
 (ii) $d(x,y)(s+t) \leq R(d(x,z)(s), d(z,y)(t))$ whenever $s \geq \lambda_1(x,z), t \geq \lambda_1(z,y)$ and $s + t \geq \lambda_1(x,y)$.

The triangle inequality (3) resembles the Menger triangle inequality in a probabilistic metric space (PM–space). The following two-place functions, which are frequently used in the study of PM–spaces, are possible choices for L and R:

$$T_1(a,b) = \mathrm{Max}(a+b-1, 0) \quad (\mathrm{Max}(\mathrm{Sum}-1, 0),$$
$$T_2(a,b) = ab \quad (\mathrm{Product}),$$
$$T_3(a,b) = \mathrm{Min}(a,b) \quad (\mathrm{Min}),$$
$$T_4(a,b) = \mathrm{Max}(a,b) \quad (\mathrm{Max}),$$
$$T_5(a,b) = a+b-ab \quad (\mathrm{Sum–Product}),$$
$$T_6(a,b) = \mathrm{Min}(a+b, 1) \quad (\mathrm{Min}(\mathrm{Sum}, 1)).$$

The above T–functions are listed in increasing order of strenght h in the sense that $T_i(a,b) \geq T_j(a,b)$ for all $a, b \in [0,1]$ (abbreviated $T_i \geq T_j$) if $i \geq j$.

LEMMA 1.4. *In the fuzzy metric space (X, d, L, R) with $R = \mathrm{Max}$, the triangle inequality* (3)(ii) *in Definition* 1.3 *is equivalent to the triangle inequality:*

$$(1.1)\quad \rho_\alpha(x,y) \leq \rho_\alpha(x,z) + \rho_\alpha(z,y) \quad \text{for all } \alpha \in (0,1] \quad \text{and} \quad x, y, z \in X.$$

THEOREM 1.5. *Let (X, d, L, R) be a fuzzy metric space with $\lim_{a\to 0_+} R(a,a) = 0$. Then the family $\beta = \{U(\epsilon,\alpha) : \epsilon > 0, 0 < \alpha \leq 1\}$ of sets $U(\epsilon,\alpha) = \{(x,y) \in X \times X : \rho_\alpha(x,y) < \epsilon\}$ form a base for a Hausdorff uniformity on $X \times X$. Moreover, the sets $\mathcal{N}_X(\epsilon,\alpha) = \{y \in X : \rho_\alpha(x,y) < \epsilon\}$ form a base for a Hausdorff topology on X and this topology is metrizable.*

DEFINITION 1.6. The convergence in a fuzzy metrix space (X, d, L, R) is defined by $\lim_n x_n = x$ if and only if $\lim_n d(x_n, x) = \overline{0}$.

From the definition of the convergence in G and Theorem 1.5, it follows that in the fuzzy metric space (X, d, L, R) with $\lim_{a\to 0} R(a,a) = 0$, the limit is uniquely determined and all subsequences of a convergent sequence are convergent as well.

DEFINITION 1.7. A sequence $\{x_n\}$ in X is called a Cauchy sequence if $\lim_{m,n} d(x_m, x_n) = \overline{0}$.

A fuzzy metric space X is complete if every Cauchy sequence in X converges. From the inequality (1.1) in Lemma 1.4, it follows that in the fuzzy metric space (X, d, L, Max), every convergent sequence is also a Cauchy sequence.

2. Main results

Throughout this section, we denote by $\mathbb{R}$ the set of real numbers, and assume that $k : (0,1) \to (0,\infty)$ is a non-increasing function satisfying the following condition:

$$M = \sup_{r\in(0,1)} k(r) < \infty \tag{2.1}$$

The following theorem generalizes [11, Theorem 3.1].

THEOREM 2.1. *Let $(X_i, d_i, L, \mathrm{Max})$ be two complete fuzzy metric spaces such that $\lim_t d_i(x,y)(t) = 0$ for all $x, y \in X_i$, $i = 1, 2$. Let D be a non-empty subset of X_1 and let $h : D \to X_1, g : h(D) \to X_1$ be two functions and g be surjective. Let $f : X_1 \to X_2$ be a closed mapping. Let $\theta : X_1 \to \mathbb{R}$ and $\phi : f(X_1) \to \mathbb{R}$ be lower semi-continuous functions, each bounded from below. Let $\{S_\ell\}_{\ell\in I}$ be a family of set-valued mappings $S_\ell : h(D) \to 2^X 1\backslash\{\phi\}$. Suppose further that if for each $x \in D$ and given constans $a, b, c > 0$,*

$$(g\circ h)(x) \notin \bigcap_{\ell\in I} S_\ell(h(x)),$$

then there exists an $\ell_0 \in I$ and a $y \in S_{\ell_0}(h(x)) \setminus \{(g\circ h)(x))\}$ such that

$$\max\{\rho_{1\alpha}((g\circ h)(x), y), c\rho_{2\alpha}(f((g\circ h)(x), f(y))\}$$
$$\leq k(r).\min\{a(\theta(g\circ h)(x)) - \theta(y)), b(\phi(f((g\circ h)(x))) - \phi(f(y))\}$$

for all $\alpha \in (0,1]$ *and* $r \in (0,1)$, *where* $\rho_{i\alpha}(x,y)$ *are the right end-points of the* α*-level interval of* $d_i(x,y)$, $i = 1,2$. *Then there exists a coincidence point* $u \in X_1$ *of* $g \circ h$ *and* $\{S_\ell\}_{\ell \in I}$, *that is, there exists a* $u \in X_1$ *such that*

$$(g \circ h)(u) \in \bigcap \ell \in IS_\ell(h(u)).$$

Proof. Since $\lim_t d_i(x,y)(t) = 0$ for $x, y \in X_i$, $i = 1,\ 2$, it follows that $\rho_{i\alpha}(x,y) < \infty$ for all $\alpha \in (0,1]$, $i = 1,2$. Hence by Lemma 1.4, we define a partial ordering "$\leq$" on X_1 as follows

$$x \leq y \quad \text{if and only if } \max\{\rho_{1\alpha}(x,y), c\rho_{2\alpha}(f(x), f(y))\} \leq k(r).\min\{a(\theta(x) - \theta(y)), b(\phi(f(x)) - \phi(f(y)))\} \tag{2.2}$$

for all $\alpha \in (0,1]$ and fixed constans $a, b, c > 0$.

It follows from (2.2) that if $x \leq y$ for $x, y \in X_1$, then we have

$$\theta(y) \leq \theta(x) \text{ and } \theta(f(y)) \leq \theta(f(x)). \tag{2.3}$$

On the other hand, the reflexivity and anti-symmetry of "$\leq$" are obvious. Now we prove the transitivity of "$\leq$". If $x \leq y$ and $y \leq z$ for $x, y, z \in X_1$ then, by (2.3), we have

$$\theta(z) \leq \theta(y) \leq \theta(x) \text{ and } \phi(f(y)) \leq \phi(f(x)).$$

Further, by (2.2) we have

$$\max\{\rho_{1\alpha}(x,y), c\rho_{2\alpha}(f(x), f(y))\} \leq k(r).\min\{a(\theta(x) - \theta(y)), b(\phi(f(x)) - \phi(f(y))\}$$

and

$$\max\{\rho_{1\alpha}(y,z), c\rho_{2\alpha}(f(y), f(z)\} \leq k(r).\min\{a(\theta(y) - \theta(z)), b(\phi(f(y)) - \phi(f(z)))\}$$

for all $\alpha \in (0,1]$, $r \in (0,1)$ and fixed constans $a, b, c > 0$. Thus we obtain

$$\begin{aligned} \rho_{1\alpha}(x,z) &\leq \rho_{1\alpha}(x,y) + \rho_{1\alpha}(y,z) \\ &\leq \max\{\rho_{1\alpha}(x,y), c\rho_{2\alpha}(f(x), f(y))\} + \max\{(\rho_{1\alpha}(y,z), c\rho_{2\alpha}(f(y), f(z))\} \\ &\leq k(r).\min\{a(\theta(x) - \theta(y)), b(\phi(f(x)) - \phi(f(y)))\} \\ &\quad + k(r).\min\{a(\theta(y) - \theta(z)), b(\phi(f(y)) - \phi(f(z)))\} \\ &= k(r).\min\{a(\theta(x) - \theta(z)), b(\phi(f(x)) - \phi(f(z)))\} \end{aligned} \tag{2.4}$$

and

$$c\rho_{2\alpha}(x,z) \leq k(r).\min\{a(\theta(x) - \theta(z)), b(\phi(f(x)) - \phi(f(z)))\} \tag{2.5}$$

for all $\alpha \in (0,1]$, $r \in (0,1)$ and fixed constans $a, b, c > 0$, that is, $x \leq z$. This shows that "$\leq$" is a partial ordering on X_1.

Now we prove that there exists a maximal element in X_1. Let $\{x_\mu\}_{\mu\in I}$ be any totally ordered subset of $(X_1, \leq)$, where I is an indexing set. Define

$$x_\mu \leq x_\nu \quad \text{if and only if } \mu \leq \nu.$$

Then $(I, \leq)$ is a directed set and $\{\theta(x_\mu)\}_{\mu\in I}$, $\{\phi(f(x_\mu))\}_{\mu\in I}$ are monotonically decreasing nets in $\mathbb{R}$. By the boundedness from below of θ and ϕ, there exist finite numbers $\gamma, \delta \geq 0$ such that $(x_\mu) \geq \gamma$ and $\phi(f(x_\mu)) \geq \delta$. Hence for all $\lambda > 0$ and $\epsilon > M\lambda. \min\{a, b\}$, there exists $\mu_0 \in I$ such that $\mu \geq \mu_0$ implies

$$\gamma \leq \theta(x_\mu) < \gamma + \lambda, \qquad \delta \leq \phi(\delta(x_\mu)) < \delta + \lambda,$$

where M is the constant as defined by (2.1). Thus for any $\mu, \nu \in I$ with $\mu_0 \leq \mu \leq \nu$, we have

$$\begin{aligned} 0 \leq \theta(x_\mu) - \theta(x_\nu) \leq \lambda, &\quad 0 \leq \phi(f(x_\mu)) - \phi(f(x_\nu)) \leq \lambda, \\ \rho_{1\alpha}(x_\mu, x_\nu) &\leq \max\{\rho_{1\alpha}(x_\mu, x_\nu), c\rho_{2\alpha}(f(x_\mu), f(x_\nu))\} \\ &\leq k(r). \min\{a(\theta(x_\mu) - \theta(x_\nu)), b(\phi(f(x_\mu)) - \phi(f(x_\nu)))\} \\ &\leq M. \min\{a, b\} < \epsilon, \end{aligned}$$

and

$$\rho_{2\alpha}(f(x_\mu), f(x_\nu)) \leq k(r). \min\{a(\theta(x_\mu) - \theta(x_\nu)), b(\phi(f(x_\mu)) - \phi(f(x_\nu)))\} < \epsilon$$

for all $\alpha \in (0, 1]$ and $r \in (0, 1)$. Thus (x_μ) is a Cauchy net in X_1, while $\{f(x_\mu)\}$ is a Cauchy net in X_2. By completeness, there exist $\overline{x} \in X_1$ and $\overline{y} \in X_2$ such that $X_\mu \to \overline{x}$ and $f(x_\mu) \to \overline{y}$. Since f is a closed mapping, $f(\overline{x}) = \overline{y}$. From the lower semi-continuity of θ and ϕ, it follows that

$$\theta(\overline{x}) \leq \liminf_\mu \theta(x_\mu) = \lim_\mu \theta(x_\mu) = \gamma \leq \theta(x_\mu) \tag{2.6}$$

and

$$\begin{aligned} \theta(f(\overline{x})) \leq{}& \liminf_\mu \phi(f(x_\mu)) = \lim_\mu \phi(f(x_\mu)) \\ &= \delta \leq \phi(f(x_\mu)) \quad \text{for all } \mu \in I. \end{aligned} \tag{2.7}$$

Next we show that $\overline{x}$ is an upper bound of $\{x_\mu\}_{\mu\in I}$. In fact, for any $\mu, \nu \in I$ with $\mu \leq \nu$, we have, by (2.6) and (2.7), that

$$\begin{aligned} &\max\{\rho_{1\alpha}(x_\mu, x_\nu), c\rho_{2\alpha}(f(x_\mu), f(x_\nu))\} \\ &\qquad \leq k(r). \min\{a(\theta(x_\mu) - \theta(x_\nu)), b(\phi(f(x_\mu) - \phi(f(x_\nu)))\} \\ &\qquad \leq k(r), \min\{a(\theta(x_\mu) - \gamma), b(\phi(f(x_\mu) - \delta)\} \end{aligned}$$

for all $\alpha \in (0, 1]$ and $r \in (0, 1)$. Taking limits with respect to ν, we obtain

$$\max\{\rho_{2\alpha}(x_\mu,\overline{x}), c\rho_{2\alpha}(f(x_\mu), f(\overline{x}))\}$$
$$\le k(r).\min\{a(\theta(x_\mu)-\gamma), b(\phi(f(x_\mu))-\delta)\}$$
$$\le k(r).\min\{a(\theta(x_\mu)-\theta(\overline{x})), b(\phi(f(x_\mu))-\phi(f(\overline{x})))\}$$

for all $\alpha \in (0,1]$ and $r \in (0,1)$. This implies that $x_\mu \le \overline{x}$ for $\mu \in I$. Therefore, $\overline{x}$ is an upper bound of $\{x_\mu\}_{\mu\in I}$. Hence by Zorn's Lemma, $(X_1, \le)$ has a maximal element $z \in X_1$.

Finally, we prove the existence of a coincidence point $u \in X_1$ of g and $\{S_\ell\}_{\ell\in I}$. In fact, since $g : h(D) \to X_1$ is surjective, there exists a $u \in D$ such that $(g \circ h)(u) = z$. Suppose that $(g\circ h)(u) \notin \cap_{\ell\in I} S_\ell(h(u))$. Then, by assumption, there exist an $\ell_0 \in I$ and a $y \in S_{\ell_0}(h(u)) - \{g \circ h(u)\}$ such that

$$\max\{\rho_{1\alpha}((g\circ h)(u), y), c\rho_{2\alpha}(f((g\circ h)(u), f(y))\}$$
$$\le k(r).\min\{a(\theta(g\circ h)(u)-\theta(y)), b(\phi(f((g\circ h)(u)))-\phi(f(y)))\}$$

for all $\alpha \in (0,1]$ and $r \in (0,1)$, and so $(g\circ h)(u) \le y$. But, since $(g\circ h)(u) = z$ is a maximal element in X_1, we obtain

$$z = (g\circ h)(u) = y \in S_{\ell_0}(h(u)) - \{(g\circ h)(u)\},$$

which is a contradiction. Therefore we have

$$(g\circ h)(u) \in \bigcap_{\ell\in I} S_\ell(h(u)),$$

that is, u is a coincidence point of $g\circ h$ and $\{(S_\ell)\circ h\}_{\ell\in I}$. ∎

As a consequence of Theorem 2.1, we have the following:

COROLLARY 2.2. *Let $(X_i, d_i, L, \mathrm{Max})$, D, h, g, f, θ, ϕ be as in Theorem 2.1 for $i = 1,2$. Let $S : h(D) \to X_1 - \{\phi\}$ be a set-valued mapping. Suppose that for each $x \in D$ and fixed constans $a,b,c > 0$, $(g\circ h)(x) \notin S(h(x))$. Then there exists a $y \in S(h(x))$ such that:*

$$\max\{\rho_{1\alpha}((g\circ h)(x), y), c\rho_{2\alpha}(f((g\circ h)(x), f(y))\}$$
$$\le k(r).\min\{a(\theta((g\circ h)(x)-\theta(y)), b(\phi(f(g\circ h)(x)-\phi(f(y)))\}$$

for all $\alpha \in (0,1]$ and $r \in (0,1)$. Then there exists a $u \in X_1$ such that $(g\circ h)(u) \in S(h(u))$.

COROLLARY 2.3 *Let $(X_1, d_1, L, \mathrm{Max})$, f, θ, ϕ be as in Theorem 2.1 for $i = 1$, 2. Let $S : X_1 \to X_1$ be a mapping such that for each $x \in X_1$ and fixed constans $a,b,c > 0$,*

$$\max\{\rho_{1\alpha}(x, S(x)), c\rho_{2\alpha}(f(x), f(S(x)))\}$$
$$\le k(r).\min\{a(\theta(x)-\theta(S(x))), b(\phi(f(S(x))))\}$$

for all $\alpha \in (0,1]$, and $r \in (0,1)$. Then S has a fixed point in X_1.

Proof. The result follows from Corollary 2.2 with $D = X_1$, $h = g = I$ (the identity mapping). ∎

The following theorem is a generalization of [11, Theorem 3.4].

THEOREM 2.4. *Let (X, d, L, Max) be a complete fuzzy metric space such that $\lim_t d(x,y)(t) = 0$ for all $x, y \in X$. Let D be a non-empty subset of X and let $h : D \to X$. Let $g : h(D) \to X$ be a surjective function. Let $\phi : X \to \mathbb{R}$ ba a lower semi-continuous function, bounded from below and let $\{S_\ell\}_{\ell \in I}$ be a family of set-valued mappings $S_\ell : h(D) \to 2^X - \{\phi\}$. Suppose that for each $x \in D$ with $(g \circ h)(x) \notin \cap_{\ell \in I} S_\ell(h(x))$, there exists an $\ell_0 \in I$ and a $y \in S_{\ell_0}(h(x())- \{(g \circ h)(x)\}$ such that*

$$\rho_\alpha((g \circ h)(x), y) \leq k(r).(\phi((g \circ h)(x)) - \phi(y))$$

for all $\alpha \in (0,1]$ and $r \in (0,1)$. Then there exists a coincidence point $u \in X$ of $g \circ h$ and $\{S_\ell \circ h\}_{\ell \in I}$, that is, there exists a $u \in X$ such that

$$(g \circ h)(u) \in \cap_{\ell \in I} S_\ell(h(u)).$$

Proof. The result follows from Theorem 2.1 with $X_1 = X_2 = X$, $d_1 = d_2 = d$, $\theta = \phi$, $a = b = c = 1$ and $f = I$ (the identity mapping). ∎

As a direct consequence of Theorem 2.4, we have the following:

COROLLARY 2.5. *Let (X, d, L, Max), D, h, g, ϕ be as in Theorem 2.4.*

Let $S : h(D) \to 2^X - \{\phi\}$ be a set-valued mapping. Suppose that if for each $x \in D$ with $(g \circ h)(x) \notin S(h(x))$, there exists a $y \in S(x)$ such that:

$$\rho_\alpha((g \circ h)(x), y) \leq k(r).(\phi(g \circ h(x)) - \phi(y))$$

for all $\alpha \in (0,1]$ and $r \in (0,1)$. Then there exists a $u \in D$ such that $(g \circ h)(u) \in S(h(u))$.

COROLLARY 2.6. *Let (X, d, L, Max), and ϕ be as in Theorem 2.4. Let $S : X \to X$ be a mapping such that for each $x \in X$,*

$$\rho_\alpha(x, S(x)) \leq k(r).(\phi(x) - \phi(S(x)))$$

for all $\alpha \in (0,1]$ and $r \in (0,1)$. Then S has a fixed point in X.

Proof. The result follows from Corollary 2.5 with $D = X$ and $g = h = I$ (the identity mapping). ∎

REMARK 2.7 (i) The Downing–Kirk fixed point theorem [5] in a fuzzy metric space is obtained as a special case of Corollary 2.3 if $k(r) = 1$ for all $r \in (0,1)$, $b = 1$ and $\theta = \phi \circ f$. Caristi's fixed point theorem [3] in a fuzzy metric space is a special case of Corollary 2.6 if $k(r) = 1$ for all $r \in (0,1)$.

(ii) Since the usual metric space is a special case of a fuzzy metric space (see [13]), therefore when X is a complete metric space, the corresponding results of [3 – 5] and [16, 17] may be recovered as a special cases of our results.

3. A variational principle

In this section, we study a more general type of Ekeland's variational principle [6] in a fuzzy metric space.

THEOREM 3.1. *Let (X, d, L, Max) be a complete fuzzy metric space such that $lim_t d(x,y)(t) = 0$ for all $x, y \in X$. Let $f : X \to X$ be a continuous mapping and let $\phi : f(X) \to \mathbb{R}$ be a lower semi-continuous function, bounded from below. Suppose that for any $\epsilon > 0$, there exists a $u \in X$ such that:*

$$\phi(f(u)) \le \inf_{x \in X} \phi(f(x)) + \epsilon. \tag{3.1}$$

If $k : (0,1) \to (0,\infty)$ is a non-increasing function satisfying the condition (3.1), then there exists an $x_0 \in X$ such that:

(1) $\rho_\alpha(f(x_0), f(u)) \le k(r).(\phi(f(u)) - \phi(f(x_0)))$ for all $\alpha \in (0,1]$ and $r \in (0,1)$,

(2) $\rho_\alpha(f(x_0), f(u)) \le k(r)$ for all $\alpha \in (0,1]$ and $r \in (0,1)$,

(3) for any $w \in X$, $w \neq x_0$, there exists an $r_0 \in (0,1)$ such that $\rho_\alpha(f(x_0), f(w)) > k(r_0).(\phi(f(x_0) - \phi(f(w))$ for all $\alpha \in (0,1]$.

Proof. Note that $\rho_\alpha(f(x), f(y)) < \infty$ for all $\alpha \in (0,1]$ and $x, y \in X$. Denote by $X_f = \{x \in X : \rho_\alpha(f(x), f(u)) \le k(r).(\phi(f(u)) - \phi(f(x)))\}$ for all $\alpha \in (0,1]$ and $r \in (0,1)$. Observe that $u \in X_f$, so $X_f \neq \Phi$. We shall show that X_f is closed.

Let $\{x_n\}$ be a sequence in X_f such that $x_n \to \overline{x}$. Then, since f is continuous and $\lim_n \rho_\alpha(f(x_n), f(\overline{x})) = \rho_\alpha(f(\lim_n x_n),\ f(\overline{x})) = \rho_\alpha(f(\overline{x}), f(\overline{x})) = 0$ for all $\alpha \in (0,1]$, then from (1.1) of Lemma 1.4 it follows that $\rho_\alpha(f(\overline{x}), f(u)) \le \lim_n \sup \rho_\alpha(f(x_n), (f(u))$ for all $\alpha \in (0,1]$. So by lower semi-continuity of ϕ we have

$$\begin{aligned} \rho_\alpha(f(\overline{x}), f(u)) &\le \limsup_n \rho_\alpha(f(x_n), f(u)) \\ &\le \limsup_n k(r).(\phi(f(u)) - \phi(f(x_n))) \\ &= k(r).(\phi(f(u)) - \liminf_n(\phi(f(x_n))) \\ &\le k(r).(\phi(f(u)) - \phi(f(\overline{x}))) \end{aligned} \tag{3.2}$$

for all $\alpha \in (0,1]$ and $r \in (0,1)$. This implies that $\overline{x} \in X_f$ and X_f is closed. Therefore $(X_f, d, L, \mathrm{Max})$ is a complete fuzzy metric space.

Now we define a partial ordering "$\leq$" on X_f by

$$x \leq y \quad \text{if and only if } \rho_\alpha(f(x), f(y)) \leq k(r).(\phi(f(x)) - \phi(f(y))) \tag{3.3}$$

for all $\alpha \in (0,1]$ and $r \in (0,1)$. Then $(X_f, \leq)$ has a maximal element, say, x_0 in X. Thus we have

$$\rho_\alpha(f(x_0), f(u)) \leq k(r)(\phi(f(u)) - \phi(f(x_0))) \tag{3.4}$$

for all $\alpha \in (0,1]$ and $r \in (0,1)$, and this implies that the assertion (1) is true.

By the condition (3.1) we have

$$0 \leq \phi(f(u)) - \phi(f(x_0)) \leq \phi(f(u)) - \inf_{x \in X} \phi(f(x)) \leq \epsilon. \tag{3.5}$$

Thus, by (1), we have

$$\rho_\alpha(f(x_0), f(u)) \leq \epsilon.k(r)$$

for all $\alpha \in (0,1]$ and $r \in (0,1)$, and this shows that the assertion (2) is true.

Suppose that the assertion (3) is not true. Then there exists a non-increasing function $k : (0,1) \to (0,\infty)$ such that for each $x \in X$ there exists a $w \in X$, $f(w) \neq f(x)$ and

$$\rho_\alpha(f(x), f(w)) \leq k(r).(\phi(f(x)) - \phi(f(w)))$$

for all $\alpha \in (0,1]$ and $r \in (0,1)$. Define $S : X_f \to X_f$ by $S(f(x)) = f(w)$. Then S satisfies the following condition

$$\rho_\alpha(f(x), S(f(x))) \leq k(r).(\phi(f(x)) - \phi(S(f(x))))$$

that is,

$$\rho_\alpha(x', S(x')) \leq k(r).(\phi(x') - \phi(S(x')))$$

for all $\alpha \in (0,1]$, $r \in (0,1)$ and $f(x) = x' \in X$. Hence, by Corollary 2.6, S has a fixed point in X_f, a contradiction with the definition of S. Therefore the assertion (3) is true. ∎

By choosing, in the above theorem, f to be an identity mapping on X we recover the following result of Jung et.al [11, Theorem 4.1] which, in turn, generalizes among others certain results of [6, Theorem 1.1] and [12, Theorem 5].

COROLLARY 3.2. *Let (X, d, L, Max) be a complete fuzzy metric space such that $\lim_t d(x,y)(t) = 0$ for all $x, y \in X$ and let $\phi : X \to \mathbb{R}$ be a lower semi-continuous function, bounded from below. Suppose that for any $\epsilon > 0$ there exists a $u \in X$ such that:*

$$\phi(u) \leq \inf_{x \in X} \phi(x) + \epsilon. \tag{3.1}$$

If $k : (0,1) \to (0,\infty)$ is a non-increasing function satisfying the condition (3.1), then there exists an $x_0 \in X$ such that:

(1) $\rho_\alpha(x_0, u) \leq k(r).(\phi(u) - \phi(x_0))$ *for all* $\alpha \in (0,1]$ *and* $r \in (0,1)$,
(2) $\rho_\alpha(x_0, u) \leq \epsilon.k(r)$ *for all* $\alpha \in (0,1]$ *and* $r \in (0,1)$,
(3) *for any* $w \in X$, $w \neq x_0$, *there exists an* $r_0 \in (0,1)$ *such that*

$$\rho_\alpha(x_0, w) > k(r_0)(\phi(x_0) - \phi(w)) \text{ for all } \alpha \in (0,1].$$

4. An equivalence

THEOREM 4.1. *Theorem* 2.4 *and Corollary* 3.2 *are equivalent.*

Proof. By taking $f = I$ (the identity mapping) and using the proof techniques of Theorem 3.1, it follows immediately that Theorem 2.4 implies Corollary 3.2.

Conversely, for any $x^* \in X$, $\phi(x^*) \neq \infty$ if $\phi(x^*) = \inf_{x \in X} \phi(x)$. So, since $h : D \to X$ and $g : h(D) \to X$ is surjective, by assumptions of Theorem 2.4, there exists an $u \in D$ such that $g(h(u)) = x^*$. The latter implies that

$$\phi(g(h(u))) = \phi(x^*) \leq \phi(h(y)) \tag{4.1}$$

for all $y \in \cap_{\ell \in I} S_\ell(h(u))$. If $g(h(u)) \notin \cap_{\ell \in I} S_\ell(h(u))$ then, by Theorem 2.4, there exists an $\ell_0 \in I$ and a $y_0 \in S_{\ell_0}(h(u)) - (g \circ h)(u)$ such that

$$\rho_\alpha(g(h(u)), y_0) \leq k(r).(\phi(g(h(u))) - \phi(h(u_0))) \tag{4.2}$$

for all $\alpha \in (0,1]$ and $r \in (0,1)$. Thus, it follows, from (4.1) and (4.2), that

$$\rho_\alpha(g(h(u)), h(y_0)) = 0$$

for all $\alpha \in (0,1]$. Thus $g(h(u)) = h(y_0) \in S_{\ell_0}(h(u)) - \{g(h(u))\}$, a contradiction.

Therefore $g(h(u)) \in \cap_{\ell \in I} S_\ell(h(u))$, and Theorem 2.4 is proven.

If $\phi(x^*) > \inf_{x \in X} \phi(x)$, let $\epsilon = \phi(x^*) - \inf_{x \in X} \phi(x)$. Then, by Corollary 3.2, for any non-increasing function $k : (0,1) \to (0,\infty)$ satisfying the condition (2.1), there exists an $x_0 \in X$ such that for any $w \in X$, $w \neq x_0$, there exists an $r_0 \in (0,1)$ such that

$$\rho_\alpha(x_0, w) > k(r_0).(\phi(x_0) - \phi(w)) \tag{4.3}$$

for all $\alpha \in (0,1]$. Since the function $g : h(D) \to X$ is surjective, there exists a $u \in D$ such that $g(h(u)) = x_0$. If $g(h(u)) \notin \cap_{\ell \in I} S_\ell(h(u))$, by Theorem 2.4, there exists an $\ell_0 \in I$ and a $y_0 \in S_\ell(h(u)) - \{g(h(u))\}$ such that

$$\rho_\alpha(g(h(u)), y_0) \leq k(r).(\phi(g(h(u)) - \phi(y_0)))$$

for all $\alpha \in (0,1]$ and $r \in (0,1)$, or equivalently,

$$\rho_\alpha(x_0, y_0) \leq k(r).(\phi(x_0) - \phi(y_0)) \tag{4.4}$$

for all $\alpha \in (0,1]$ and $r \in (0,1)$. Since $y_0 \in S_{\ell_0}(h(u)) - \{g(h(u))\} = S_{\ell_0}(h(u)) - \{h(x_0)\}$, we have $y_0 \neq x_0$. Thus (4.4) contradicts with (4.3). Therefore $g(h(u)) \in \cap_{\ell \in I} S_\ell(h(u))$. ∎

REMARK 4.2. Theorem 4.1 includes [11, Theorem 5.1] and improves [12, Theorem 6] , which, in turn, generalize the corresponding results of [4], [7] and [16].

5. Applications to Menger Spaces

DEFINITION 5.1. Let X be a non-empty set. For each pair $(x, y) \in X \times X$, consider a left continuous distribution function F_{xy} such that:

(1) $F_{xy}(t) = 1$ for all $t > 0$ if and only if $x = y$,
(2) $F_{xy}(0) = 0$,
(3) $F_{xy} = F_{yx}$ for all $x, y \in X$,
(4) $F_{xy}(s + r) \geq \Delta(F_{xz}^{(s)}, F_{zy}^{(r)})$ for all $x, y, z, \in X$,

where $\Delta : [0,1] \times [0,1] \to [0,1]$ is a t-norm. Then we call (X, F, Δ) a Menger space (see [18]).

Kaleva and Seikkala [13], have shown for a given Menger space (X, F, Δ) that (X, d, L, R) is a fuzzy metric space with $d : X \times X \to G$ defined by

$$d(x,y)(t) = \begin{cases} 0 & \text{if } t < t_{xy}, \\ 1 - F_{xy}(t) & \text{if } t \geq t_{xy}, \end{cases} \tag{5.1}$$

where $t_{xy} = \sup\{t : F_{xy}(t) = 0\}$ and

$$L(a, b) = 0. \quad R(a, b) = 1 - \Delta(1 - a, 1 - b). \tag{5.2}$$

In this space, we have

$$\begin{aligned} \rho_\alpha(x, y) &= \sup\{t : d(x, y)(t) \geq \alpha\} \\ &= \sup\{t : F_{xy}(t) \leq 1 - \alpha\} \end{aligned} \tag{5.3}$$

for all $\alpha \in (0,1]$. Therefore, as in [8], we have evidently the following equivalent assertion: "For any $\epsilon > 0$ and any $\alpha \in (0,1]$, there exists an $N > 0$ such that $F_{x_m x_n}(\epsilon) \geq 1 - \alpha$, whenever $m > n > N$ "if and only if" for any $\epsilon > 0$, there exists an $N > 0$ such that $\sup\{t : F_{x_m x_n}(t) \leq 1 - \alpha\} = \rho_\alpha(x_m, x_n) \leq \epsilon$, whenever $m > n > N$, $\alpha \in (0,1]$".

Thus $\{x_n\}$ is a Cauchy sequence in the Menger space (X, F, Δ) if and only if $\{x_n\}$ is a Cauchy sequence in the corresponding fuzzy metric space (X, d, L, R) obtained by (5.1) and (5.2). Therefore (X, F, Δ) is complete if and only if (X, d, L, R) is complete. Thus, applying Theorem 2.1 to (X, d, L, R) and using (5.3), we have the following result.

THEOREM 5.2 *Let (X_i, F_i, Δ_i) be two complete Menger spaces, where $\Delta_i =$ Min, $i = 1, 2$. Let D be a non-empty subset of $X_1, h : D \to X$ be a surjective function. Let $f : X_1 \to X_2$ be a closed mapping and let $\phi : f(X_1) \to \mathbb{R}$ be a lower semi-continuous function, bounded from below. Let $\{S_\ell\}_{\ell \in I}$ be a* *family of set-valued mappings* $S_\ell : h(D) \to 2^X 1 \setminus \{\emptyset\}$. *Suppose further that for each* $x \in D$ *and any given constant* $c > 0$ *with*

$$(g \circ h)(x) \notin \cap_{\ell \in I} S_\ell(h(x)),$$

there exist an $\ell_0 \in I$ *and a* $y \in S_{\ell_0}(h(x)) - \{g(h(u))\}$ *such that*

$$\max\{\sup\{t : F_{1(g\circ h)(x)y}(t) \le 1-\alpha\}, c.\sup\{t : F_{2f((g\circ h)(x))f(y)}(t) \le 1-\alpha\}\} \le k(r).(\phi(f(g\circ h)(x))) - \phi(f(y)))$$

for all $\alpha \in (0,1]$. *Then there exists a coincidence point* $u \in X_1$ *of* $(g \circ h)$ *and* $\{S_\ell \circ h\}_{\ell \in I}$, *that is, there exists a* $u \in X_1$ *such that*

$$(g \circ h)(u) \in \cap_{\ell \in I} S_\ell(h(u))$$

Similarly, we obtain the following:

COROLLARY 5.3. *Let* (X_i, F_i, Δ_i), D, f, h, g, ϕ *be as in Theorem 5.2 for* $i = 1, 2$. *Let* $S : h(D) \to 2^X 1 \setminus \{\emptyset\}$ *be a set-valued mapping. Suppose that for each* $x \in D$ *and any given constant* $c > 0$ *with* $(g \circ h)(x) \notin S(h(x))$, *there exists a* $y \in S(h(x))$ *such that:*

$$\max\{\sup\{t : F_{1(g\circ h)(x)y}(t) \le \alpha\}, c.\sup\{t : F_{2f(g\circ h)(x)f(y)}(t) \le 1-\alpha\}\} \le k(r).(\phi(f(g\circ h)(x)) - \phi(f(y)))$$

for all $\alpha \in (0,1]$ *and* $r \in (0,1)$. *Then there exists a* $u \in D$ *such that* $(g \circ h)(u) \in S(h(u))$.

THEOREM 5.4. *Let* (X, F, Δ) *be a complete Menger space with* $\Delta =$ Min. *Let* D *be a non-empty subset of* $X, h : D \to X$ *and let* $g : h(D) \to X$ *be a surjective function. Let* $\phi : X \to \mathbb{R}$ *be a lower semi-continuous function, bounded from below and let* $\{S_\ell\}_{\ell \in I}$ *be a family of set-valued mappings* $S_\ell : h(D) \to 2^X \setminus \{\emptyset\}$. *Suppose that for each* $x \in D$ *with* $(g\circ h)(x) \notin \cap_{\ell \in I} S_\ell(h(x))$, *there exists an* $\ell_0 \in I$ *and a* $y_0 \in S_{\ell_0}(h(x)) - \{(g \circ h)(x)\}$ *such that*

$$\sup\{t : F_{(g\circ h)(x)y}(t) \le 1-\alpha\} \le k(r)\{\phi((g\circ h)(x)) - \phi(y)\}$$

for all $\alpha \in (0,1]$ *and* $r \in (0,1)$. *Then there exists a coincidence point* $u \in X$ *of* $g \circ h$ *and* $(S_\ell \circ h)_{\ell \in I}$, *that is, there exists a* $u \in X$ *such that*

$$(g \circ h)(u) \in \cap_{\ell \in I} S_\ell(h(u)).$$

COROLLARY 5.5. *Let* (X, F, Δ), D, h, g *be as in Theorem* 5.4. *Let* $S : h(D) \to 2^X \setminus \{\emptyset\}$ *be a set-valued mapping. Suppose that for each* $x \in D$, $(g \circ h)(x) \in S(h(x))$, *and there exists a* $y \in S(h(x))$ *such that*

$$\sup\{t : F_{(g\circ h)(x)y}(t) \le 1-\alpha\} \le k(r).(\phi((g\circ h)(x) - \phi(y))$$

for all $\alpha \in (0,1]$. *Then there exists a* $u \in D$ *such that* $(g \circ h)(u) \in S(h(u))$.

Finally we have the following:

THEOREM 5.6. *Let (X, F, Δ) be a complete Menger space with $\Delta = \text{Min}$, and let $\phi : X \to \mathbb{R}$ be a lower semi-continuous function, bounded from below. Suppose that for any $\epsilon > 0$, there exists a $u \in X$ such that*

$$\phi(u) \leq \inf_{x \in X} \phi(x) + \epsilon.$$

If $k : (0,1) \to (0,\infty)$ is a non-increasing function satisfying the condition (2.1), then there exists an $x_0 \in X$ such that

(1) $\sup\{t : F_{fx_0fu}(t) \leq 1-\alpha\} \leq k(r).(\phi(f(u))-\phi(f(x_0))$ *for all* $\alpha \in (0,1]$ *and* $r \in (0,1)$,

(2) $\sup\{t : F_{fx_0fu}(t) \leq 1-\alpha\} \leq k(r)$ *for all* $\alpha \in (0,1]$ *and* $r \in (0,1)$,

(3) *for any* $w \in X$, $w \neq x_0$, *there exists an* $r_0 \in (0,1)$ *such that*

$$\sup\{t : F_{f(x_0)f(w)}(t) \leq 1-\alpha\} > k(r_0).(\phi(f(x_0))-\phi(f(w)))\ \textit{for all}\ \alpha \in (0,1].$$

REMARK 5.7 Our results improve and include the corresponding results in [10], [11] and [12]. In particular, by choosing f to be an identity mapping on X in Theorem 5.6, we recover [12, Theorem 6.6].

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