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MALLIAVIN CALCULUS IN CONSTRUCTION OF HEDGING PORTFOLIO FOR THE HESTON MODEL OF A FINANCIAL MARKET

Dedicated to Professor Kazimierz Urbanik

Abstract. We present a method of theoretical and numerical construction of the hedging (replicating) portfolio for a given derivative financial instrument for the Heston model of a financial market. The stochastic Heston model is defined by an appropriate system of Itô type stochastic differential equations. We use a methodology based on an application of the Clark–Ocone–Haussmann formula, leading to closed formulae for optimal replicating strategies. We show how to use it in computer oriented applications.

1. Introduction

Some papers indicating on a possibility to apply Malliavin calculus in calculations of functionals on classes of solutions to stochastic differential equations (SDE – for short) modelling certain financial mathematics problems has appeared lately. It is possible, for example, to combine this idea with computer Monte Carlo simulations techniques in order to obtain efficient algorithms for calculation of the so called Greecs, i.e. certain functionals on the class of processes solving the Black–Scholes system of linear SDEs commonly used in mathematical finance (see [1]).

This approach applies also to derivation of useful formulae describing optimal portfolio process in some stochastic optimization problems playing an important role in mathematical finance. In this paper we construct optimal replicating portfolio for the Heston model of financial market, in a mathematical framework proposed in [5].

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We also compare quantitatively replicating portfolios for the Black–Scholes and Heston models, using our approximate computer algorithms based on calculation of the Malliavin derivatives of appropriate stochastic processes and application of the Monte Carlo simulation techniques.

2. Financial market models

Following recent monographs [5] and [7] we briefly recall here necessary notions and facts from stochastic finance (all necessary definitions, formal assumptions, etc., omitted here can be found there).

Let $W = \{W(t) : t \in [0,T]\}$ denote a given Brownian motion process on $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}$, where $\{\mathcal{F}_t\}_{t\in[0,T]}$ is the augmentation by the null sets in $\mathcal{F}_T^W$ of the filtration $\{\mathcal{F}_t^W\}_{t\in[0,T]}$ generated by $W = W(t)$. Let us suppose that all processes considered here are well defined on the spaces $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, *adapted* to this filtration and have continuous paths, what means that they are progressively measurable. We need also another probability space $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$. Operations $\mathbb{E}$, $\mathbb{E}_t$ and $\widetilde{\mathbb{E}}$, $\widetilde{\mathbb{E}}_t$ denote expectation and expectation conditioned to $\mathcal{F}_t$, on both these spaces, respectively.

We are interested in two special cases of financial market models presented e.g. in [5] or [7] in a more general framework, i.e. in models consisting of a *money market* (or *bond*) $S_0 = S_0(t)$ and one *stock* $S_1 = S_1(t)$. In this one dimensional case we get a stochastic model consisting of a system of 2 linear Itô SDEs

$$S_0(t) = S_0(0) + \int_0^t r(s)\, S_0(s)\, ds, \tag{2.1}$$

$$S_1(t) = S_1(0) + \int_0^t \mu(s)\, S_1(s)\, ds + \int_0^t \sigma(s)\, S_1(s)\, dW(s), \tag{2.2}$$

where processes $r = r(t)$, $\mu = \mu(t)$ are $\mathbf{L}^1(\Omega \times [0,T])$–integrable, process $\sigma = \sigma(t)$ is $\mathbf{L}^2(\Omega \times [0,T])$–integrable, all defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$.

In this paper we intend to compare in some sense two different models of a financial market: the commonly used Black–Scholes model and, starting attract attention, more general (nonlinear with respect to stochastic volatility), Heston model.

The *Black–Scholes* model is defined by a system of 2 linear SDEs with constant coefficients

$$S_0(t) = S_0(0) + r \int_0^t S_0(s)\, ds, \tag{2.3}$$

(2.4) $$S_1^B(t) = S_1^B(0) + \mu \int_0^t S_1^B(s)\, ds + \sigma \int_0^t S_1^B(s)\, dW(s),$$

for $t \in [0,T]$ and given $S_0(0) > 0$, $S_1(0) > 0$.

The *Heston model* (see [2]) is described by a system of 3 SDEs

$$S_0(t) = S_0(0) + \int_0^t r^H(s)\, S_0(s)\, ds,$$

(2.5) $$S_1^H(t) = S_1^H(0) + \int_0^t \mu^H(s)\, S_1^H(s)\, ds + \int_0^t \sqrt{V(s)}\, S_1^H(s)\, dW(s),$$

(2.6) $$V(t) = V(0) + \kappa \int_0^t (\alpha - V(s))\, ds + \gamma \int_0^t \sqrt{V(s)}\, dW(s),$$

for $t \in [0,T]$, where $V(0) > 0$, $S_1^H(0) > 0$ are given, κ, α are positive constants, and T, $S_0(0)$ are the same as above in the Black–Scholes model. We assume that processes $r^H = r^H(t)$ and $\mu^H = \mu^H(t)$ are adapted to $\{\mathcal{F}_t\}$ and are $\mathbf{L}^1(\Omega \times [0,T])$–integrable.

REMARK 2.1. Coefficients of equations (2.5), (2.6) describing Heston model satisfy standard growth and Lipschitz conditions, i.e. assumptions of classical theorems on existence and uniqueness of solutions of Itô stochastic differential equations. Repeating an argument from [3] on Bessel processes, one can check that under our assumptions the nonnegative process $V = V(t)$ solving (2.6) is well defined.

REMARK 2.2. From the fact that in the Heston model we have the same copy of Brownian motion $W = \{W(t) : t \in [0,T]\}$ in (2.5) and in (2.6), it follows that process $\{\sqrt{V(t)} : t \in [0,T]\}$ is adapted to $\{\mathcal{F}_t\}$ and the Heston model can be considered as a special case of model (2.1)–(2.2).

One of our main goals is to check what is the difference between replicating portfolios for this 2 models approximating each other in some sense, so we propose the following definition.

DEFINITION 2.1. Models defined by stochastic equations (2.4) and (2.5)–(2.6) are called comparable iff $\mu^H(t) \equiv \mu$ and σ and $V = V(t)$ are related through the equality

(2.7) $$\sigma^2 = \mathbb{E}\Big(\frac{1}{T}\int_0^T V(s)\, ds\Big).$$

Quantitative information on the quality of this criterion is provided by Figures 5.1, 5.2 obtained from computer visualizations of solutions to underlying systems of SDEs.

In a framework given by the model (2.1)–(2.2), let us introduce a few more necessary processes and stochastic equations. *Discount process* is given by $\beta(t) = 1/S_0(t)$. For $n \in \{0,1\}$ let $\eta_n = \eta_n(t)$ denote the number of shares of bond and stock, respectively, so the value of the investor's holdings at time t is $\pi_0(t) + \pi_1(t)$, where $\pi_0(t) \stackrel{\text{df}}{=} \eta_0(t)S_0(t)$, $\pi_1(t) \stackrel{\text{df}}{=} \eta_1(t)S_1(t)$. The process $\pi = (\pi_0, \pi_1) = \{(\pi_0(t), \pi_1(t)) : t \in [0,T]\}$ with values in $\mathbb{R}^2$ is called a *portfolio process.* We assume that the process $\pi = \pi(t)$ is adapted to filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ and such that all integrals in (2.8) are well defined and finite.

With these processes we associate two other processes: the *gains process* $G = G(t)$

$$dG(t) \stackrel{\text{df}}{=} \eta_0(t)\; r(t)S_0(t)\; dt + \eta_1(t)\left[dS_1(t) + S_1(t)\delta(t)\; dt\right], \quad \text{with} \quad G(0) = 0,$$

and – playing an important role – the *wealth process* $X \equiv X^{x,c,\pi} = \{X^{x,c,\pi}(t) : t \in [0,T]\}$

$$X(t) \stackrel{\text{df}}{=} x - \int_0^t c(s)\; ds + G(t),$$

with $x > 0$ denoting *initial value of an investment*, $c = c(t)$ describing the *consumption process* and $\delta = \delta(t)$ – *dividend rate process.*

REMARK 2.3. It can be checked that the wealth process X satisfies on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ the following SDE

$$X(t) = x + \int_0^t [X(s)r(s) - c(s)]ds + \int_0^t \pi_1(s)[\mu(s) + \delta(s) - r(s)]\; ds + \int_0^t \pi_1(s)\sigma(s)dW(s)]. \tag{2.8}$$

Let us define a process called the *market price of risk*

$$\theta(t) \stackrel{\text{df}}{=} [\mu(t) + \delta(t) - r(t)]/\sigma(t),$$

and next, two other processes

$$\widetilde{W}(t) \stackrel{\text{df}}{=} W(t) + \int_0^t \theta(s)\; ds,$$

$$Z(t) \stackrel{\text{df}}{=} \mathbb{E}_t\left[\exp\left\{-\int_0^t \theta(s)\; dW(s) - \frac{1}{2}\int_0^t \theta^2(s)\; ds\right\}\right].$$

Assuming that the process $Z = Z(t)$ is a martingale on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, finally we introduce a probability measure on Ω, given by

$$\widetilde{\mathbb{P}}(A) \stackrel{\text{df}}{=} \int_A Z(T)\, d\mathbb{P}, \quad \text{for } A \in \mathcal{F}_T.$$

REMARK 2.4. From Girsanov theorem's now follows that $\widetilde{W} = \widetilde{W}(t)$ is a Brownian motion process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \widetilde{\mathbb{P}})$.

So, applying Itô formula and taking into account the discounting process $\beta = \beta(t)$, for the wealth process $X = \{X(t)\}$ we derive the following new SDE

$$\beta(t)X(t) = x - \int_0^t \beta(u)\, c(u)\, du + \int_0^t \beta(u)\, \pi_1(u)\sigma(u)\, d\widetilde{W}(u),$$

on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \widetilde{\mathbb{P}})$.

Finally we introduce the *state price density process*

$$\widetilde{H}(t) \stackrel{\text{df}}{=} \beta(t)\, \exp\Big\{ -\int_0^t \theta(s)\, dW(s) - \tfrac{1}{2}\int_0^t \theta^2(s)\, ds\Big\},$$

getting, after an application of the Itô formula, another SDE for the wealth process X

$$\text{(2.9)} \quad \widetilde{H}(t)\, X(t) = x - \int_0^t \widetilde{H}(u)\, c(u)\, du + \int_0^t \widetilde{H}(u)\, [\sigma(u)\pi_1(u)\sigma(u) - X(u)\theta(u)]\, dW(u),$$

but this time on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, which is more practically useful.

3. Malliavin calculus in construction of replicating portfolios

First we recall (see Theorem 3.5 in [5]) a theorem on the existence of replicating portfolio.

THEOREM 3.1. *Let B denote a nonnegative, $\mathcal{F}_T$-measurable random variable. If $c = c(t)$ is a consumption process and initial wealth x satisfies the condition:*

$$x = \mathbb{E}\Big[\int_0^T \widetilde{H}(u)c(u)du + \widetilde{H}(T)\, B\Big],$$

then there exists a portfolio $\pi = (\pi_0, \pi_1)$ such that corresponding wealth process $X^{x,c,\pi} = X^{x,c,\pi}(t)$ satisfies following conditions

$$X^{x,c,\pi}(0) = x,$$
$$X^{x,c,\pi}(T) = B,$$

$$X^{x,c,\pi}(t) \geq 0, \quad a.s.$$
$$X^{x,c,\pi}(t) = \pi_0(t) + \pi_1(t),$$

and can be described for all $t \in [0,T]$ by the following equality

$$\widetilde{H}(t)X^{x,c,\pi}(t) = \mathbb{E}_t\Big[\int_t^T \widetilde{H}(u)c(u)du + \widetilde{H}(T)X^{x,c,\pi}(T)\Big].$$

We can say more.

REMARK 3.1. There exists a stochastic process $\psi = \psi(t)$ such that

$$\sigma(t)\pi_1(t) = \frac{\psi(t)}{\widetilde{H}(t)} + X(t)\theta(t). \tag{3.1}$$

This process can be derived from the relation

$$\mathbb{E}_t\Big[\int_0^T \widetilde{H}(u)c(u)du + \widetilde{H}(T)X^{x,c,\pi}(T)\Big] = x + \int_0^t \psi(u)dW(u).$$

Our aim is to construct optimal replicating portfolios for both models presented in Section 2 by (2.4) and (2.5)–(2.6), and to compare them quantitatively with the use of approximate computer methods based on calculation of the Malliavin derivatives of appropriate stochastic processes and application of the Monte Carlo simulation techniques.

Our main tool is the *Clark–Ocone–Haussmann formula.* To present it briefly let us assume that for a given random variable F we look for a stochastic process $\psi = \psi(t)$ on $t \in [0,T]$ such that

$$F = \mathbb{E}[F] + \int_0^T \psi(t)\ dW(t). \tag{3.2}$$

The answer gives the following theorem

THEOREM 3.2. *Let F denote a random variable belonging to the space $\mathbf{D}^{1,1}$. Then*

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}_t[\mathbb{D}_t F]dW(t), \tag{3.3}$$

$$\mathbb{E}_t[F] = \mathbb{E}[F] + \int_0^t \mathbb{E}_s[\mathbb{D}_s F]dW(s), \tag{3.4}$$

where $\mathbf{D}^{1,1}$ denotes a Sobolev type Banach space of random variables $\mathbf{L}^1$-integrable with their first Malliavin derivatives $\mathbb{D}_t[F]$, (see [8] and [1]).

From equality (3.3) it follows directly, that for the process $\psi = \psi(t)$ from (3.2) we have $\psi(t) = \mathbb{E}_t[\mathbb{D}_t F]$, what means that the process $\psi = \psi(t)$ is just a Malliavin derivative of the random variable F.

Now we are in a position to propose two constructive theorems describing replicating portfolio. Thanks to linearity of SDE (2.9) describing wealth process and linear dependence of this process on a consumption process c, we consider separately 2 problems: first – with $c = c(t) \equiv 0$, and second – with $B = 0$ *a.s.*

THEOREM 3.3. *Let us suppose that all assumptions of Theorem* 3.1 *are satisfied. If* $c = c(t) \equiv 0$, *then from the condition* $\widetilde{H}(T)\, B \in \mathbf{D}^{1,1}$ *it follows, that the portfolio replicating* B *has the form*

$$(3.5)\ \pi_1(t) = \frac{1}{\widetilde{H}(t)\,\sigma(t)}\,\mathbb{E}_t\left[\widetilde{H}(T)\,\mathbb{D}_t B\right] - \frac{1}{\widetilde{H}(t)\,\sigma(t)}\,\mathbb{E}_t\Big[\widetilde{H}(T)\,B\,\Big(\int_t^T \mathbb{D}_t r(s)ds + \int_t^T \mathbb{D}_t\theta(s)d\widetilde{W}(s)\Big)\Big].$$

If additionally $r = r(t)$ *and* $\theta = \theta(t)$ *are deterministic functions, then portfolio* π_1 *takes the form*

$$(3.6)\quad \pi_1(t) = \exp\Big(-\int_t^T r(s)ds\Big)\,\sigma(t)^{-1}\,\widetilde{\mathbb{E}}_t[\mathbb{D}_t B] = \exp\Big(-\int_t^T (r(s) + \tfrac{1}{2}\theta^2(s))ds - \int_t^T \theta(s)dW(s)\Big)\,\sigma(t)^{-1}\,\mathbb{E}_t[\mathbb{D}_t B].$$

Proof. Let $F \overset{\text{df}}{=} \widetilde{H}(T)\,B$. If $F \in \mathbf{D}^{1,1}$, then thanks to Clark–Ocone–Haussmann formula $\psi(t) = \mathbb{E}[\mathbb{D}_t F]$. Replacing process ψ in (3.1) by $\mathbb{E}[\mathbb{D}_t F]$, we get

$$(3.7)\qquad \pi_1(t) = \frac{\mathbb{E}_t\,[\mathbb{D}_t F] + \mathbb{E}_t\,[F]\,\theta(t)}{\widetilde{H}(t)\,\sigma(t)}.$$

After further calculations we get

$$(3.8)\qquad \pi_1(t) = \frac{1}{\widetilde{H}(t)\,\sigma(t)}\Big\{\mathbb{E}_t\left[\widetilde{H}(T)\,\mathbb{D}_t B\right] + \mathbb{E}_t\left[B\mathbb{D}_t\,\widetilde{H}(T)\right] + \theta(t)\,\mathbb{E}_t\left[\widetilde{H}(T)\,B\right]\Big\},$$

and since $\mathbb{D}_t\widetilde{H}(s) = -\widetilde{H}(s)\Big(\theta(t) + \int_t^s \mathbb{D}_t r(u)du + \int_t^s \mathbb{D}_t\theta(u)d\widetilde{W}(u)\Big)\,I_{[0,s]}(t)$, then the formula (3.8) transforms into (3.5). □

THEOREM 3.4. *Let us suppose that all assumptions of Theorem* 3.1 *are satisfied. If* $B = 0$ *a.s, then for* $\int_0^T \widetilde{H}(s)\,c(s)\,ds \in \mathbf{D}^{1,1}$, *the replicating portfolio takes the form*

$$\pi_1(t) = \frac{1}{\widetilde{H}(t)\,\sigma(t)}\,\mathbb{E}_t\Big[\int_t^T \widetilde{H}(s)\,\mathbb{D}_t\,c(s)\,ds\Big] - \frac{1}{\widetilde{H}(t)\,\sigma(t)}\,\mathbb{E}_t\Big[\int_t^T \widetilde{H}(s)c(s)\Big(\int_t^S \mathbb{D}_t\,r(u)\,du + \int_t^S \mathbb{D}_t\,\theta(u)\,d\widetilde{W}(u)\Big)\,ds\Big]. \tag{3.9}$$

Proof. Since $B = 0$, so we get

$$X(t) = \frac{1}{\widetilde{H}(t)}\mathbb{E}\Big[\int_t^T \widetilde{H}(s)\,c(s)\,ds\Big].$$

If $F \overset{\text{df}}{=} \int_0^T \widetilde{H}(s)\,c(s)\,ds$ belongs to $\mathbf{D}^{1,1}$, then $\psi(t) = \mathbb{E}[\mathbb{D}_t F]$ and as a consequence we get

$$\pi_1(t) = \frac{\mathbb{E}_t\,[\mathbb{D}_t F] + \mathbb{E}_t\,\Big[\int_t^T \widetilde{H}(s)\,c(s)\,ds\Big]\,\theta(t)}{\widetilde{H}(t)\,\sigma(t)},$$

where

$$\mathbb{D}_t F = \int_t^T (c(s)\,\mathbb{D}_t\,\widetilde{H}(s) + \widetilde{H}(s)\,\mathbb{D}_t\,c(s))ds,$$

what ends the proof. □

REMARK 3.2. The method of construction of replicating portfolio obtained here by (3.8) is much better suited for numerous applications then that presented in [9]. It was successfully applied to computer construction of portfolios of different kinds and can be applied in some other even more general situations, e.g. for construction of optimal strategies solving various optimal consumption and investment problems.

4. Replication of European call option

Malliavin calculus makes it possible to replicate practically any derivative instrument on a given financial market. Let us recall that *European call option* for a stock price $S(t)$ given by SDE (2.2) for $t \in [0,T]$, is described by the random variable

$$B \overset{\text{df}}{=} \max\{S(T) - K, 0\}, \tag{4.1}$$

were $K > 0$ is a given constant called a *striking price*.

Our aim is to construct with the use of Theorems 3.3 and 3.4 a replicating portfolio for a random variable B given by (4.1) for 2 models: Black–Scholes model, and Heston model introduced in Section 2. Notice, that it is clear that in order to describe any portfolio $\pi(t) = (\pi_0(t), \pi_1(t))$ on a market with 1 stock with a price $S(t)$, it is enough to compute $\eta_1(t)$ such that $\pi_1(t) = \eta_1(t)S(t)$.

THEOREM 4.1. *Let $S_1^B = S_1^B(t)$ and $S_1^H = S_1^H(t)$ be given by* (2.4) *and* (2.5)–(2.6), *respectively. Let B be defined by* (4.1) *in both these cases. Then, we get*

$$\eta_1^B(t) \equiv \eta_1(S_1^B(t)) = \frac{\beta(T)}{\beta(t)} \frac{1}{\sigma S_1^B(t)} \widetilde{\mathbb{E}}_t \left[\sigma S_1^B(T) \boldsymbol{I}_{(K,\infty)}(S_1^B(T))\right] = \widetilde{\mathbb{E}}_t \left[\frac{\beta(T) S_1^B(T)}{\beta(t) S_1^B(t)} \boldsymbol{I}_{(K,\infty)}(S_1^B(T))\right] \tag{4.2}$$

for Black–Scholes model, and

$$\eta_1^H(t) \equiv \eta_1(S^H(t), V(t)) = a(t) + b(t) - c(t), \tag{4.3}$$

for Heston model, where

$$a(t) = \widetilde{\mathbb{E}}_t \left[\frac{\beta(T) S_1^H(T)}{\beta(t) S_1^H(t)} \boldsymbol{I}_{(K,\infty)}(S_1^H(T))\right],$$

$$b(t) = \frac{1}{\sqrt{V(t)}} \widetilde{\mathbb{E}}_t \left[\frac{\beta(T) S_1^H(T)}{\beta(t) S_1^H(t)} \boldsymbol{I}_{(K,\infty)}(S_1^H(T)) \Big(-\int_t^T \sqrt{V(s)}\, \mathbb{D}_t \sqrt{V(s)}\, ds + \int_t^T \mathbb{D}_t \sqrt{V(s)}\, dW(s)\Big)\right],$$

$$c(t) = \frac{1}{\sqrt{V(t)}} \widetilde{\mathbb{E}}_t \left[\frac{\beta(T) \max\{S_1^H(T) - K, 0\}}{\beta(t) S_1^H(t)} \Big(-\int_t^T \theta(s)\, \mathbb{D}_t\, \theta(s)\, ds + \int_t^T \mathbb{D}_t\, \theta(s)\, dW(s)\Big)\right].$$

Proof. With the use of known rules (see [8] or [1]) allowing for calculation of Malliavin derivatives of random variables, processes defined by stochastic integrals or solving Itô SDEs, we get step by step the following results:

first, the Malliavin derivative of B given by (4.1):

$$\mathbb{D}_t B = \mathbb{D}_t S(T) \boldsymbol{I}_{(K,\infty)}(S(T)); \tag{4.4}$$

second, the Malliavin derivative of S_1^B solving SDE (2.4):

$$\mathbb{D}_t S_1^B(u) = S_1^B(u) S_1^B(t)^{-1} \sigma S_1^B(t) I_{[0,u]}(t) = \sigma S_1^B(u) I_{[0,u]}(t); \tag{4.5}$$

third, the Malliavin derivative of the solution $S(t)$ of (2.2):

$$\mathbb{D}_t S(u) = S(u) I_{[0,u]}(t) \Big(\int_t^u \mathbb{D}_t \mu(s) ds - \int_t^u \sigma(s) \mathbb{D}_t \sigma(s) ds + \int_t^u \mathbb{D}_t \sigma(s) dW(s)\Big); \tag{4.6}$$

fourth, Malliavin derivative for $V(t)$ in Heston model

(4.7) $\mathbb{D}_t V(s) =$

$$\beta\sqrt{V(t)}\,\exp\Big(-\kappa(s-t)-\frac{\beta^2}{8}\int_t^s\frac{1}{V(u)}\,du+\frac{\beta}{2}\int_t^s\frac{1}{\sqrt{V(u)}}dW(u)\Big)\,I_{[0,s]}(t).$$

Now, from (3.6), thanks to (4.4) and (4.5), we get directly (4.2).

Similarly, taking (4.4), substituting (4.7) for $\mathbb{D}_t\sigma(s)$ and $\mathbb{D}_t\mu(s)\equiv 0$ in (4.6) and putting this into (3.5), we get finally (4.3). □

5. Computer experiment

Using well developed numerical and statistical methods of approximation of SDEs of different kinds and computer construction and visualization of their solutions (see [4], [6]), and extending them for approximate construction of Malliavin derivatives of underlying random variables and processes, it is possible – with the use of Theorem 4.1 – to compute replicating portfolios (4.2) and (4.3) for stochastic Black–Scholes and Heston models of financial market, introduced in Section 2. On Figures 5.1, 5.2 we see two very similar – connected through relation (2.7) – visualizations of processes describing these models, but on Figures 5.3, 5.4 portfolios replicating random variable B defined by (4.1) for two different values 1.0, 1.2 of striking price K and interpreted as an European call option are seemingly quite different for these two models (there, N^1 stands for $\eta_1^B(t)$, N^2 – for $\eta_1^H(t)$, with assumption that $\pi_0(t)+\pi_1(t)=\eta_0(t)S_0(t)+\eta_1(t)S_1(t)$, with $\eta_0(t)=1-\eta_1(t)$ for both models).

Let us recall that *p–quantile line* $q_p^X = q_p^X(t)$ for given stochastic process $X = X(t)$ and fixed $p\in(0,1)$ is a function defined by the formula

$$\mathbb{P}\{X(t)\le q_p^X(t)\}=p,\quad \text{for } t\in[0,T].$$

Let us also remark, that „closeness" of processes $S_1^B = S_1^B(t)$ and $S_1^H = S_1^H(t)$ related by (2.7) can also be understood as closeness of their densities, quantile lines and other numeric quantities such as e.g. values of underlying cylindrical (Wiener) measures. In our computer experiment differences between values of all numerical quantities obtained for both models were on a same level as errors induced by approximate methods applied to the construction of these processes.

REMARK 5.1. To the best of our knowledge, the results of computer experiments presented here are the first of this kind in accessible literature. The main, apparently important for applications conclusion, one can derive from obtained results, is that for two different stochastic models of financial market, one with constant and second with stochastically varying volatility, which have very similar statistical properties, corresponding strategies replicating a given derivative instrument can be visibly (surprisingly) different.

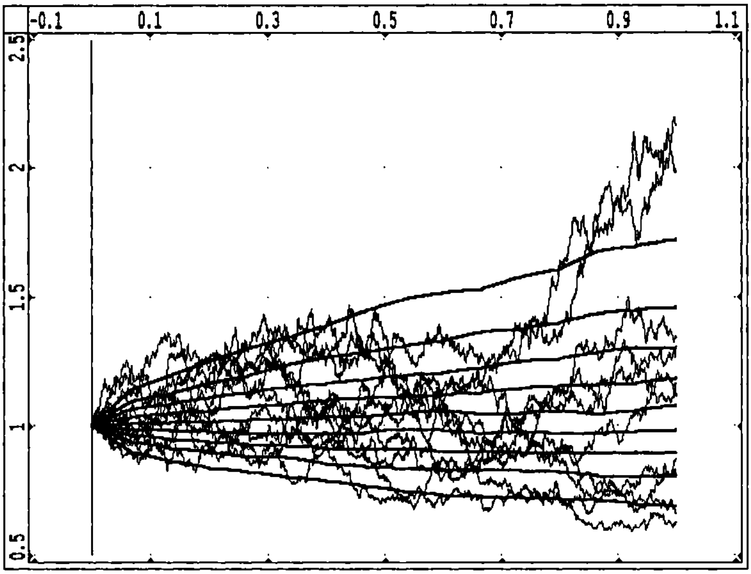

Figure 5.1. Solution of SDE (2.4) modelling Black–Scholes stock price evolution, represented by 10 exemplary trajectories and 9 p–quantile lines for $p \in \{0.1, 0.2, \ldots, 0.9\}$.

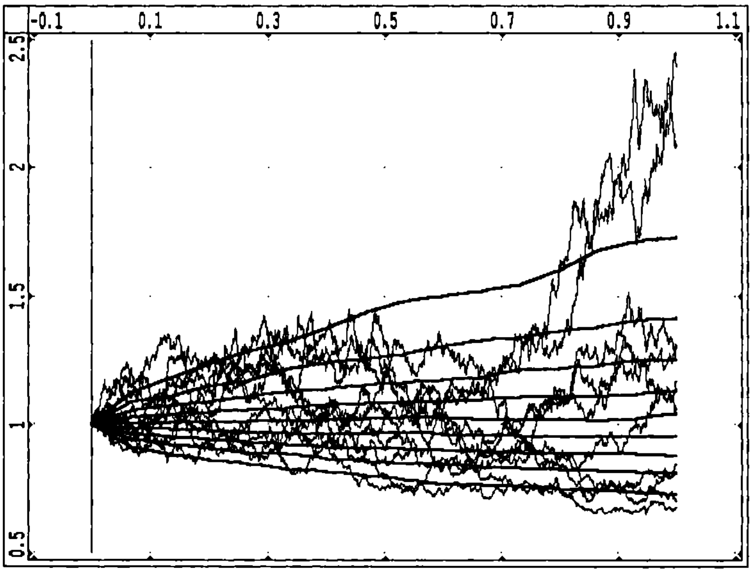

Figure 5.2. Solution of SDE (2.5)–(2.6) modelling Heston stock price evolution, represented by 10 exemplary trajectories and 9 quantile lines for $p \in \{0.1, 0.2, \ldots, 0.9\}$.

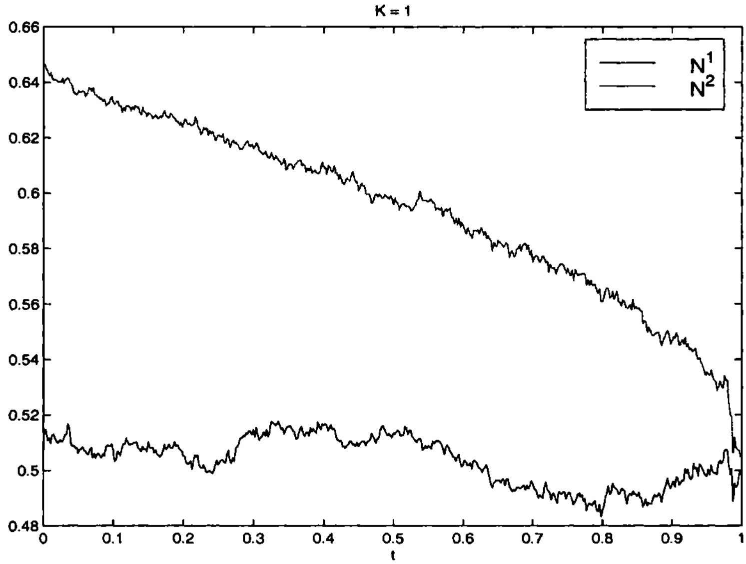

Figure 5.3. Comparison of replicating strategies for Black–Scholes and Heston European call option price models ($K = 1.0$).

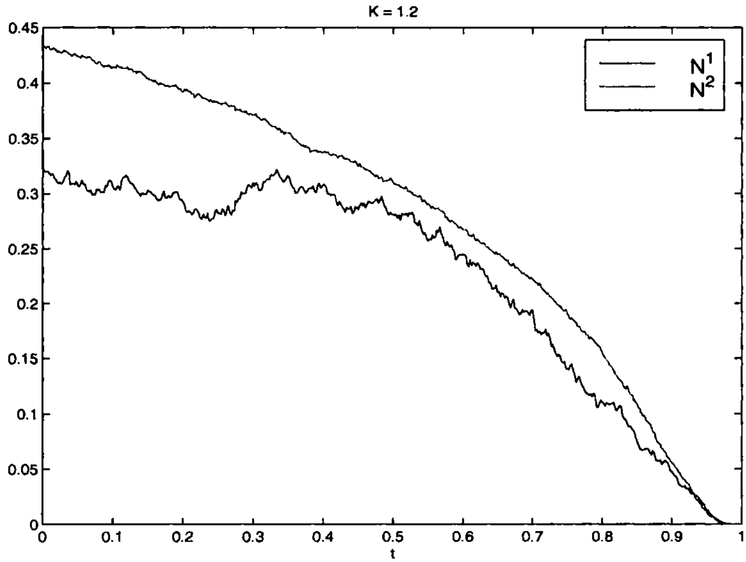

Figure 5.4. Comparison of replicating strategies for Black–Scholes and Heston European call option price models ($K = 1.2$).

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