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PREDICTION INTERVALS FOR STATIONARY TIME SERIES USING THE SIEVE BOOTSTRAP METHOD

Dedicated to Professor Kazimierz Urbanik

Abstract. We consider the problem of constructing prediction intervals for future observations of stationary time series. Our approach relies on the sieve bootstrap procedure introduced by Bühlmann (1997, 1998) which is asymptotically valid for the rich class of linear stationary processes which can be inverted and represented as an autoregressive processes of order infinity ($AR(\infty)$). We extend the results obtained earlier by Stine (1987) for autoregressive time series of known order. A more traditional Gaussian strategy is also presented. We verify accuracy of the proposed methods via numerical comparison including both Gaussian and non-Gaussian data.

1. Introduction

Forecasting of the future values is one of the most popular application of time series modeling. Typically, in such situation we consider construction of "the best" (in some sense) predictors. In order to verify the accuracy of the forecast we need to define the error of prediction, which can be treated as a measure of uncertainty of the forecast. A close related problem is the construction of prediction intervals for future observations. For this purpose, for Gaussian data we may use well-known strategy. On the other hand, we cannot expect the Gaussian prediction intervals to perform very well for non-Gaussian series. In this context, more general bootstrap-based method may be proposed.

During the last years Efron's bootstrap has become a powerful tool for estimating certain statistical characteristics. Unfortunately, in the time series context there is not such a unique resampling procedure as for the independent set-up. Roughly speaking, two different approaches have been proposed, i.e. "model-based" and "model-free" approach, according to whether

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or not the dependence mechanism in the time series is known. A review of resampling techniques for stationary time series is presented for instance by Carlstein (1992).

A vast literature contributes to the special case where the observed time series is an autoregressive process of known order p ($AR(p)$). In particular Stine (1987) applied bootstrap algorithm for such models in order to obtain prediction intervals.

Recently Bühlmann (1997, 1998) has proposed a new resampling method called sieve bootstrap which is a purely nonparametric scheme, i.e. has the advantage that no particular finite parametric model for data is assumed. This approach is based on Grenander's (1981) method of sieves whose main idea is the approximation of an infinite-dimensional, nonparametric model by a sequence of finite-dimensional parametric models. Moreover Bühlmann (1997) showed that for many linear processes the stationary sieve bootstrap for $AR(\infty)$ models has generally a better performance than other nonpararmetric resampling technique called blockwise bootstrap introduced by Künsch (1989).

In this article the problem of obtaining prediction intervals for stationary time series is addressed via Bühlmann sieve bootstrap scheme. Two bootstrap-based methods are proposed, i.e. hybrid bootstrap and bootstrap-t. These approaches can be applicable for wide class of stationary processes and thus are generalization of Stine's results. The efficiency of prediction intervals is verified via computer simulations made both for Gaussian and non-Gaussian data and including comparison with optimal Gaussian strategy of constructing prediction intervals.

The article is organized as follows. In section 2 we briefly present Bühlman's sieve bootstrap scheme. In section 3 we describe construction of the best h-step linear predictor for future value of time series and focus on useful, from practical point of view, recursive computational algorithm. Constructions of prediction intervals, including traditional Gaussian approach and new, based on the sieve bootstrap method are given in section 4. Section 5 is devoted to numerical study.

2. The sieve bootstrap procedure

Consider $\{X_t\}_{t\in Z}$ – real valued, a zero mean stationary process. The sieve bootstrap procedure is valid for the rich subclass of linear stationary processes $X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$ which can be inverted and represented as an autoregressive process of order infinity ($AR(\infty)$). Therefore, we have the following representation

$$\sum_{j=0}^{\infty} \phi_j X_{t-j} = \varepsilon_t \ , \ \phi_0 = 1, \tag{1}$$

where $\{\varepsilon_t\}_{t\in Z}$ is a sequence of uncorrelated random variables with expectation $E[\varepsilon_t] = 0$ and $\sum_{j=0}^{\infty} \phi_j{}^2 < \infty$. The representation (1) for purely stochastic processes is guaranteed by additional assumptions of invertibility (Anderson, 1971, Theorem 7.6.9).

Assume now that we have observations $X_1, \ldots, X_n$ being realizations of the process $\{X_t\}_{t\in Z}$. First and fundamental step in the sieve bootstrap method is fitting an autoregressive process of order $p = p(n)$ increasing "sufficiently slow" as the sample n increases, i.e. $p(n) \to \infty$ $(n \to \infty)$ and $p(n) = o(n)$. Various techniques have been proposed in literature for selecting an appropriate AR model. We follow Bühlmann's suggestion and choose the approximating order $p = p(n)$ using Akaike information criterion (AIC) in an increasing range $[1, C_n]$ with C_n growing as n increasing. In practice $C_n = 10 \log_{10}(n)$ for the increasing range is used.

We estimate autoregressive coefficients $\phi_1, \ldots, \phi_{p(n)}$ basing on observations $\{X_t\}_{t=1}^n$. The estimates $\hat{\phi}_p = (\hat{\phi}_1, \ldots, \hat{\phi}_{p(n)})'$ can be computed by the familiar Yule-Walker method (Brockwell and Davies, 1987, p. 232–233).

$$\hat{\Gamma}_p \hat{\phi}_p = -\hat{\gamma}_p,$$

where $\Gamma_p = [\hat{\gamma}(i-j)]_{i,j=1}^p$ and $\hat{\gamma}_p = (\hat{\gamma}(1), \hat{\gamma}(2), \ldots, \hat{\gamma}(p))'$, and $\hat{\gamma}$ is the estimate of autocovariance function defined as

$$\hat{\gamma}(j) = \frac{1}{n} \sum_{t=1}^{n-|j|} X_{t,n} X_{t+|j|,n}, \quad |j| \le n-1.$$

From computational point of view it is more convenient to calculate the Yule-Walker coefficients using the recursive Durbin-Levinson algorithm:

$$\hat{\phi}_{11} = \frac{\hat{\gamma}(1)}{\hat{\gamma}(0)}, \qquad \hat{\nu}_1 = \hat{\gamma}(0)[1 - \hat{\phi}_{11}^2],$$

$$\hat{\phi}_{mm} = \Big(\hat{\gamma}(m) - \sum_{j=1}^{m-1} \hat{\phi}_{m-1,j}\, \hat{\gamma}(m-j)\Big) / \hat{\nu}_{m-1},$$

$$\begin{bmatrix} \hat{\phi}_{m,1}, \\ \vdots \\ \hat{\phi}_{m,m-1} \end{bmatrix} = \begin{bmatrix} \hat{\phi}_{m-1,1} \\ \vdots \\ \hat{\phi}_{m-1,m-1} \end{bmatrix} - \hat{\phi}_{mm} \begin{bmatrix} \hat{\phi}_{m-1,m-1} \\ \vdots \\ \hat{\phi}_{m-1,1} \end{bmatrix},$$

$$\hat{\nu}_m = \hat{\nu}_{m-1}(1 - \hat{\phi}_{mm}^2).$$

Using above recursions bypasses the matrix inversion required in the direct computation of $\hat{\phi}_p$. Next we compute residuals

$$\hat{\varepsilon}_{t,n} = \sum_{j=0}^{p(n)} \hat{\phi}_{j,n}\, X_{t-j}, \quad \hat{\phi}_{0,n} = 1, \quad t = p+1, \ldots, n.$$

Then we construct the resampling. For this purpose let us center residuals

$$\tilde{\varepsilon}_{t,n} = \hat{\varepsilon}_{t,n} - \frac{1}{n-p} \sum_{t=p+1}^{n} \hat{\varepsilon}_{t,n}, \quad t = p+1, \ldots, n,$$

and next we draw residuals $\tilde{\varepsilon}_t^*$ from the empirical cumulative distribution function of $\{\tilde{\varepsilon}_{t,n}\}_{t=p+1}^n$, i.e. $\tilde{\varepsilon}_t^*$ i.i.d. $\sim \hat{F}_{\varepsilon,n}$, where

$$\hat{F}_{\varepsilon,n}(u) = \frac{1}{n-p} \sum_{t=p+1}^{n} 1_{[\tilde{\varepsilon}_{t,n} \le u]}.$$

Finally the bootstrap replicate $\{X_t^*\}$ is defined by the recursion

$$\sum_{j=0}^{p(n)} \hat{\phi}_{j,n} X_{t-j}^* = \tilde{\varepsilon}_t^*, \quad \hat{\phi}_{0,n} = 1. \tag{2}$$

In practice we generate the bootstrap process $\{X_t^*\}$ starting the recursion with some starting values, e.g. equal to some resampled innovations $\tilde{\varepsilon}_t^*$.

The obtained bootstrap replicate can be used for various purposes. Particularly, if we have any statistics $T_n = T_n(X_1, \ldots, X_n)$ being a measurable function of n observations, we can define the bootstrapped statistics T_n^* by the plug-in principle: $T_n^* = T_n(X_1^*, \ldots, X_n^*)$. Bühlmann (1997) has proved consistency of the sieve bootstrap method for various kinds of statistics T_n including arithmetic mean and some class of nonlinear estimators.

It is worth noting that sieve bootstrap scheme can be also extended to non-stationary time series of the form $Y_t = s(t) + Z_t, \ t \in Z$, where $s(t)$ is a deterministic trend and Z_t is a stationary $AR(\infty)$ noise process with mean zero (Bühlmann (1998)).

3. The best linear predictor

It is known (Brockwell and Davies, 1987, p. 159-162) that for a zero mean stationary process, the best linear combination of $1, X_1, \ldots, X_n$ for predicting X_{n+h}, $(h \ge 1)$ is the projection of X_{n+h} onto the closed linear subspace $\overline{sp}\{X_1, \ldots, X_n\}$. "The best" denotes here element with minimum mean-square distance from X_{n+h}. Thus, we can denote the h-step predictor as

$$P_n X_{n+h} = P_{\overline{sp}\{X_1,\ldots,X_n\}} X_{n+h}.$$

The above predictor can be found from appropriate prediction equations. However in order to save the time of computation we use rather one of the recursive methods. The best linear h-step predictor $P_n X_{n+h}$ can easily be found using the innovations algorithm (Brockwell and Davies, 1987, p.167-

168), i.e.

$$P_n X_{n+h} = \sum_{j=h}^{n+h-1} \theta_{n+h-1,j}(X_{n+h-j} - \hat{X}_{n+h-j}), \tag{3}$$

where the one-step predictors $\hat{X}_{m+1}$ are given by

$$\hat{X}_{m+1} = \begin{cases} 0 & , m = 0 \\ \sum_{j=1}^{m} \theta_{mj}(X_{m+1-j} - \hat{X}_{m+1-j}) & , m \geq 1 \end{cases}$$

and coefficients θ_{mj} are computed from recursive equations

$$\begin{cases} v_0 = \gamma(0), \\ \theta_{m,m-k} = v_k^{-1}(\gamma(m-k) - \sum_{j=0}^{k-1} \theta_{k,k-j}\theta_{m,m-j}v_j), \ k = 0, 1, \ldots, m-1, \\ v_m = \gamma(0) - \sum_{j=0}^{n-1} \theta_{n,n-j}^2 v_j. \end{cases}$$

Moreover the prediction mean squared error (PMSE), which measures the uncertainty of the corresponding forecast can be expressed as

$$\sigma_n^2(h) := E(X_{n+h} - P_n X_{n+h})^2 = \gamma(0) - \sum_{j=h}^{n+h-1} \theta_{n+h-1,j}^2 \, v_{n+h-1-j}. \tag{4}$$

4. Prediction intervals

We consider now problem of constructing prediction intervals for future value X_{n+h} which is closely related to finding the best linear predictor and corresponding PMSE.

A prediction interval is a random interval based on past observations $\underline{X} = (X_1, X_2, \ldots, X_n)$ and constructed for future value X_{n+h}, $h \geq 1$. More precisely, we define $(1-2\alpha)$-prediction intervals for X_{n+h} as

$$I(h, \underline{X}) = [L(\underline{X}), R(\underline{X})],$$

so that $P(L(\underline{X}) \leq X_{n+h} \leq R(\underline{X})) = 1 - 2\alpha$.

For a stationary Gaussian process the construction of prediction intervals can simply be done using the fact (Brockwell and Davies, 1987, p.175) that the prediction error $\Delta_n(h) := X_{n+h} - P_n X_{n+h}$ is normally distributed with mean zero and variance $\sigma_n^2(h)$. Assuming that the true autocovariance function of the model $\gamma(\cdot)$ is known we can calculate the best linear h-step predictor $P_n X_{n+h}$ and corresponding PMSE $\sigma_n^2(h)$ from the innovations algorithm presented above and then $(1-2\alpha)$-Gaussian prediction interval is given by

$$I_G(h) = [P_n X_{n+h} - \Phi_{1-\alpha}\, \sigma_n(h), P_n X_{n+h} + \Phi_{1-\alpha}\, \sigma_n(h)], \tag{5}$$

where $\Phi_{1-\alpha}$ denotes the $(1-\alpha)$-quantile of the standard normal distribution.

For non-Gaussian time series we can obtain prediction intervals approximating unknown distribution of the prediction error $\Delta_n(h)$ with the aid of bootstrap method. This approach, for autoregressive time series of known order p ($AR(p)$) has been proposed by Stine (1987). We extend this procedure to the case of $AR(\infty)$ processes using Bühlman's sieve bootstrap scheme.

In the case of construction of bootstrap prediction intervals to determine predictors and prediction mean squared errors we will use the innovations algorithm for the autocovariance function of the approximating $AR(p(n))$ model. Such estimates of optimal predictors and prediction errors will be denoted respectively as $\hat{P}_n X_{n+h}$ and $\hat{\sigma}_n^2(h)$. In practice we may calculate the autocovariance of the AR model with the aid of algorithm for causal $ARMA(p,q)$ processes (Brockwell and Davies, 1987, p.91-97) which will be given in details in section 5. It is worth noting that autocovariance of the fitted model at lags $0, 1, \ldots, p(n)$ coincide with the corresponding sample autocovariances.

The main idea of using bootstrap in the context of forecasting is to generate the replicate $X_1^*, \ldots, X_n^*$ of the observed series $X_1, \ldots, X_n$ and next extend this replicate into the future time $n+h$. In the sieve bootstrap method we can easily generate the extended replicate by recursion (2). Then, we can approximate unknown distribution of the prediction error $\Delta_n(h) = X_{n+h} - \hat{P}_n X_{n+h}$ by the corresponding bootstrap distribution of $\Delta_n^*(h) := X_{n+h}^* - \hat{P}_n^* X_{n+h}^*$, where $P_n^* X_{n+h}^*$ denotes the best linear predictor for X_{n+h}^* calculated from $X_1^*, \ldots, X_n^*$. This strategy is known as hybrid bootstrap (Shao and Tu, 1995). Denoting q_α^* and $q_{1-\alpha}^*$ as respectively, α and $1-\alpha$ quantile of distribution $\Delta_n^*(h)$ we obtain $(1-2\alpha)$-bootstrap prediction interval

$$I_B(h) = [\hat{P}_n X_{n+h} + q_\alpha^*, \hat{P}_n X_{n+h} + q_{1-\alpha}^*]. \tag{6}$$

In practice we estimate q_α^* and $q_{1-\alpha}^*$ as empirical quantiles, basing on B-bootstrap replicates, where B is sufficiently large.

Another algorithm for obtaining bootstrap prediction intervals, which we may use is so-called bootstrap-t. For the independent set-up the method was originally suggested by Efron but has become more popular and attractive since Hall (1988) showed its good second-order properties. The idea of the bootstrap-t is in brief to estimate the percentiles of the studentized statistics T_n by bootstraping. The unknown distribution of a studentized statistics

$$T_n(h) = \frac{X_{n+h} - \hat{P}_n X_{n+h}}{\hat{\sigma}_n(h)}$$

is therefore estimated by the bootstrap distribution of

$$T_n^*(h) = \frac{X_{n+h}^* - \hat{P}_n^* X_{n+h}^*}{\hat{\sigma}_n^*(h)}.$$

Finally, bootstrap-t prediction interval is given by

$$I_{B-t}\ (h) = [\hat{P}_n X_{n+h} + t_\alpha^*\, \hat{\sigma}_n(h), \hat{P}_n X_{n+h} + t_{1-\alpha}^*\, \hat{\sigma}_n(h)], \tag{7}$$

where t_α^* and $t_{1-\alpha}^*$ are corresponding quantiles of $T_n^*(h)$ which we may estimate as before using sufficiently large number of bootstrap replications.

Described above construction of hybrid and bootstrap-t prediction intervals (given by (6) and (7)) requires theoretical justification. In other words, we have to prove the consistency of the proposed intervals, i.e. $P(X_{n+h} \in I(h)) \to 1 - 2\alpha$. Such result was given by Stine (1987) in the case of bootstrap prediction intervals for $AR(p)$ models. In this article we verify accuracy of the bootstrap intervals via computer experiments described in section 5. Theoretical considerations concerned with consistency will be given in the another paper.

5. Simulations

5.1. Introduction

We investigate our procedures of construction prediction intervals on some simulated examples. For this purpose the following models are considered:

(M1) $ARMA(1,1)$, $X_t = 0.8X_{t-1} - 0.6\varepsilon_{t-1} + \varepsilon_t$, where ε_t i.i.d. $\sim N(0,1)$.

(M2) $AR(48)$, $X_t = \sum_{j=1}^{48} \phi_j X_{t-j} + \varepsilon_t$, $\phi_j = (-1)^{j+1} 7.5/(j+1)^3$ $(j = 1, \ldots, 48)$, where ε_t i.i.d. $\sim (logN(0,1) - \sqrt{e})/\sqrt{e(e-1)}$.

(M3) Ornstein-Uhlenbeck process, i.e. X_t is a stationary Gaussian and Markovian process with expectation 0 and covariance function $K(h) = \alpha e^{-\beta|h|}$, where $\alpha = 1$, $\beta = 0.01$.

The data sets of sample $n = 100$ are generated according to M1-M3. Ornstein-Uhlenbeck process is simulated using the formula: $X_t = \sqrt{\alpha}\, e^{-\beta t} W(e^{2\beta t})$, where W is the standard Wiener process.

We choose the order $p(n)$ of the approximating autoregressive process by minimizing the AIC (Akaike Information Criteria) in a range $1 \le p \le 10 \log_{10}(n)$. The results of the data-driven choices of $\hat{p}_{AIC}$ based on 100 realizations of the model are given in table.

	$E(\hat{p}_{AIC})$	$SD(\hat{p}_{AIC})$	$Min(\hat{p}_{AIC})$	$Max(\hat{p}_{AIC})$
M1	2.35	2.16	1	15
M2	2.79	2.58	1	17
M3	2.02	2.77	1	20

Prediction intervals are constructed using Gaussian (I_G), hybrid bootstrap (I_B) and sieve bootstrap-t (I_{B-t}) approaches. The optimal Gaussian prediction intervals are computed assuming that the true autocovariance function of the model is known. For bootstrap intervals to determine predictors and prediction mean squared errors we use the autocovariance function of the approximating $AR(p(n))$ model. Because in practical application the true autocovariance function is unknown in the case of Ornstein-Uhlenbeck process (model M3) we construct also Gaussian prediction intervals based on autocovariance of the fitted autoregressive process. In practice the values of the autocovariance function $\gamma(\cdot)$ for $ARMA$ processes can be found from the following algorithm (Brockwell and Davies, 1987, p.91-97) . If we assume that X_t is the causal $ARMA(p,q)$ process denoted as $\phi(B)\,X_t = \theta(B)\,Z_t$, where $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$, $\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$, B is the backward shift operator ($B\,X_t = X_{t-1}$) and $\{Z_t\}$ is white noise with zero mean and variance σ^2. Then, to determine autocovariance function of X_t the following equations are used:

$$\begin{cases} \gamma(k) - \phi_1\gamma(k-1) - \cdots - \phi_p\gamma(k-p) = \sigma^2 \sum_{k\le j\le q} \theta_j\psi_{j-k}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le k < \max(p, q+1) \\ \gamma(k) - \phi_1\gamma(k-1) - \cdots - \phi_p\gamma(k-p) = 0, \;\; k \ge \max(p, q+1) \end{cases}$$

where ψ_j satisfy

$$\begin{cases} \psi_j - \sum\limits_{0<k\le j} \phi_k\psi_{j-k} = \theta_j,\; 0 \le j < \max(p, q+1) \\ \psi_j - \sum\limits_{0<k\le p} \phi_k\psi_{j-k} = 0, \;\; j \ge \max(p, q+1) \end{cases}$$

where $\theta_0 = 1, \theta_j = 0$ for $j > q$ and $\phi_j = 0$ for $j > p$. In practice, first we find $\gamma(0), \ldots, \gamma(p)$ from the equations with $k = 0, 1, \ldots, p$ and then use the subsequent equations to compute $\gamma(p+1), \gamma(p+2), \ldots$ recursively.

5.2. Results

Figures 1–6 show prediction intervals for all considered models constructed for nominal coverage equal 95%.

The accuracy of presented prediction intervals is studied in terms of empirical coverage. More precisely, we repeat the procedure of construction intervals and check coverage frequencies for each $h = 1, \ldots, 10$. For bootstrap intervals we use B=200 replicates. In our study the nominal coverage equals 80% and 95%. Tables 1-3. contain coverage percentages with esitmated standard errors in parentheses (in percentages) which were obtained by simulating 100 different realizations of models $M1 - M3$. For all methods the average lengths of intervals are also included.

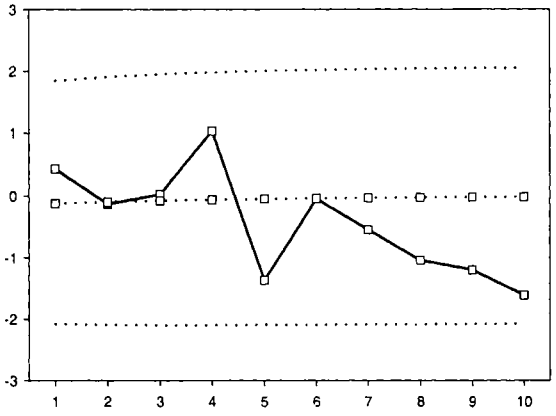

Figure 1. Gaussian prediction intervals for M1: true future values (solid line with squares), predictors (dotted line with squares), Gaussian intervals (dotted line).

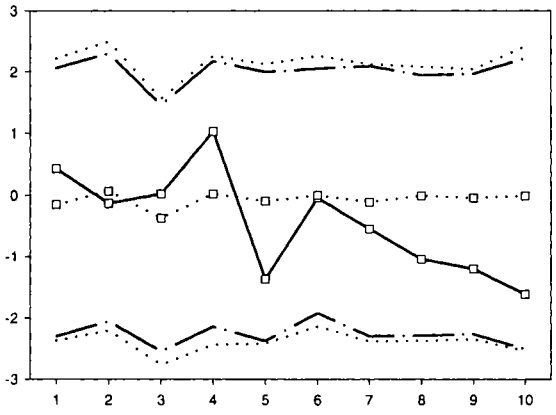

Figure 2. Bootstrap prediction intervals for M1: true future values (solid line with squares), predictors (dotted line with squares), hybrid sieve bootstrap interval (dashdot line), sieve bootstrap-t (dotted line).

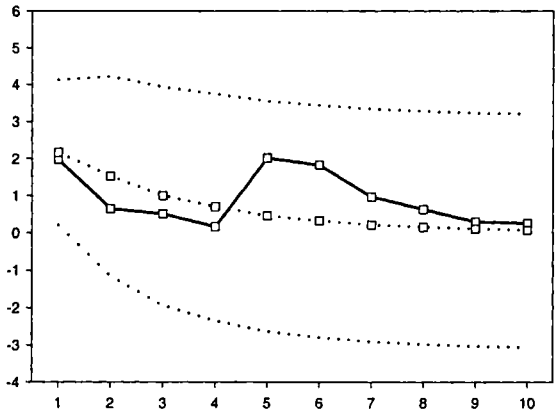

Figure 3. Gaussian prediction intervals for M2: true future values (solid line with squares), predictors (dotted line with squares), Gaussian intervals (dotted line).

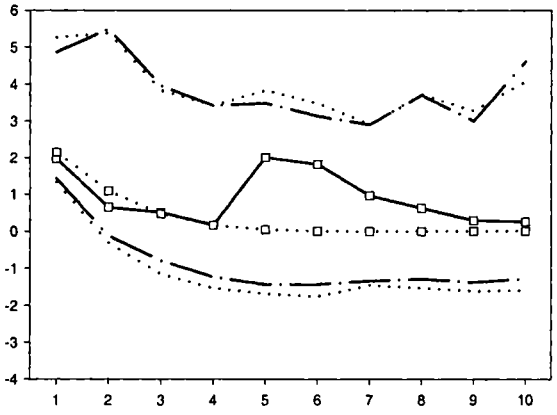

Figure 4. Bootstrap prediction intervals for M2: true future values (solid line with squares), predictors (dotted line with squares), hybrid sieve bootstrap interval (dashdot line), sieve bootstrap-t (dotted line).

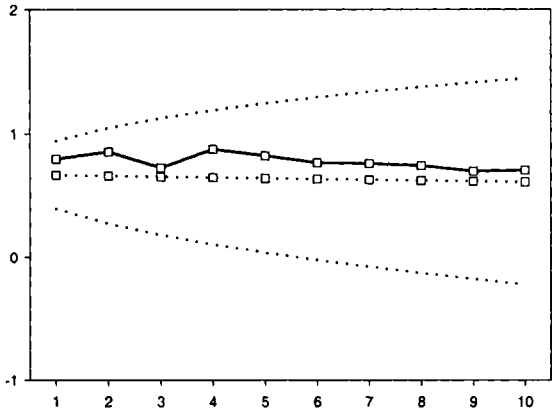

Figure 5. Gaussian prediction intervals for M3: true future values (solid line with squares), predictors (dotted line with squares), Gaussian intervals (dotted line).

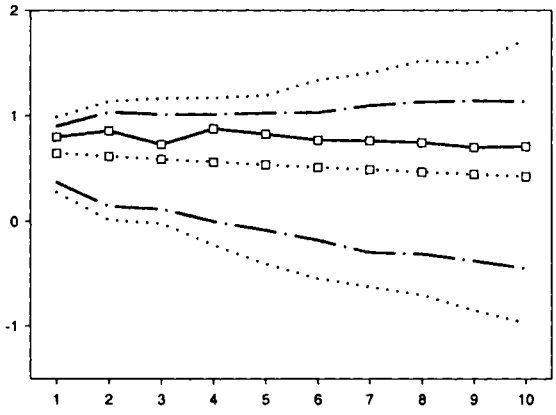

Figure 6. Bootstrap prediction intervals for Ornstein-Uhlenbeck process: true future values (solid line with squares), predictors (dotted line with squares), hybrid sieve bootstrap interval (dashdot line), sieve bootstrap-t (dotted line).

	Gaussian		*hybrid*		*bootstrap-t*	
h	coverage	E(length)	coverage	E(length)	coverage	E(length)
1	93%(2.55)	3.92	90%(3.00)	3.83	94%(2.37)	3.94
2	96%(1.96)	4.00	95%(2.18)	3.93	95%(2.18)	4.04
3	93%(2.55)	4.05	92%(2.71)	3.97	92%(2.71)	4.10
4	94%(2.37)	4.08	92%(2.71)	3.96	92%(2.71)	4.07
5	92%(2.71)	4.10	89%(3.13)	3.99	91%(2.86)	4.13
6	95%(2.18)	4.11	94%(2.37)	4.00	94%(2.37)	4.11
7	97%(1.71)	4.12	93%(2.55)	4.06	94%(2.37)	4.21
8	96%(1.96)	4.12	93%(2.55)	3.99	94%(2.37)	4.11
9	96%(1.96)	4.13	92%(2.71)	4.00	92%(2.71)	4.11
10	98%(1.40)	4.13	95%(2.18)	4.04	96%(1.96)	4.15

Table 1. Empirical coverage for M1. Nominal coverage = 95%

	Gaussian		*hybrid*		*bootstrap-t*	
h	coverage	E(length)	coverage	E(length)	coverage	E(length)
1	93%(2.55)	2.56	74%(4.39)	1.71	80%(4.00)	1.82
2	91%(2.86)	3.51	77%(4.21)	2.55	78%(4.14)	2.78
3	91%(2.86)	3.84	74%(4.39)	2.87	78%(4.14)	3.12
4	90%(3.00)	3.98	76%(4.27)	2.99	80%(4.00)	3.28
5	92%(2.71)	4.05	76%(4.27)	3.01	82%(3.84)	3.31
6	87%(3.36)	4.08	72%(4.49)	3.02	75%(4.33)	3.33
7	90%(3.00)	4.09	73%(4.44)	3.03	79%(4.07)	3.35
8	92%(2.71)	4.10	77%(4.21)	3.04	83%(3.76)	3.35
9	89%(3.13)	4.10	77%(4.21)	3.05	78%(4.14)	3.38
10	91%(2.86)	4.10	76%(4.27)	3.06	82%(3.84)	3.40

Table 2. Empirical coverage for M2. Nominal coverage = 80 %

	Optimal Gaussian		*Gaussian*		*hybrid*		*bootstrap-t*	
h	coverage	E(length)	coverage	E(length)	coverage	E(length)	coverage	E(length)
1	96%(1.96)	0.55	93%(2.55)	0.59	87%(3.36)	0.50	90%(3.00)	0.59
2	94%(2.37)	0.78	86%(3.47)	0.82	86%(3.47)	0.71	90%(3.00)	0.85
3	93%(2.55)	0.95	88%(3.25)	0.98	86%(3.47)	0.85	91%(2.86)	1.06
4	94%(2.37)	1.09	88%(3.25)	1.10	79%(4.07)	0.96	90%(3.00)	1.23
5	95%(2.18)	1.21	86%(3.47)	1.19	84%(3.67)	1.04	91%(2.86)	1.38
6	96%(1.96)	1.32	87%(3.36)	1.27	84%(3.67)	1.10	91%(2.86)	1.51
7	98%(1.40)	1.42	88%(3.25)	1.34	84%(3.67)	1.17	93%(2.55)	1.63
8	96%(1.96)	1.51	89%(3.13)	1.40	85%(3.57)	1.23	93%(2.55)	1.75
9	96%(1.96)	1.59	89%(3.13)	1.45	85%(3.57)	1.27	92%(2.71)	1.85
10	97%(1.71)	1.67	87%(3.36)	1.49	82%(3.84)	1.31	91%(2.86)	1.93

Table 3. Empirical coverage for M3. Nominal coverage = 95 %

5.3. Summary

We observe that for Gaussian series both bootstrap-t and optimal Gaussian prediction intervals perform similarly and outperfom hybrid bootstrap method. On the other hand for non-Gaussian data Gaussian intervals have tendency to exceed the nominal coverage what we can see especially for nominal coverage equals 80 %. In these cases the average length of Gaussian interval is greater than corresponding bootstrap intervals. Moreover for non-Gaussian models hybrid bootstrap also seems to be less effective than bootstrap-t method.

For Ornstein-Uhlenbeck process (model M3) the optimal Gaussian prediction intervals are the most effective. Moreover, we can see that for both nominal coverage levels (80% and 95%) bootstrap-t intervals perform better than both hybrid bootstrap and Gaussian intervals based on estimated autocovariance.

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