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ORTHOGONAL SETS IN EFFECT ALGEBRAS

Abstract. We show that for a lattice effect algebra two conceptions of completeness (σ-completeness) coincide. Moreover, a separable effect algebra is complete if and only if it is σ-complete. Further, in an Archimedean atomic lattice effect algebra to every nonzero element x there is a $\oplus$-orthogonal system G of not necessary different atoms such that $x = \bigoplus G$. A lattice effect algebra E is complete if and only if every block of E is complete. Every atomic Archimedean lattice effect algebra is a union of atomic blocks, since each of its elements is a sum of a $\oplus$-orthogonal system of atoms.

1. Introduction and basic definitions

Effect algebras (introduced by Foulis D.J. and Bennett M.K. in [3] (1994)) generalize orthoalgebras (including orthomodular lattices) and MV-algebras [2], providing an instrument for studying quantum effects that may be unsharp.

DEFINITION 1.1. A structure $(E; \oplus, 0, 1)$ is called an *effect algebra* if $0, 1$ are two distinguished elements and $\oplus$ is a partially defined binary operation on E which satisfies the following conditions for any $a, b, c \in E$:

(Ei) $b \oplus a = a \oplus b$, if $a \oplus b$ is defined,
(Eii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$, if one side is defined,
(Eiii) for every $a \in P$ there exists a unique $b \in P$ such that $a \oplus b = 1$, we put $a' = b$,
(Eiv) if $1 \oplus a$ is defined then $a = 0$.

In every effect algebra $(E; \oplus, 0, 1)$ the partial binary operation $\ominus$ and the partial order $\leq$ can be defined by

$$a \leq c \text{ and } c \ominus a = b \quad \text{if and only if} \quad a \oplus b \text{ is defined and } a \oplus b = c\,.$$

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If E with the defined partial order is a lattice then $(E;\oplus,0,1)$ is called a *lattice effect algebra*. Examples of lattice effect algebras are direct products or horizontal sums of an orthomodular lattice and MV-algebra or horizontal sum of two MV-algebras.

In [7] compatibility of two elements of an effect algebra E was introduced. We say that $a, b \in E$ are *compatible* (written $a \leftrightarrow b$) if there exist $u, v, w \in P$ such that $a = u \oplus w$, $b = v \oplus w$ and $u \oplus w \oplus v$ is defined. If E is a lattice effect algebra then $a \leftrightarrow b$ if and only if $(a \vee b) \ominus a = b \ominus (a \wedge b)$. A lattice effect algebra in which every pair $a, b \in E$ is compatible is called a *Boolean effect algebra* ([12]).

2. $\bigoplus$-orthogonal systems of elements

Let $F = \{a_1, a_2, \dots, a_n\}$ be a finite sequence of not necessarily different elements of E. If $a_1 \oplus a_2 \oplus \dots \oplus a_n$ (remind that here $\oplus$ is commutative and associative) exists, then F is called *orthogonal* and the element $a_1 \oplus a_2 \oplus \dots \oplus a_n$ is denoted by $\bigoplus F$.

An arbitrary system $G = \{a_\varkappa\}_{\varkappa \in H}$ of not necessarily different elements of E is called $\oplus$-*orthogonal* if and only if for every finite set $K \subseteq H$ the element $\bigoplus\{a_\varkappa \mid \varkappa \in K\}$ exists in E. For a $\oplus$-orthogonal system G the element $\bigoplus G$ exists if and only if $\bigvee\{\bigoplus\{a_\varkappa \mid \varkappa \in K\} \mid K \subseteq H \text{ is finite}\}$ exists in E. We say that an orthogonal system G_1 is a subsystem of $G = (u_\varkappa)_{\varkappa \in H}$ if and only if there is $H_1 \subseteq H$ such that $G_1 = (u_\varkappa)_{\varkappa \in H_1}$ (written $G_1 \subseteq G$).

Assume that $(E;\oplus,0,1)$ is a lattice effect algebra and $P = \{x_n\}_{n=1}^{\infty}$ is an arbitrary sequence of elements of E. Let for $n = 2, 3, \dots$, $x_n^* := \left(\bigvee_{k=1}^{n} x_k\right) \ominus \left(\bigvee_{k=1}^{n-1} x_k\right)$ and $x_1^* = x_1$. Then for $n = 1, 2 \dots$ we have $\bigoplus_{k=1}^{n} x_k^* = \bigvee_{k=1}^{n} x_k$. Moreover, for every finite $Q \subseteq P$, $\bigoplus Q$ exists and $\bigoplus Q \le \bigoplus_{k=1}^{n_0} x_k^*$ for some $n_0 \in N$ (N is the set of all positive integers). It follows that $P^* = (x_n^*)_{n=1}^{\infty}$ is $\oplus$-orthogonal and $\bigvee P$ exists if and only if $\bigoplus P^*$ exists, in which case $\bigvee P = \bigoplus P^*$. We have proved the following lemma.

LEMMA 2.1. *For a lattice effect algebra $(E;\oplus,0,1)$ the following conditions are equivalent:*

(i) *For every at most countable set $P \subseteq E$, $\bigvee P$ exists in E.*

(ii) *For every at most countable $\oplus$-orthogonal system G of elements of E, $\bigoplus G$ exists in E.*

A lattice effect algebra $(E;\oplus,0,1)$ is called *σ-complete* if E is a σ-complete lattice (equivalently, for every at most countable set $P \subseteq E$, $\bigvee P$

exists in E). E is called *complete* if E is a complete lattice (equivalently, for every $P \subseteq E$, $\bigvee P$ exists in E).

DEFINITION 2.2. An effect algebra $(E; \oplus, 0, 1)$ is called *Archimedean* if for no nonzero element $e \in E$, $ne := e \oplus e \oplus \ldots \oplus e$ (n-times) exists for every $n \in N$. An Archimedean effect algebra is called *separable* if every $\oplus$-orthogonal system of elements of E is at most countable.

Note that every complete effect algebra is Archimedean ([11]).

THEOREM 2.3. *Let $(E; \oplus, 0, 1)$ be a separable σ-complete effect algebra. Then to every set $P \subseteq E$ there is an at most countable set P_1 such that $\bigvee P_1 = \bigvee P$.*

Proof. Let $\emptyset \neq P \subseteq E$. Let $\mathcal{E} = \{\alpha \subseteq P \mid \alpha \text{ is finite}\}$ and for every $\alpha \in \mathcal{E}$ $x_\alpha := \bigvee \alpha$. Let $\alpha_1 \in \mathcal{E}$. If for every $x \in P$ $x \leq x_{\alpha_1}$ then $x_{\alpha_1} = \bigvee P$. Assume that there is an $\alpha \in \mathcal{E}$ such that $x_\alpha \not\leq x_{\alpha_1}$ and put $\alpha_2 := \alpha_1 \cup \alpha$. If $\beta \in \mathcal{E}$ such that $x_\beta \not\leq x_{\alpha_2}$ then we put $\alpha_3 := \alpha_2 \cup \beta, \ldots$. Let $y_\omega := \bigvee_{n=1}^{\infty} x_{\alpha_n}$. If for every $\alpha \in \mathcal{E}$ we have $x_\alpha \leq y_\omega$ then $y_\omega := \bigvee P$. Assume that there is an $\alpha_\omega \in \mathcal{E}$ such that $x_{\alpha_\omega} \not\leq y_\omega$ and let us put $y_{\omega+1} = y_\omega \vee x_{\alpha_\omega}$, hence $y_{\omega+1} \ominus y_\omega \neq 0$. If for every countable transfinite number $\varkappa$ we have $y_{\varkappa+1} > y_\varkappa$ then the system $\{y_{\varkappa+1} \ominus y_\varkappa \mid \varkappa < \varkappa_0\}$, where $\varkappa_0$ is the first uncountable transfinite ordinal number, is an uncountable $\oplus$-orthogonal system of elements of E, a contradiction. We conclude that there is an at most countable set P_1 of elements of E such that $\bigvee P_1 = \bigvee P$.

COROLLARY 2.4. *A separable effect algebra $(E; \oplus, 0, 1)$ is complete if and only if E is σ-complete.*

LEMMA 2.5. *Assume that $(E; \oplus, 0, 1)$ is a lattice effect algebra, $x \in E$ and $\mathcal{U} \subseteq E$ is such that for all $u \in \mathcal{U}$, $u \leq x$. Then $\bigwedge\{x \ominus u \mid u \in \mathcal{U}\} = 0 \Rightarrow \bigvee \mathcal{U} = x$.*

Proof. Let $d \in E$ such that $u \leq d$ for every $u \in \mathcal{U}$. Then $u \leq x \wedge d \leq x$, which gives $x \ominus (x \wedge d) \leq x \ominus u$. It follows that $x \ominus (x \wedge d) = 0$ and hence $x = x \wedge d$. We conclude that $x \leq d$ and thus $x = \bigvee \mathcal{U}$.

THEOREM 2.6. *For an Archimedean lattice effect algebra $(E; \oplus, 0, 1)$ the following conditions are equivalent:*

(i) *For every non-empty subset P of E, $\bigvee P$ exists in E.*
(ii) *For every $\oplus$-orthogonal system G of elements of E, $\bigoplus G$ exists in E.*

Proof. (i)$\Rightarrow$(ii): This is obvious.

(ii)$\Rightarrow$(i): Assume that $\emptyset \neq P \subseteq E$ and $\bigvee P \neq 0$. Let $\mathcal{V} = \{v \in E \mid x \leq v$ for every $x \in P\}$. Let $\mathcal{M} = \{(u_\varkappa)_{\varkappa \in H} \mid (u_\varkappa)_{\varkappa \in H}$ is a $\oplus$-orthogonal system of nonzero elements of E such that for every finite $K \subseteq H$ and every $v \in \mathcal{V}$, $\bigoplus_{\varkappa \in K} u_\varkappa \leq v\}$. Evidently $\mathcal{M}$ is a poset in which every chain has an upper bound. By Zorn's Lemma there exists a maximal element $(u_\varkappa)_{\varkappa \in H_0} \in \mathcal{M}$. Put $G = \{ \bigoplus_{\varkappa \in K} u_\varkappa \mid K \subseteq H_0, K \text{ is finite}\}$. Assume that there is an $e \in E \setminus \{0\}$ such that $e \leq v \ominus g$ for every $v \in \mathcal{V}$ and every $g \in G$. It follows that $e \oplus g \leq v$ for every $g \in G$ and every $v \in \mathcal{V}$. By the maximality of $(u_\varkappa)_{\varkappa \in H_0}$ we obtain that there is a $\varkappa_1 \in H_0$ such that $e = u_{\varkappa_1}$. Moreover, for every $g \in G$ and every $v \in V$ we have $e \oplus g \in G$ which implies that $e \leq v \ominus (e \oplus g)$ and hence $(e \oplus e) \oplus g \leq v$. By induction we obtain that $ne = e \oplus e \oplus \ldots \oplus e$ (n-times) exists for every $n \in N$, a contradiction. We conclude that $\bigwedge\{v \ominus g \mid v \in \mathcal{V}, g \in G\} = 0$. By (ii), $\bigoplus_{\varkappa \in H_0} u_\varkappa = \bigvee G = g_0$ exists in E. Evidently $g_0 \leq v$ for every $v \in \mathcal{V}$. Let $d \leq v$ for every $v \in \mathcal{V}$. Then $g_0 \vee d \leq v$ which gives $(g_0 \vee d) \ominus g \leq v \ominus g$ for every $g \in G$. Hence $\bigwedge\{(g_0 \vee d) \ominus g \mid g \in G\} = 0$, which by Lemma 2.5 implies that $g_0 = \bigvee G = d \vee g_0$. Thus $d \leq g_0$ and hence $g_0 = \bigwedge \mathcal{V} = \bigvee P$.

Throughout the proof of the next theorem we will use the symbol $G \leq v$ if and only if $g \leq v$, for every $g \in G$. Similarly we shall write $x \leftrightarrow G$ if and only if $x \leftrightarrow g$, for every $g \in G$.

THEOREM 2.7. *An Archimedean lattice effect algebra $(E; \oplus, 0, 1)$ is complete if and only if every block of E is complete.*

P r o o f. Assume first that E is complete. By [10], Corollary 4.4, if $M \subseteq E$ is a block and $P \subseteq M$ is such that $\bigvee P$ and $\bigwedge P$ exist in E then $\bigwedge P, \bigwedge P \in M$. Hence M is a complete lattice.

Assume now that every block of E is a complete lattice. Let $(u_\varkappa)_{\varkappa \in H}$ be a $\oplus$-orthogonal system of elements of E. Let $G := \{ \bigoplus_{\varkappa \in K} \mid K \subseteq H$, K is finite$\}$ and let $x, y \in G$. Then there are finite sets $K_1, K_2 \subseteq H$ such that $x = \bigoplus_{\varkappa \in K_1} u_\varkappa$ and $y = \bigoplus_{\varkappa \in K_2} u_\varkappa$. Let $u = \bigoplus_{\varkappa \in K_1 \setminus K_2} u_\varkappa$, $w = \bigoplus_{\varkappa \in K_1 \cap K_2} u_\varkappa$, $v = \bigoplus_{\varkappa \in K_2 \setminus K_1} u_\varkappa$. Then $x = u \oplus w$ and $y = w \oplus v$ and $u \oplus w \oplus v = \bigoplus_{\varkappa \in K_1 \cup K_2} u_\varkappa$, which gives $x \leftrightarrow y$ ([7], [8]). We conclude that G is a set of mutually compatible elements. Let $\mathcal{D} = \{D \subseteq E \mid D$ is a set of mutually compatible upper bounds of $G\}$. Hence for $D \in \mathcal{D}$ and $d_1, d_2 \in D$ we have $G \leq d_1$, $G \leq d_2$ and $d_1 \leftrightarrow d_2$which implies that $G \cup D$ is a set of mutually compatible elements. Moreover, $\mathcal{D}$ with partial order $D_1 \leq D_2$ if and only if $D_1 \subseteq D_2$ is a poset in which every chain $\mathcal{D}_1 \subseteq \mathcal{D}$ has an upper bound $\bigcup \mathcal{D}_1$. By Zorn's Lemma there exists a maximal element $D_0 \in \mathcal{D}$. since $G \cup D_0$ is a set of mutually compatible elements, there is a block $M \subseteq E$ such that $G \cup D_0 \subseteq M$. M

is complete, therefore there is $v_0 = \bigwedge D_0$ in M. Let $w \in E$ be such that $G \leq w$. Then $G \leq v_0 \wedge w \leq v_0 \leq D_0$. It follows that $v_0 \wedge w \leftrightarrow D_0$. By the maximality of D_0 we obtain that $v_0 \wedge w \in D_0$ which gives $v_0 \leq v_0 \wedge w$ and hence $v_0 \leq w$. This proves that $\bigvee G = v_0$ in E. We conclude that there exists $\bigoplus_{\varkappa \in H} u_\varkappa$ and $\bigoplus_{\varkappa \in H} u_\varkappa = v_0$. By Theorem 2.6, for every non-empty subset P of E, $\bigvee P$ exists in E, which implies that E is complete.

Assume that L is a lattice (not necessarily complete). $F \subseteq L$ is called a *full sub-lattice* of L if for all $P, Q \subseteq F$ such that $\bigvee P$ and $\bigwedge Q$ exist in L we have $\bigvee P, \bigwedge Q \in F$.

Assume that $(E; \oplus, 0, 1)$ is a lattice effect algebra. Let us put

$E_S := \{w \in E \mid w \wedge w' = 0\}$, the *set of sharp elements* of E,

$B := \{x \in E \mid x \leftrightarrow y \text{ for all } y \in E\}$, the *compatibility center* of E,

$C(E) := \{z \in E \mid x = (x \wedge z) \vee (x \wedge z') \text{ for all } x \in E\}$, the *center* of E,

M denotes an arbitrary maximal set of mutually compatible elements of E, a *block* of E.

THEOREM 2.8. *Let $(E; \oplus, 0, 1)$ be a lattice effect algebra and let $(a_\varkappa)_{\varkappa \in H}$ be a $\oplus$-orthogonal system such that $\bigoplus_{\varkappa \in H} a_\varkappa$ exists in E. Let $D \in \{E_S, B, C(E), M\}$. Then*

(i) *$(a_\varkappa \in D$ for all $\varkappa \in H) \Rightarrow \bigoplus_{\varkappa \in H} a_\varkappa \in D$,*

(ii) *E is σ-complete $\Rightarrow$ D is σ-complete,*

(iii) *E is complete $\Rightarrow$ D is complete.*

Proof. In [15] it was shown that if $x \leftrightarrow a$ for all $a \in A \subseteq E$ and $\bigvee A$ exists in E then $x \leftrightarrow \bigvee A$. Moreover, by [10] for every finite K, $a_\varkappa \leftrightarrow x$ for every $\varkappa \in K$ implies $\bigoplus_{\varkappa \in K} a_\varkappa \leftrightarrow x$. It follows that B and M are full sub-lattices of E. Further E_S is a full sublattice of E (see [15]) and hence also $C(E) = E_S \cap B$ (see [14]) is a full sublattice of E. Thus every $D \in \{E_S, B, C(E), M\}$ is a full sublattice of E which proves (ii) and (iii). Moreover, every $D \in \{E_S, B, C(E), M\}$ is a sub-effect algebra of E (see [10]–[15]), which gives that if $K \subseteq H$ is finite and $a_\varkappa \in D$ for every $\varkappa \in H$ then $\bigoplus_{\varkappa \in K} a_\varkappa \in D$. and thus also $\bigoplus_{\varkappa \in H} a_\varkappa = \bigvee \{ \bigoplus_{\varkappa \in K} a_\varkappa \mid K \subset H \text{ is finite}\} \in D$.

Note that every M is a maximal Boolean sub-effect algebra of E, and hence $B = \bigcap \{M \mid M \text{ is a block in } E\}$ is also a Boolean sub-effect algebra of E, E_S is an orthomodular lattice (see [15]) and hence $C(E) = B \cap E_S$ is a Boolean algebra.

3. Atoms and finite elements in effect algebras

An element p of an effect algebra $(E; \oplus, 0, 1)$ is called an *atom* if $0 \neq b \leq p \Rightarrow b = p$. E is called *atomic* if for every $x \in E \setminus \{0\}$ there is an atom $p \in E$ such that $p \leq x$. An element $u \in E$ is called *finite* if there is a finite sequence $\{p_1, p_2, \dots, p_n\}$ of not necessarily different atoms of E such that $u = p_1 \oplus p_2 \oplus \dots \oplus p_n$.

THEOREM 3.1. *Let $(E; \oplus, 0, 1)$ be an Archimedean atomic lattice effect algebra. Then for every $x \in E \setminus \{0\}$*

(i) *there is a $\oplus$-orthogonal system $(a_\varkappa)_{\varkappa \in H}$ of atoms of E such that $x = \bigoplus\limits_{\varkappa \in H} a_\varkappa$,*

(ii) $x = \bigvee\{u \in E \mid u \leq x, u$ *is finite*$\}$.

Proof. Assume that $x \in E \setminus \{0\}$.

(i) Put $\mathcal{M} = \{(a_\varkappa)_{\varkappa \in H} \mid (a_\varkappa)_{\varkappa \in H}$ is a $\oplus$-orthogonal system of atoms such that $\bigoplus\limits_{\varkappa \in K} a_\varkappa \leq x$ for every finite $K \subseteq H\}$. Then $\mathcal{M}$ is a poset in which every chain has an upper bound and hence by Zorn's Lemma there is a maximal element $(a_\varkappa)_{\varkappa \in H_0} \in \mathcal{M}$. Now in much the same way as in the proof of Theorem 2.6, the assumption that there is an $e \in E$ such that $e \leq x \ominus \bigoplus\limits_{\varkappa \in K} a_\varkappa$ for every finite $K \subseteq H_0$ implies that $e = 0$. By Lemma 2.5, we conclude that $\bigvee\{\bigoplus\limits_{\varkappa \in K} a_\varkappa \mid K \subseteq H_0 \text{ is finite}\} = x$.

(ii) As we can see above, for every $K \subseteq H_0$ the element $u_K = \bigoplus\limits_{\varkappa \in K} a_\varkappa$ is finite and $x = \bigvee\{u_K \mid K \subseteq H_0 \text{ is finite}\}$, which gives $x = \bigvee\{u \in E \mid u \leq x$, u is finite$\}$.

Assume that $(E; \oplus, 0, 1)$ is a lattice effect algebra. $G \subseteq E$ is a *set of mutually compatible elements* if and only if for every pair $x, y \in G$, $x \leftrightarrow y$. If E is a set of mutually compatible elements then E is called a *Boolean effect algebra* [8], [12] [13]. Every set $G \subseteq E$ of mutually compatible elements of a lattice effect algebra E is a subset of a maximal set of mutually compatible elements called a *block*. Every lattice effect algebra E is a union of its blocks. In fact blocks are maximal Boolean sub-effect algebras of E.

It has been shown in [10] that, for elements of a lattice effect algebra, $x \leftrightarrow z$ and $y \leftrightarrow z$ implies that $x \vee y \leftrightarrow z$ and if $x \oplus y$ exists then also $x \oplus y \leftrightarrow z$. Moreover, if $z \leftrightarrow x$ for every $x \in A$ and $\bigvee A$ exists in E then $z \leftrightarrow \bigvee A$.

It is worth noting that every Boolean effect algebra can be organized into an *MV-algebra* and vice versa (see [8], [13]). Thus every Boolean effect algebra is called also an MV-effect algebra (D. Foulis).

THEOREM 3.2. *Let $(E;\oplus,0,1)$ be an Archimedean atomic lattice effect algebra. Let $A := \{p \in E \mid p$ is an atom$\}$ and $\mathcal{A} := \{A_\varkappa \subseteq A \mid A_\varkappa$ is a maximal set of mutually compatible atoms$\}$. Let $\mathcal{M} := \{M_\varkappa \subseteq E \mid M_\varkappa \supseteq A_\varkappa \in \mathcal{A}$, $M_\varkappa$ is a block of $E\}$. Then $E = \bigcup \mathcal{M}$.*

Proof. Let $0 \neq x \in E$. By Theorem 3.1, there is a $\oplus$-orthogonal system $(a_\gamma)_{\gamma\in G}$ of atoms such that $x = \bigoplus_{\gamma\in G} a_\gamma$. Let $A_\varkappa \in \mathcal{A}$ be such that for every $\gamma \in G$, $a_\gamma \in A_\varkappa$ and let $M_\varkappa \in \mathcal{M}$ be such that $A_\varkappa \subseteq M_\varkappa$. For every $y \in M_\varkappa$ we have $y \leftrightarrow a_\gamma$ for every $\gamma \in G$ and hence $y \leftrightarrow x = \bigvee\{\bigoplus_{\gamma\in K} a_\gamma \mid K \subseteq H$, K is finite$\}$. By the maximality of $M_\varkappa$ we conclude that $x \in M_\varkappa$.

COROLLARY 3.3. *Let $(E;\oplus,0,1)$ be an Archimedean atomic lattice effect algebra. Let the set of all atoms of E be mutually compatible. Then E is a Boolean effect algebra.*

QUESTION. Does there exist an atomic Archimedean non-orthomodular lattice effect algebra which has a non-atomic block?

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