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HEDGING IN THE CRR MODEL UNDER CONCAVE TRANSACTION COSTS

Dedicated to Professor Kazimierz Urbanik

Abstract. In the paper hedging a contingent claim in the Cox-Ross-Rubinstein model under concave transaction costs is studied. Sufficient conditions for the optimality of the replicating strategy for the European option are given. The problem of describing of portfolios which allow, starting from a given moment to hedge a contingent claim is considered for both the European and American options.

1. Introduction

In the paper we consider a discrete time financial market where two assets are given for trading, a riskless bond and a risky stock whose price is characterized by the so-called Cox-Ross-Rubinstein (CRR) model. Transfers of wealth from one asset to another take place only at the discrete moments and the concave transaction costs for these transfers are incurred. We continue to study the problems from [1], [2], [3] where the CRR model with proportional transaction costs was considered.

We show that under some mild assumptions a replicating strategy is optimal for a special class of European options. Next, we prove that if the transaction costs are sufficiently small, a replicating strategy is optimal for any European option. Moreover, for both European and American option simple descriptions of the set of capitals which are sufficient, starting from a given moment to hedge a contingent claim are given.

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2. The model

Let $(\Omega, \mathcal{F}, P)$ be a probability space such that $\Omega = \{a, b\}^T$ where $-1 < a < 0$ and $b > 0$. We consider a market with two assets, a risky stock and a riskless bond with the constant price assumed to be equal to one. We assume that all assets are infinitely divisible.

Throughout this paper (in)equalities or other statements depending on $\omega \in \Omega$ if not stated otherwise will be understood in the P almost sure sense. Sometimes we shall emphasize it and write that such (in)equalities or statements are up to P null events.

Let s_t denote the price of the stock at time $t, t = 0, \dots, T$. We assume that s_t satisfies the following recursive formula:

$s_{t+1} = (1 + \eta_{t+1})s_t, t = 0, \dots, T-1, s_0 \in \mathbf{R}^+ \backslash \{0\}$ where $\eta_t, t = 1, \dots, T$ is a sequence of i.i.d. random variables such that $P(\eta_t = a) + P(\eta_t = b) = 1$ and $0 < P(\eta_t = a) < 1$ for each $t = 1, \dots, T$.

The above recursive formula for the price of the stock characterizes so called Cox-Ross-Rubinstein model.

For any $\omega = (\omega_1, \dots, \omega_T) \in \Omega$, we put $\omega_0^e = (e, \omega_2, \dots, \omega_T)$ and $\omega_t^e = (\omega_1, \dots, \omega_t, e, \omega_{t+2}, \dots, \omega_T)$ for $t = 1, \dots, T-1, \quad e = a, b$.

Let $\mathbf{F} = \{\mathcal{F}_t, t = 0, \dots, T\}$ be a family of increasing sub σ-algebras such that $\mathcal{F}_t = \sigma(s_u, 0 \le u \le t), t = 0, \dots, T$. We assume that $\mathcal{F} = \mathcal{F}_T$.

For any $\mathcal{F}_{t+1}$ measurable random variable p_{t+1} we define $\mathcal{F}_t$ measurable random variables p_t^a and p_t^b as follows: $p_t^a(\omega) = p_{t+1}(\omega_t^a(\omega))$ and $p_t^b(\omega) = p_{t+1}(\omega_t^b(\omega)), t = 0, \dots, T-1$.

Moreover, let $p_{t-1}^{e_1 e_2} = (p_t^{e_1})_{t-1}^{e_2}$ for $e_1, e_2 \in \{a, b\}$ and $t = 1, \dots, T-1$

Selling bonds worth $B(z)$ we get stocks worth z, and selling stocks worth $A(z)$ we get bonds worth z, where $z \ge 0$.

The functions A and B, defined on $\mathbf{R}^+$ are concave, nondecreasing and satisfy $B(0) = 0, A(0) = 0$. Moreover, there exist right-hand derivatives in zero $B'(0), A'(0)$ and $\min\{B(z), A(z)\} \ge z$ for $z \ge 0$.

Let $B'(0) = 1 + \lambda$ and $A'(0) = \frac{1}{1-\mu}$.

Let us introduce the function τ as follows:

$$\tau(z) = \begin{cases} B(z) \text{ if } z \ge 0 \\ -A^{-1}(-z) \text{ if } z < 0, \end{cases}$$

$\tau(z)$ can be interpreted as the cost of getting stocks worth z (z negative means that we sell $|z|$ stocks).

REMARK 2.1. Let $p, q \in \mathbf{R}$. Then, $\tau(p) + \tau(q) \ge \tau(p+q)$.

For any $\mathcal{F}_t$ measurable r.v. p_t we define a function Ψ_{p_t} as follows:
$\Psi_{p_t}(z) = \tau((p_{t-1}^b - z)s_{t-1}^b) - \tau((p_{t-1}^a - z)s_{t-1}^a), z \in \mathbf{R}$ for each $t = 1, \dots, T$.

A trading strategy (x,y) is a pair of processes $\{(x_t,y_t),t=0,\ldots,T\}$ where x_t,y_t are $\mathcal{F}_t$ measurable for each $t=0,\ldots,T$. Here, x_t,y_t denote numbers of units of bonds and stocks respectively, held by the seller of the option at time t (after transaction at this moment). Moreover, for a strategy (x,y) let $x_{-1},y_{-1}\in\mathbf{R}$ denote initial holdings (in units) of bonds and shares of the stock respectively.

A trading strategy (x,y) is said to be self-financing if :

$x_t-x_{t-1}+\tau((y_t-y_{t-1})s_t)\leq 0,\ t=0,\ldots,T.$

The above inequality means that at every trading moment, the sales must finance the purchases.

Denote the set of all self-financing, trading strategies by $\mathcal{A}$.

3. European options

A European contingent claim (or a European option) f is a pair $f=(f_1,f_2)$ of $\mathcal{F}_T$ measurable random variables. Here, f_1,f_2 denote number of units of bonds and stocks respectively, that are paid to the buyer of the option at time T.

We say that a self-financing, trading strategy (x,y) hedges a contingent claim f if $f_1-x_{T-1}+\tau((f_2-y_{T-1})s_T)\leq 0$.

We say that a trading strategy $(x,y)\in\mathcal{A}$ is replicating for a contingent claim f if:

$x_t-x_{t-1}+\tau((y_t-y_{t-1})s_t)=0,\ t=0,\ldots,T$ and $(x_T,y_T)=(f_1,f_2)$.

For any $(p,q)\in\mathbf{R}^2$ we define a set $C_{(p,q)}$ as follows:

$C_{(p,q)}=\{(u,v)\in\mathbf{R}^2: p-u+\tau(q-v)\leq 0\}.$

REMARK 3.1. For any $(p,q)\in\mathbf{R}^2$ if $(u,v)\in C_{(p,q)}$ then $C_{(u,v)}\subseteq C_{(p,q)}$.

The seller's price of a European option f is defined by:

$\pi(f)=\inf\{x_0+\tau(y_0s_0),(x,y)\in\mathcal{A}$ and hedges $f\}$.

Given an option f, we say that a hedging strategy $(x,y)\in\mathcal{A}$ is optimal if for any other hedging strategy $(\overline{x},\overline{y})\in\mathcal{A}$ we have $C_{(\overline{x}_0,\overline{y}_0s_0)}\subseteq C_{(x_0,y_0s_0)}$.

It is not difficult to see that if a strategy $(x,y)\in\mathcal{A}$ is optimal for an option f then the value $x_0+\tau(y_0s_0)$ connected with this strategy is equal to the price of f.

For any $\omega\in\Omega$ and $t=0,\ldots,T-1$ we define sets $H_f(t)(\omega)$ and $H'_f(t)(\omega)$ as follows:

$H_f(t)(\omega)=\{(u,v)\in\mathbf{R}^2:$ there exists $(x,y)\in\mathcal{A}$ such that $(x_{t-1},y_{t-1}s_t)(\omega)=(u,v)$ and $P[f_1-x_{T-1}+\tau((f_2-y_{T-1})s_T)\leq 0\mid\mathcal{F}_t](\omega)=1\}$.

$H_f(t)$ is a set of pre-transaction portfolios that at time t guarantee hedging the claim f at time T.

$H'_f(t)(\omega)=\{(u,v)\in\mathbf{R}^2:$ there exists $(x,y)\in\mathcal{A}$ such that $(x_t,y_ts_t)(\omega)=(u,v)$ and $P[f_1-x_{T-1}+\tau((f_2-y_{T-1})s_T)\leq 0\mid\mathcal{F}_t](\omega)=1\}$.

$H'_f(t)$ is a set of post-transaction portfolios that at time t guarantee hedging the claim f at time T.

Moreover, let $H_f(T) = C_{(f_1, f_2 s_T)}$. $H_f(T)$ is a set of pre-transaction portfolios that at the moment T guarantee hedging the claim $f(T)$ at time T.

For each $t = 0, \ldots, T-1$ we have the following fact:

LEMMA 3.2. *Let u, v be real numbers and $\omega \in \Omega$ be fixed. If $(u, vs_t(\omega)) \in H_f(t)(\omega)$ and $H'_f(t)(\omega) \subseteq C_{(u, vs_t(\omega))}$ then $H_f(t)(\omega) = C_{(u, vs_t(\omega))}$.*

Proof. To simplify notation we will omit the dependency on ω in this proof.

It is not difficult to see that $C_{(u,vs_t)} \subseteq H_f(t)$. We only have to prove that $H_f(t) \subseteq C_{(u,vs_t)}$. Let u_1, v_1 be real numbers such that $(u_1, v_1 s_t) \in H_f(t)$. Then, there exists $(u_2, v_2) \in \mathbf{R}^2$ such that $(u_2, v_2 s_t) \in H'_f(t)$ and $u_2 - u_1 + \tau((v_2 - v_1)s_t) \leq 0$. Since $H'_f(t)$ is contained in $C_{(u,vs_t)}$ we have an inequality $u - u_2 + \tau((v - v_2)s_t) \leq 0$.

The last two inequalities imply $u - u_1 + \tau((v - v_2)s_t) + \tau((v_2 - v_1)s_t) \leq 0$.

Thus, by Remark 2.1 we get $u - u_1 + \tau((v - v_1)s_t) \leq 0$.

Therefore $(u_1, v_1 s_t) \in C_{(u,vs_t)}$ and finally we obtain $H_f(t) \subseteq C_{(u,vs_t)}$.

The proof is therefore completed. ∎

Now for each $t = 1, \ldots, T$ we define a set Π_t consisting of a special type of pairs of random variables.

Let $\Pi_t, t = 1, \ldots, T$ denote a set of all pairs of random variables $(p(s_t), q(s_t))$ such that q is a nondecreasing real function and there exists a random variable $w(s_{t-1})$ such that

$$\tau((q^b_{t-1} - w(s_{t-1}))s^b_{t-1}) - \tau((q^a_{t-1} - w(s_{t-1}))s^a_{t-1}) = p^a_{t-1} - p^b_{t-1}$$

and $q^a_{t-1} \leq w(s_{t-1}) \leq q^b_{t-1}$.

Using [4] we have the following fact:

THEOREM 3.3. *For any European option f there exists a replicating strategy. If $\frac{1+b}{1+a} > \frac{1+\lambda}{1-\mu}$ then a replicating strategy is unique.*

Moreover, if $f \in \Pi_T$ and a strategy $(x, y) \in \mathcal{A}$ is replicating for f, then (x, y) satisfies the inequalities $y^a_t \leq y_t \leq y^b_t$ for each $t = 0, \ldots, T-1$.

From now on in this paper we make the following assumption:

(3.1) $b - \lambda > 0$ and $\mu + a < 0$.

THEOREM 3.4. *Let f be a European option such that $f \in \Pi_T$. Then, there exists a unique replicating strategy $(x, y) \in \mathcal{A}$ which is optimal.*

Moreover, $H_f(t) = C_{(x_t, y_t s_t)}$ for each $t = 0, \ldots, T-1$.

Proof. By Theorem 3.3 there exists a unique strategy $(x, y) \in \mathcal{A}$ which is replicating for f and satisfies inequalities $y^a_t \leq y_t \leq y^b_t$ for each $t = 0, \ldots, T-1$. It is clear that $C_{(x_t, y_t s_t)} \subseteq H_f(t)$ for each $t = 0, \ldots, T-1$ and

$C_{(x_T, y_T s_T)} = H_f(T)$. Suppose, for some $t = 0, \ldots, T-1$ we have $C_{(x_{t+1}, y_{t+1} s_{t+1})} = H_f(t+1)$.

From now on we fix an $\omega \in \Omega$ in this proof.

Let u, v be real numbers such that $(u, vs_t) \in H_f'(t)$. Then, by definition of $H_f'(t)$ we have the inequalities:

(3.2) $x_t^a - u + \tau((y_t^a - v)s_t^a) \leq 0,$

(3.3) $x_t^b - u + \tau((y_t^b - v)s_t^b) \leq 0.$

Since the strategy (x, y) is replicating we have:

(3.4) $x_t^a - x_t + \tau((y_t^a - y_t)s_t^a) = 0,$

(3.5) $x_t^b - x_t + \tau((y_t^b - y_t)s_t^b) = 0.$

We will prove now that

(3.6) $x_t - u + \tau((y_t - v)s_t) \leq 0.$

There are two cases:

1. $v \leq y_t$.

From (3.3) and (3.5) we have

(3.7) $x_t - u + \tau((y_t^b - v)s_t^b) - \tau((y_t^b - y_t)s_t^b) \leq 0.$

Since $y_t \leq y_t^b$ we have the following inequalities:

$(y_t - v)s_t^b \leq \tau((y_t^b - v)s_t^b) - \tau((y_t^b - y_t)s_t^b)$, $\tau((y_t - v)s_t) \leq \frac{1+\lambda}{1+b}(y_t - v)s_t^b$.

Therefore, by (3.1) we obtain $\tau((y_t^b - v)s_t^b) - \tau((y_t^b - y_t)s_t^b) \geq \tau((y_t - v)s_t)$.

The above inequality and (3.7) imply (3.6).

2. $v \geq y_t$.

From (3.2) and (3.4) we have

(3.8) $x_t - u + \tau((y_t^a - v)s_t^a) - \tau((y_t^a - y_t)s_t^a) \leq 0.$

Since $y_t^a \leq y_t$ we have the following inequalities:

$\tau((y_t^a - y_t)s_t^a) - \tau((y_t^a - v)s_t^a) \leq (v - y_t)s_t^a$, $\frac{1-\mu}{1+a}(v - y_t)s_t^a \leq -\tau((y_t - v)s_t)$.

Therefore, by (3.1) we obtain $\tau((y_t^a - v)s_t^a) - \tau((y_t^a - y_t)s_t^a) \geq \tau((y_t - v)s_t)$.

From the above inequality and (3.8) we get (3.6).

By (3.6) we have $H_f'(t) \subseteq C_{(x_t, y_t s_t)}$. Therefore since $(x_t, y_t s_t) \in H_f(t)$, by Lemma 3.2 we see that $H_f(t) = C_{(x_t, y_t s_t)}$. By backward induction, we get $H_f(t) = C_{(x_t, y_t s_t)}$, P *a.s.* for each $t = 0, \ldots, T-1$. For any other hedging strategy $(\overline{x}, \overline{y}) \in \mathcal{A}$ we have $(\overline{x}_0, \overline{y}_0 s_0) \in H_f(0)$. Therefore the strategy (x, y) is optimal by Remark 3.1.

The proof is now completed. ∎

3.1. Small transaction costs

In this subsection we will show that if the transaction costs are sufficiently small, i.e.

(3.9) $\min\{1+b, \frac{1}{1+a}\} > \frac{1+\lambda}{1-\mu}$

then, for any European contingent claim f there exists an optimal, self-financing, trading strategy which replicates the portfolio (f_1, f_2) at time T.

We have the following fact:

THEOREM 3.5. *If the condition* (3.9) *is satisfied, then for any European option f there exists a replicating strategy $(x, y) \in \mathcal{A}$ which is optimal.*

Moreover, $H_f(t) = C_{(x_t, y_t s_t)}$ for each $t = 0, \ldots, T-1$.

Proof. Let f be a given European option. By Theorem 3.3 there exists a unique strategy $(x, y) \in \mathcal{A}$ which is replicating for f. It is clear that $C_{(x_t, y_t s_t)} \subseteq H_f(t)$ for each $t = 0, \ldots, T-1$ and $C_{(x_T, y_T s_T)} = H_f(T)$. Suppose, for some $t = 0, \ldots, T-1$ we have $C_{(x_{t+1}, y_{t+1} s_{t+1})} = H_f(t+1)$.

Since (x, y) is replicating the following inequalities hold:

(3.10) $x_t^a - x_t + \tau((y_t^a - y_t)s_t^a) = 0,$

(3.11) $x_t^b - x_t + \tau((y_t^b - y_t)s_t^b) = 0.$

From now on we fix an $\omega \in \Omega$ in this proof.

Let u, v be real numbers such that $(u, v s_t) \in H_f'(t)$. Then, by definition of $H_f'(t)$ we have the inequalities:

(3.12) $x_t^a - u + \tau((y_t^a - v)s_t^a) \le 0,$

(3.13) $x_t^b - u + \tau((y_t^b - v)s_t^b) \le 0.$

We will prove now that

(3.14) $x_t - u + \tau((y_t - v)s_t) \le 0.$

There are a six cases:

1. $v \le y_t \le y_t^b$.

In this case the proof of (3.14) is analogous to the proof of (3.6) in case 1 of Theorem 3.4.

2. $y_t^b \le v \le y_t$.

From (3.11) and (3.13) we get

(3.15) $x_t - u + \tau((y_t^b - v)s_t^b) - \tau((y_t^b - y_t)s_t^b) \le 0.$

From the properties of the functions A and B we have the following inequalities:

$\tau((y_t^b - v)s_t^b) - \tau((y_t^b - y_t)s_t^b) \ge (1-\mu)(y_t - v)s_t^b,$

$(y_t - v)s_t^b \ge \frac{1+b}{1+\lambda}\tau((y_t - v)s_t).$

Therefore, by (3.9) we obtain:

$\tau((y_t^b - v)s_t^b) - \tau((y_t^b - y_t)s_t^b) \ge \tau((y_t - v)s_t).$

From the last inequality and (3.15) we get (3.14).

3. $v \le y_t^b \le y_t$.

By (3.11) and (3.13) we get (3.15).

By (3.9) we have $(b-\lambda)(y_t^b - v) \ge 0 \ge ((1+\lambda) - (1-\mu)(1+b))(y_t - y_t^b)$.

From the above we get:

$$(1+b)(y_t^b - v) - (1-\mu)(1+b)(y_t^b - y_t) \ge (1+\lambda)(y_t - v). \tag{3.16}$$

By the properties of the function τ we have:

$\tau((y_t^b - v)s_t^b) \ge (1+b)(y_t^b - v)s_t$,

$\tau((y_t^b - y_t)s_t^b) \le (1-\mu)(1+b)(y_t^b - y_t)s_t$,

$\tau((y_t - v)s_t) \le (1+\lambda)(y_t - v)s_t$.

By the last three inequalities and (3.16) we obtain:

$\tau((y_t^b - v)s_t^b) - \tau((y_t^b - y_t)s_t^b) \ge \tau((y_t - v)s_t)$.

From the last inequality and (3.15) we get (3.14).

4. $y_t^a \le y_t \le v$.

In this case the proof of (3.14) is analogous to the proof of (3.6) in case 2 of Theorem 3.4.

5. $y_t \le v \le y_t^a$.

From (3.10) and (3.12) we have

$$x_t - u + \tau((y_t^a - v)s_t^a) - \tau((y_t^a - y_t)s_t^a) \le 0. \tag{3.17}$$

From the properties of the functions A and B we have the following inequalities:

$\tau((y_t^a - v)s_t^a) - \tau((y_t^a - y_t)s_t^a) \ge (1+\lambda)(y_t - v)s_t^a$,

$(y_t - v)s_t^a \ge \frac{1+a}{1-\mu}\tau((y_t - v)s_t)$.

Therefore, by (3.9) we obtain $\tau((y_t^a - v)s_t^a) - \tau((y_t^a - y_t)s_t^a) \ge \tau((y_t - v)s_t)$.

From the last inequality and (3.17) we get (3.14).

6. $y_t \le y_t^a \le v$.

From (3.10) and (3.12) we get (3.17).

By (3.9) we have $(a+\mu)(v - y_t^a) \le 0 \le ((1-\mu) - (1+\lambda)(1+a))(y_t^a - y_t)$.

From the above we get:

$$(1+a)(y_t^a - v) - (1+\lambda)(1+a)(y_t^a - y_t) \ge (1-\mu)(y_t - v). \tag{3.18}$$

By the properties of the function τ we have:

$\tau((y_t^a - v)s_t^a) \ge (1+a)(y_t^a - v)s_t$,

$\tau((y_t^a - y_t)s_t^a) \le (1+\lambda)(1+a)(y_t^a - y_t)s_t$,

$\tau((y_t - v)s_t) \le (1-\mu)(y_t - v)s_t$.

By the last three inequalities and (3.18) we obtain

$\tau((y_t^a - v)s_t^a) - \tau((y_t^a - y_t)s_t^a) \ge \tau((y_t - v)s_t)$.

From the last inequality and (3.17) we get (3.14).

By (3.14) we have $H_f'(t) \subseteq C_{(x_t,y_ts_t)}$. Therefore since $(x_t, y_ts_t) \in H_f(t)$, by Lemma 3.2 we see that $H_f(t) = C_{(x_t,y_ts_t)}$. By backward induction, we have $H_f(t) = C_{(x_t,y_ts_t)}$, P *a.s.* for each $t = 0, \dots, T-1$. For any other hedging strategy $(\overline{x}, \overline{y}) \in \mathcal{A}$ we have $(\overline{x}_0, \overline{y}_0 s_0) \in H_f(0)$. Therefore the strategy (x, y) is optimal by Remark 3.1.

The proof is therefore completed. ∎

4. American options

We define an American option (or an American contingent claim) φ as a pair $\{\varphi(t) = (\varphi_1(t), \varphi_2(t)), t = 0, \dots, T\}$ of $\mathbf{F}$ adapted processes. Here, $\varphi_1(t), \varphi_2(t)$ denote number of units of bonds and stocks respectively, that are paid to the option's buyer assuming he exercises the option at time t.

We say that a strategy $(x, y) \in \mathcal{A}$ hedges an American contingent claim φ if $\varphi_1(t) - x_{t-1} + \tau((\varphi_2(t) - y_{t-1})s_t) \le 0$ for each $t = 0, \dots, T$.

The seller's price of an American option φ is defined by:

$\pi(\varphi) = \inf\{x_0 + \tau(y_0 s_0), (x, y) \in \mathcal{A} \text{ and hedges } \varphi\}$.

Given an option φ, we say that a hedging strategy $(x, y) \in \mathcal{A}$ is optimal if for any other hedging strategy $(\overline{x}, \overline{y}) \in \mathcal{A}$ we have $C_{(\overline{x}_0, \overline{y}_0 s_0)} \subseteq C_{(x_0, y_0 s_0)}$.

It is not difficult to see that if a strategy $(x, y) \in \mathcal{A}$ is optimal for an option φ then the value $x_0 + \tau(y_0 s_0)$ connected with this strategy is equal to the price of φ.

For any $\omega \in \Omega$ and $t = 0, \dots, T-1$ we define sets $H_\varphi(t)(\omega)$ and $H_\varphi'(t)(\omega)$ as follows:

$H_\varphi(t)(\omega) = \{(u, v) \in \mathbf{R}^2$: there exists $(x, y) \in \mathcal{A}$ such that $(x_{t-1}, y_{t-1}s_t)(\omega) = (u, v)$ and $P[\varphi_1(n) - x_{n-1} + \tau((\varphi_2(n) - y_{n-1})s_n) \le 0 \mid \mathcal{F}_t](\omega) = 1$ for each $n = t+1, \dots, T\}$.

$H_\varphi(t)$ is a set of pre-transaction portfolios that at time t guarantee hedging the claim $\varphi(n)$ at time n for each $n = t+1, \dots, T$.

$H_\varphi'(t)(\omega) = \{(u, v) \in \mathbf{R}^2$: there exists $(x, y) \in \mathcal{A}$ such that $(x_t, y_ts_t)(\omega) = (u, v)$ and $P[\varphi_1(n) - x_{n-1} + \tau((\varphi_2(n) - y_{n-1})s_n) \le 0 \mid \mathcal{F}_t](\omega) = 1$ for each $n = t+1, \dots, T\}$.

$H_\varphi'(t)$ is a set of post-transaction portfolios that at time t guarantee hedging the claim $\varphi(n)$ at time n for each $n = t+1, \dots, T$.

Moreover, let $H_\varphi(T) = C_{(\varphi_1(t), \varphi_2(t)s_T)}$. $H_\varphi(T)$ is a set of pre-transaction portfolios that at the moment T guarantee hedging the claim $\varphi(T)$ at time T.

Let Γ denote a set of all functions γ which satisfy the following conditions:

(C1) $z_2 - z_1 \le \gamma(z_2) - \gamma(z_1) \le (1+\lambda)(z_2 - z_1)$, for any real, nonnegative z_1, z_2 such that $z_1 \le z_2$.

(C2) $z_2 - z_1 \ge \gamma(z_2) - \gamma(z_1) \ge (1-\mu)(z_2 - z_1)$, for any real, nonpositive z_1, z_2 such that $z_1 \le z_2$.

(C3) $\gamma(z) \leq \tau(z)$, for any $z \in \mathbf{R}$.
(C4) $\gamma(0) = 0$.

LEMMA 4.1. *For any $\gamma \in \Gamma$ and u, $v \in \mathbf{R}$, if $uv \leq 0$ then $\gamma(u) + \gamma(v) \geq \gamma(u+v)$.*

Proof. Without any loss of generality we assume that $u \geq 0$ and $v \leq 0$
We have two cases.

1. $u+ v \geq 0$.

By (C1) we have $\gamma(u) - \gamma(u+v) \geq -v$. From (C2) and (C4) we get $-\gamma(v) \leq -v$. Combining the last two inequalities we obtain $\gamma(u) + \gamma(v) \geq \gamma(u+v)$.

2. $u+ v \leq 0$.

By (C2) we have $\gamma(u+v) - \gamma(v) \leq u$. From (C1) and (C4) we get $\gamma(u) \geq u$. Combining the last two inequalities we obtain $\gamma(u) + \gamma(v) \geq \gamma(u+v)$.

The proof is therefore completed. ∎

For any $(p_1, p_2) \in \mathbf{R}^2$ we define sets $\overline{\partial}C_{(p_1,p_2)}$ and $\underline{\partial}C_{(p_1,p_2)}$ as follows:
$\overline{\partial}C_{(p_1,p_2)} = \{(u,v) \in \mathbf{R}^2 : p_1 - u + \tau(p_2 - v) = 0 \text{ and } v > p_2\}$,
$\underline{\partial}C_{(p_1,p_2)} = \{(u,v) \in \mathbf{R}^2 : p_1 - u + \tau(p_2 - v) = 0 \text{ and } v < p_2\}$.

For any $p = (p_1, p_2) \in \mathbf{R}^2$ and $q = (q_1, q_2) \in \mathbf{R}^2$ we define a set $V(p,q)$ as follows:

$V(p,q) = \{(c,d) \in \partial(C_p \cap C_q)$: for any $(u,v) \in \partial(C_p \cap C_q)$ if $v > d$ then $(u,v) \in \overline{\partial}C_p \cup \overline{\partial}C_q$ and if $v < d$ then $(u,v) \in \underline{\partial}C_p \cup \underline{\partial}C_q\}$. Here $\partial(C_p \cap C_q)$ denotes a boundary of $C_p \cap C_q$, i.e. $\partial(C_p \cap C_q) = (C_p \cap C_q) \backslash int(C_p \cap C_q)$.

REMARK 4.2. The set $V(p,q)$ is non-empty.

LEMMA 4.3. *For any $(c,d) \in V(p,q)$ there exists $\nu \in \Gamma$ such that*
$C_p \cap C_q = \{(u,v) \in \mathbf{R}^2 : c - u + \nu(d-v) \leq 0\}$.

Proof. Let $(c,d) \in V(p,q)$. It is not difficult to see that there exists a continuous function ν such that $\partial(C_{(p_1,p_2)} \cap C_{(q_1,q_2)}) = \{(u,v) \in \mathbf{R}^2 : c - u + \nu(d-v) = 0\}$ and $C_{(p_1,p_2)} \cap C_{(q_1,q_2)} = \{(u,v) \in \mathbf{R}^2 : c - u + \nu(d-v) \leq 0\}$. Obviously, ν satisfies (C4).

By Remark 3.1 we have $\partial C_{(c,d)} \subseteq C_p \cap C_q$. Therefore for any $(u,v) \in \mathbf{R}^2$, if $c - u + \tau(d-v) = 0$ then $c - u + \nu(d-v) \leq 0$. Consequently, ν satisfies (C3).

Now, we will prove that ν satisfies (C1) and (C2). For each $z \geq 0$ there exists $\varepsilon > 0, \xi \geq -z$ and $\zeta \in \mathbf{R}$ such that $\nu(e) = \tau(e + \xi) + \zeta$ for any $e \in \langle z, z+\varepsilon \rangle$. Moreover, for each $z \leq 0$ there exists $\varepsilon > 0, \xi \geq z$ and $\zeta \in \mathbf{R}$ such that $\nu(e) = \tau(e - \xi) + \zeta$ for any $e \in \langle z - \varepsilon, z \rangle$. Thus, because ν is continuous, it satisfies (C1) and (C2) by the properties of τ.

The proof is therefore completed. ∎

For each $t = 1, \dots, T$ we have the following fact:

LEMMA 4.4. *Let* $(p_1(s_t), p_2(s_t)), (q_1(s_t), q_2(s_t)) \in \Pi_t$. *There exist random variables* $(c(s_t), d(s_t))$ *such that for any* $\omega \in \Omega$ *there is a function* $\nu_{s_t(\omega)} \in \Gamma$ *with the equality:* $C_{(p_1,p_2s_t)(\omega)} \cap C_{(q_1,q_2s_t)(\omega)} = \{(u,v) \in \mathbf{R}^2 : c(s_t)(\omega) - u + \nu_{s_t}(d(s_t)s_t - v)(\omega) \le 0\}$.

Moreover, there exists a unique r.v. $w(s_{t-1})$ *such that* $\nu_{s^b_{t-1}}((d^b_{t-1} - w(s_{t-1}))s^b_{t-1}) - \nu_{s^a_{t-1}}((d^a_{t-1} - w(s_{t-1}))s^a_{t-1}) = c^a_{t-1} - c^b_{t-1}$ *and* $d^a_{t-1} \le w(s_{t-1}) \le d^b_{t-1}$.

Proof. To simplify notation we shall write p_i, q_i respectively, instead of $p_i(s_t), q_i, (s_t)$ $i = 1, 2$. By Remark 4.2 it is easily seen that there exists a pair of random variables $(c(s_t), d(s_t))$ such that $(c(s_t), d(s_t)s_t) \in V((p_1, p_2s_t), (q_1, q_2s_t))$.

Therefore, by Lemma 4.3 for any $\omega \in \Omega$ there exists a function $\nu_{s_t(\omega)}$ such that the first equality of Lemma 4.4 holds. Moreover, it is not difficult to see that $\partial(C_{(p_1,p_2s_t)} \cap C_{(q_1,q_2s_t)}) = \{(u,v) \in \mathbf{R}^2 : c(s_t) - u + \nu_{s_t}(d(s_t)s_t - v) = 0\}$.

From now on we fix an $\omega \in \Omega$ in this proof.

Let Φ_t be a function defined as follows:
$\Phi_t(z) = \nu_{s^b_{t-1}}((d^b_{t-1} - z)s^b_{t-1}) - \nu_{s^a_{t-1}}((d^a_{t-1} - z)s^a_{t-1})$.

Since $\nu_{s^a_{t-1}}, \nu_{s^b_{t-1}} \in \Gamma$ we have $\Phi_t(z) \le (((1+a) - (1-\mu)(1+b))z + (1-\mu)(1+b)d^b_{t-1} - (1+a)d^a_{t-1})s_{t-1}$ for $z \ge \max\{d^a_{t-1}, d^b_{t-1}\}$ and $\Phi_t(z) \ge (((1+a)(1+\lambda) - (1+b))z + (1+b)d^b_{t-1} - (1+a)(1+\lambda)d^a_{t-1})s_{t-1}$ for $z \le \min\{d(s^a_{t-1}), d(s^b_{t-1})\}$.

Therefore by (3.1) we conclude that
$\lim_{z\to\infty} \Phi_t(z) = -\infty$ and $\lim_{z\to-\infty} \Phi_t(z) = \infty$.

Furthermore, since $\nu_{s^a_{t-1}}, \nu_{s^b_{t-1}} \in \Gamma$ by (3.1) it follows that the function Φ_t is strictly decreasing. Thus, because $\Phi_t(z)$ is continuous there exists a unique random variable $w(s_{t-1})$ such that $\Phi_t(w(s_{t-1})) = c^a_{t-1} - c^b_{t-1}$.

Consequently, it follows immediately that there exists a r.v. $u(s_{t-1})$ such that

$$(4.1)\quad c^e_{t-1} - u(s_{t-1}) + \nu_{s^e_{t-1}}((d^e_{t-1} - w(s_{t-1}))s^e_{t-1}) = 0, \quad e = a, b.$$

To simplify notation we shall write p^e_i, q^e_i, u, w respectively, instead of $p_i(s^e_{t-1}), q^e_i(s^e_{t-1}), u(s_{t-1}), w(s_{t-1})$ $i = 1, 2; e = a, b$.

Assume that $d^b_{t-1} < w$.

Thus, since $(c^b_{t-1}, d^b_{t-1}s^b_{t-1}) \in V((p^b_1, p^b_2s^b_{t-1}), (q^b_1, q^b_2s^b_{t-1}))$ by (4.1) it follows that $(u, ws^b_{t-1}) \in \overline{\partial}C_{(p^b_1,p^b_2s^b_{t-1})} \cup \overline{\partial}C_{(q^b_1,q^b_2s^b_{t-1})}$.

Assume that $(u, ws^b_{t-1}) \in \overline{\partial}C_{(p^b_1,p^b_2s^b_{t-1})}$.

Consequently, $w > p^b_2$ and $p^b_1 - u + \tau((p^b_2 - w)s^b_{t-1}) = 0$.

By (4.1) it is clear that $p^a_1 - u + \tau((p^a_2 - w)s^a_{t-1}) \le 0$.

Consequently,

(4.2) $\tau((p_2^b - w)s_{t-1}^b) - \tau((p_2^a - w)s_{t-1}^a) \geq p_1^a - p_1^b$.

Since $(p_1, p_2) \in \Pi_t$, there exists $z \in \langle p_2^a, p_2^b \rangle$ such that $\tau((p_2^b - z)s_{t-1}^b) - \tau((p_2^a - z)s_{t-1}^a) = p_1^a - p_1^b$.

The last equality and (4.2) imply

$\tau((p_2^b - w)s_{t-1}^b) - \tau((p_2^a - w)s_{t-1}^a) \geq \tau((p_2^b - z)s_{t-1}^b) - \tau((p_2^a - z)s_{t-1}^a)$.

Transforming equivalently we get:

(4.3) $\tau((p_2^a - z)s_{t-1}^a) - \tau((p_2^a - w)s_{t-1}^a) \geq \tau((p_2^b - z)s_{t-1}^b) - \tau((p_2^b - w)s_{t-1}^b)$.

By the inequalities $p_2^a \leq z \leq p_2^b < w$ and the properties of τ we have

$\tau((p_2^a - z)s_{t-1}^a) - \tau((p_2^a - w)s_{t-1}^a) \leq (1 + a)(w - z)s_{t-1}$,

$\tau((p_2^b - z)s_{t-1}^b) - \tau((p_2^b - w)s_{t-1}^b) \geq (1 - \mu)(1 + b)(w - z)s_{t-1}$.

From the last two inequalities, (4.3) and since $w > z$ we get $(1 + a) \geq (1 - \mu)(1 + b)$.

In case when $(u, ws_{t-1}) \in \overline{\partial} C_{(q_1^b, q_2^b s_{t-1}^b)}$ the proof is the same as above, we only write p_1^b, p_2^b respectively instead of q_1^b, q_2^b. But the inequality $(1 + a) \geq (1 - \mu)(1 + b)$ is a contradiction to (3.1) and consequently we have $w \leq d_{t-1}^b$. By a similar consideration it can be shown that $d_t^a \leq w$, the proof of which we leave to the reader. ∎

Using Lemma 4.4 we will prove the following theorem concerning American options with $\varphi(t) \in \Pi_t, t = 1, \ldots, T$.

THEOREM 4.5. *Let φ be an American option such that $\varphi(t) \in \Pi_t$, for each $t = 1, \ldots, T$. Then there exists a strategy $(x, y) \in \mathcal{A}$ which is optimal and $H_\varphi(t) = C_{(x_t, y_t s_t)}$ for each $t = 0, \ldots, T - 1$. Moreover, $(x_t, y_t) \in \Pi_t$ for each $t = 1, \ldots, T - 1$.*

Proof. We shall construct our strategy $(x, y) = \{(x_t(s_t), y_t(s_t), t = 0, \ldots, T)\}$ using backward induction. We set $(x_T, y_T) = (\varphi_1(s_T), \varphi_2(s_T))$.

It is clear that $H_\varphi(T) = C_{(x_T, y_T s_T)}$ and $(x_T, y_T) \in \Pi_T$.

Assume that for some $t = 0, \ldots, T - 1$ there exist random variables $x_{t+1}(s_{t+1}), y_{t+1}(s_{t+1})$ such that $H_\varphi(t + 1) = C_{(x_{t+1}, y_{t+1} s_{t+1})}$ and $(x_{t+1}, y_{t+1}) \in \Pi_{t+1}$.

Then, for any $(u, v) \in \mathbf{R}^2$ and $\omega \in \Omega$ we have the following equivalence: $(u, vs_t(\omega)) \in H'_\varphi(t)(\omega)$ if and only if it satisfies a system of inequalities:

$$\begin{cases} x_t^e(\omega) - u + \tau((y_t^e - v)s_t^e)(\omega) \leq 0 \\ \varphi_1(s_t^e(\omega)) - u + \tau((\varphi_2(s_t^e) - v)s_t^e)(\omega) \leq 0. \end{cases} \quad e = a, b$$

By Lemma 4.4 there exists a pair random variables $(c_{t+1}(s_{t+1}), d_{t+1}(s_{t+1}))$ such that for any $\omega \in \Omega$ there is a function $\nu_{t, s_{t+1}(\omega)} \in \Gamma$ with the following equivalence:

for any $(u, v) \in \mathbf{R}^2$ a system of inequalities:

$$\begin{cases} x_{t+1}(\omega) - u + \tau((y_{t+1} - v)s_{t+1})(\omega) \le 0, \\ \varphi_1(s_{t+1})(\omega) - u + \tau((\varphi_2(s_{t+1}) - v)s_{t+1})(\omega) \le 0 \end{cases}$$

is equivalent to an inequality $c(s_{t+1})(\omega) - u + \nu_{t,s_{t+1}}((d(s_{t+1}) - v)s_{t+1})(\omega) \le 0$.

Consequently, for any $(u, v) \in \mathbf{R}^2$ and $\omega \in \Omega$ we have the following equivalence:

$(u, vs_t(\omega)) \in H'_\varphi(t)(\omega)$ if and only if it satisfies a system of inequalities:

$$(4.4)\quad \begin{cases} c_t^b(\omega) - u + \nu_{t,s_t^b}((d_t^b - v)s_t^b)(\omega) \le 0 \\ c_t^a(\omega) - u + \nu_{t,s_t^a}((d_t^a - v)s_t^a)(\omega) \le 0. \end{cases}$$

By Lemma 4.4 there exists a unique r.v. $y_t(s_t)$ which satisfies:

$\nu_{t,s_t^b}((d_t^b - y_t)s_t^b) - \nu_{t,s_t^a}((d_t^a - y_t)s_t^a) = c_t^a - c_t^b$ and

(4.5) $d_t^a \le y_t \le d_t^b$.

Consequently, there exists a r.v. $x_t(s_t)$ such that the following equalities hold:

$$(4.6)\quad \begin{cases} c_t^b - x_t + \nu_{t,s_t^b}((d_t^b - y_t)s_t^b) = 0, \\ c_t^a - x_t + \nu_{t,s_t^a}((d_t^a - y_t)s_t^a) = 0. \end{cases}$$

By (4.6) it follows that $x_t^e - x_t + \tau((y_t^e - y_t)s_t^e) \le 0, \quad e = a, b$ and this means that our constructed strategy is self-financing.

By (4.4), (4.6) and since $\nu_{t,s_t^a}, \nu_{t,s_t^b} \in \Gamma$, using similar arguments as in Theorem 3.4 it is not difficult to show that for any $(u, v) \in \mathbf{R}^2$ and $\omega \in \Omega$ if $(u, vs_t(\omega)) \in H'_\varphi(t)(\omega)$ then $x_t(s_t(\omega)) - u + \tau((y_t(s_t) - v)s_t)(\omega) \le 0$. Consequently, $H'_\varphi(t) \subseteq C_{(x_t(s_t), y_t(s_t)s_t)}$. It is clear that $H'_\varphi(t) \subseteq H_\varphi(t)$. By (4.6) we have $(x_t, y_t s_t) \in H'_\varphi(t)$ and in consequence $(x_t, y_t s_t) \in H_\varphi(t)$. Therefore Lemma 3.2 implies $H_\varphi(t) = C_{(x_t(s_t), y_t(s_t)s_t)}$.

Suppose now that $t = 1, \dots, T-1$.

We will show that $(x_t, y_t) \in \Pi_t$.

By (4.5) we have the following inequalities $d_{t-1}^{aa} \le y_{t-1}^a \le d_{t-1}^{ba}$ and $d_{t-1}^{ab} \le y_{t-1}^b \le d_{t-1}^{bb}$.

Since $d_{t-1}^{ba} = d_{t-1}^{ab}$ we therefore have that $y_{t-1}^a \le y_{t-1}^b$. By (4.6) we get:

$c_{t-1}^{ba} - x_{t-1}^a + \nu_{t,s_{t-1}^{ba}}((d_{t-1}^{ba} - y_{t-1}^a)s_{t-1}^{ba}) = 0,$

$c_{t-1}^{ab} - x_{t-1}^b + \nu_{t,s_{t-1}^{ab}}((d_{t-1}^{ab} - y_{t-1}^b)s_{t-1}^{ab}) = 0.$

From the above equalities we get:

$$(4.7)\quad x_{t-1}^a - x_{t-1}^b = \nu_{t,s_{t-1}^{ba}}((d_{t-1}^{ba} - y_{t-1}^a)s_{t-1}^{ba}) - \nu_{t,s_{t-1}^{ab}}((d_{t-1}^{ab} - y_{t-1}^b)s_{t-1}^{ab}).$$

By the inequalities $y_{t-1}^a \le d_{t-1}^{ab} \le y_{t-1}^b$ and Lemma 4.1 we have:

$$(4.8)\quad \nu_{t,s_{t-1}^{ab}}((y_{t-1}^b - y_{t-1}^a)s_{t-1}^{ab}) \ge \nu_{t,s_{t-1}^{ba}}((d_{t-1}^{ba} - y_{t-1}^a)s_{t-1}^{ba})$$
$$- \nu_{t,s_{t-1}^{ab}}((d_{t-1}^{ab} - y_{t-1}^b)s_{t-1}^{ab}) \ge -\nu_{t,s_{t-1}^{ab}}((y_{t-1}^a - y_{t-1}^b)s_{t-1}^{ab}).$$

By (4.7), (4.8) and (C3) we get:
$\tau((y^b_{t-1} - y^a_{t-1})s^{ab}_{t-1}) \geq x^a_{t-1} - x^b_{t-1}$,
$-\tau((y^a_{t-1} - y^b_{t-1})s^{ab}_{t-1}) \leq x^a_{t-1} - x^b_{t-1}$.
Therefore since $y^a_{t-1} \leq y^b_{t-1}$ we get:
$\Psi_{y_t}(y^a_{t-1}) \geq \tau((y^b_{t-1} - y^a_{t-1})s^{ab}_{t-1}) \geq x^a_{t-1} - x^b_{t-1}$
and
$\Psi_{y_t}(y^b_{t-1}) \leq -\tau((y^a_{t-1} - y^b_{t-1})s^{ab}_{t-1}) \leq x^a_{t-1} - x^b_{t-1}$.
Consequently, since Ψ_{y_t} is continuous and decreasing we see that there exists a unique r.v. $w_{t-1}(s_{t-1})$ such that $\Psi_{y_t}(w_{t-1}) = x^a_{t-1} - x^b_{t-1}$ and $y^a_{t-1} \leq w_{t-1} \leq y^b_{t-1}$.

Thus, $(x_t(s_t), y_t(s_t)) \in \Pi_t$.

By backward induction, it follows that there exists a strategy $(x, y) \in \mathcal{A}$ such that $H_\varphi(t) = C_{(x_t, y_t s_t)}$, P *a.s.* for each $t = 0, \ldots, T-1$ and $(x_t, y_t) \in \Pi_t$ for each $t = 1, \ldots, T-1$.

The proof is therefore completed. ∎

We show below some examples of the American options with $\varphi(t) \in \Pi_t$, $t = 1, \ldots, T$.

EXAMPLE 1. Long call option with delivery.

When the stock price is K or greater, a holder of the option buys one share of the stock for the price K.

$f_1(s) = -K1_{s \geq K}$, $f_2(s) = 1_{s \geq K}$.

EXAMPLE 2. Long call option with delivery and cash settlement.

As in Example 1 a holder buys one share of the stock at the non-negative price K, he does it however when possible cash settlement is nonnegative. If it is negative he doesn't exercise the option. Note that by definition of the function A, $A(K)$ is the minimal value of the stock settlement which is required to get K bonds

$f_1(s) = -K1_{s \geq A(K)}$, $f_2(s) = 1_{s \geq A(K)}$.

EXAMPLE 3. Long call option with delivery and settlement in shares of the stock.

This case is similar to the proceeding. However now, the decision of buying one share of the stock at the non-negative price K is made when the holder's settlement in shares of the stock is nonnegative. Note that by definition of the function B, $B^{-1}(K)$ is the value of the stock settlement obtained by selling K bonds

$f_1(s) = -K1_{s \geq B^{-1}(K)}$, $f_2(s) = 1_{s \geq B^{-1}(K)}$.

EXAMPLE 4. Long put option.

When the stock price is K or lower a holder of the option sells one share of the stock for the price K

$f_1(s) = K1_{s\le K}$, $f_2(s) = -1_{s\le K}$.

4.1. Small transaction costs in case of the American option

Assuming sufficiently small transaction costs we have the following fact:

THEOREM 4.6. *If the condition* (3.9) *is satisfied, then for any American option* φ *there exists a strategy* $(x, y) \in \mathcal{A}$ *which is optimal. Moreover,* $H_\varphi(t) = C_{(x_t, y_t s_t)}$ *for each* $t = 0, \ldots, T-1$.

Proof. We set $(x_T, y_T) = (\varphi_1(T), \varphi_2(T))$. It is clear that $H_\varphi(T) = C_{(x_T, y_T s_T)}$.

Assume that for some $t = 0, \ldots, T-1$ there exists a pair of $\mathcal{F}_{t+1}$ measurable random variables (x_{t+1}, y_{t+1}) such that $H_\varphi(t+1) = C_{(x_{t+1}, y_{t+1} s_{t+1})}$.

Using Remark 4.2 and Lemma 4.3 it is not difficult to show that there exist $\mathcal{F}_{t+1}$ measurable random variables c_{t+1}, d_{t+1} such that for any $\omega \in \Omega$ there is a function $\gamma_{t+1}(\omega) \in \Gamma$ with the following equivalence:
for any $(u, v) \in \mathbf{R}^2$ a system of inequalities

$$\begin{cases} x_{t+1}(\omega) - u + \tau((y_{t+1} - v)s_{t+1})(\omega) \le 0 \\ \varphi_1(t+1)(\omega) - u + \tau((\varphi_2(t+1) - v)s_{t+1})(\omega) \le 0 \end{cases}$$

is equivalent to an inequality $c_{t+1}(\omega) - u + \gamma_{t+1}((d_{t+1} - v)s_{t+1})(\omega) \le 0$.

From now on we fix an $\omega \in \Omega$ in this proof.

Denote $\gamma_{t+1}(w_t^e)$ by γ_t^e, $e = a, b$.

For any $(u, v) \in \mathbf{R}^2$, $(u, vs_t) \in H'_\varphi(t)$ if and only if the following system of inequalities is satisfied:

$$\begin{cases} x_t^e - u + \tau((y_t^e - v)s_t^e) \le 0 \\ \varphi_1^e(t) - u + \tau((\varphi_2^e(t) - v)s_t^e) \le 0. \end{cases} \qquad e = a, b$$

Therefore for any $(u, v) \in \mathbf{R}^2$, $(u, vs_t) \in H'_\varphi(t)$ if and only if (u, v) satisfies a system of inequalities:

(4.9) $\quad c_t^a - u + \gamma_t^a((d_t^a - v)s_t^a) \le 0,$

(4.10) $\quad c_t^b - u + \gamma_t^b((d_t^b - v)s_t^b) \le 0.$

Moreover, since $\gamma_t^a, \gamma_t^b \in \Gamma$ it is not difficult to see that there exist $\mathcal{F}_t$ measurable random variables x_t, y_t such that the following equalities hold:

(4.11) $\quad c_t^a - x_t + \gamma_t^a((d_t^a - y_t)s_t^a) = 0,$

(4.12) $\quad c_t^b - x_t + \gamma_t^b((d_t^b - y_t)s_t^b) = 0.$

From (4.11) and (4.12) it follows that $x_t^e - x_t + \tau((y_t^e - y_t)s_t^e) \le 0$, $e = a, b$ and this means that our constructed strategy is self-financing.

Let u, v be real numbers such that $(u, vs_t) \in H'_\varphi(t)$. We will prove now that

(4.13) $\quad x_t - u + \tau((y_t - v)s_t) \le 0.$

From (4.10) and (4.12) we have

(4.14) $x_t - u + \gamma_t^b((d_t^b - v)s_t^b) - \gamma_t^b((d_t^b - y_t)s_t^b) \le 0.$

From (4.9) and (4.11) we have

(4.15) $x_t - u + \gamma_t^a((d_t^a - v)s_t^a) - \gamma_t^a((d_t^a - y_t)s_t^a) \le 0.$

There are a six cases:

1. $v \le y_t \le d_t^b$.

By (C1) we have $(y_t - v)s_t^b \le \gamma_t^b((d_t^b - v)s_t^b) - \gamma_t^b((d_t^b - y_t)s_t^b)$.
By the properties of B we get, $\tau((y_t - v)s_t) \le \frac{1+\lambda}{1+b}(y_t - v)s_t^b$.
Therefore, by (3.1) we obtain
$\gamma_t^b((d_t^b - v)s_t^b) - \gamma_t^b((d_t^b - y_t)s_t^b) \ge \tau((y_t - v)s_t)$.
The last inequality and (4.14) imply (4.13).

2. $d_t^b \le v \le y_t$.

By (C2) we have $\gamma_t^b((d_t^b - v)s_t^b) - \gamma_t^b((d_t^b - y_t)s_t^b) \ge (1-\mu)(y_t - v)s_t^b$.
By the properties of B we get $(y_t - v)s_t^b \ge \frac{1+b}{1+\lambda}\tau((y_t - v)s_t)$. Therefore, by (3.9) we obtain $\gamma_t^b((d_t^b - v)s_t^b) - \gamma_t^b((d_t^b - y_t)s_t^b) \ge \tau((y_t - v)s_t)$.
From the last inequality and (4.14) we get (4.13).

3. $v \le d_t^b \le y_t$.

By (3.9) we have $(b-\lambda)(d_t^b - v) \ge 0 \ge ((1+\lambda) - (1-\mu)(1+b))(y_t - d_t^b)$.
From the above we get:

(4.16) $(1+b)(d_t^b - v) - (1-\mu)(1+b)(d_t^b - y_t) \ge (1+\lambda)(y_t - v)$.

By (C1), (C2) and (C4) we have:
$\gamma_t^b((d_t^b - v)s_t^b) \ge (1+b)(d_t^b - v)s_t$,
$\gamma_t^b((d_t^b - y_t)s_t^b) \le (1-\mu)(1+b)(d_t^b - y_t)s_t$,
$\tau((y_t - v)s_t) \le (1+\lambda)(y_t - v)s_t$.
By the last three inequalities and (4.16) we obtain:
$\gamma_t^b((d_t^b - v)s_t^b) - \gamma_t^b((d_t^b - y_t)s_t^b) \ge \tau((y_t - v)s_t)$.
From the last inequality and (4.14) we get (4.13).

4. $y_t \le v \le d_t^a$.

By (C1) we have $\gamma_t^a((d_t^a - v)s_t^a) - \gamma_t^a((d_t^a - y_t)s_t^a) \ge (1+\lambda)(y_t - v)s_t^a$.
From the properties of A we get $(y_t - v)s_t^a \ge \frac{1+a}{1-\mu}\tau((y_t - v)s_t)$. Therefore, by (3.9) we obtain:
$\gamma_t^a((d_t^a - v)s_t^a) - \gamma_t^a((d_t^a - y_t)s_t^a) \ge \tau((y_t - v)s_t)$.
From the last inequality and (4.15) we get (4.13).

5. $d_t^a \le y_t \le v$.

By (C2) we have $\gamma_t^a((d_t^a - y_t)s_t^a) - \gamma_t^a((d_t^a - v)s_t^a) \leq (v - y_t)s_t^a$.

From the properties of A we get $\frac{1-\mu}{1+a}(v - y_t)s_t^a \leq -\tau((y_t - v)s_t)$.

Therefore, by (3.1) we obtain

$\gamma_t^a((d_t^a - v)s_t^a) - \gamma_t^a((d_t^a - y_t)s_t^a) \geq \tau((y_t - v)s_t)$.

From the above inequality and (4.15) we get (4.13).

6. $y_t \leq d_t^a \leq v$.

By (3.9) we have $(a+\mu)(v-d_t^a) \leq 0 \leq ((1-\mu)-(1+\lambda)(1+a))(d_t^a - y_t)$.

From the above we get:

$$(4.17)\quad (1+a)(d_t^a - v) - (1+\lambda)(1+a)(d_t^a - y_t) \geq (1-\mu)(y_t - v).$$

By (C1), (C2) and (C4) we have:

$\gamma_t^a((d_t^a - v)s_t^a) \geq (1+a)(d_t^a - v)s_t$,

$\gamma_t^a((d_t^a - y_t)s_t^a) \leq (1+\lambda)(1+a)(d_t^a - y_t)s_t$,

$\tau((y_t - v)s_t) \leq (1-\mu)(y_t - v)s_t$.

By the last three inequalities and (4.17) we obtain $\gamma_t^a((d_t^a - v)s_t^a) - \gamma_t^a((d_t^a - y_t)s_t^a) \geq \tau((y_t - v)s_t)$.

By the last inequality and (4.15) we get (4.13).

From (4.13) we have $H_\varphi'(t) \subseteq C_{(x_t, y_t s_t)}$. It is clear that $H_\varphi'(t) \subseteq H_\varphi(t)$. Therefore, since $(x_t, y_t s_t) \in H_\varphi'(t)$ we get $(x_t, y_t s_t) \in H_\varphi(t)$. Consequently, Lemma 3.2 implies $H_\varphi(t) = C_{(x_t(s_t), y_t(s_t)s_t)}$.

By backward induction it follows that there exists a strategy $(x, y) \in \mathcal{A}$ such that $H_\varphi(t) = C_{(x_t, y_t s_t)}$, P *a.s.* for each $t = 0, \ldots, T-1$.

The proof is therefore completed. ∎

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