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ON INVARIANCE OF LINEAR SUBSPACES FOR STOCHASTIC EVOLUTION EQUATIONS

Dedicated to Professor Kazimierz Urbanik

Abstract. In the paper we consider Itô equation on a Hilbert space. We give necessary and sufficient conditions ensuring the invariance property of linear subspaces of the state space by mild solutions to the equation.

1. Introduction

The paper deals with the following Itô equation on a Hilbert space H:

$$(1) \qquad dX = (AX + D(t,X))dt + B(t,X)dW(t), \quad X(t_0) = x_0$$

on a time interval $[t_0, T]$. Here $A : H \supset D(A) \to H$ is a linear operator, D and B are Lipschitz continuous and W is a cylindrical Wiener process. We study *the invariance problem* related to equation (1) for linear subspaces of H. A subspace G of H is called *invariant* for (1), if for any $t_0 \in [0,T]$ and $x_0 \in G$, the mild solution X to (1) stays in G almost surely. We give two criterions on the invariance property as necessary and sufficient conditions on the coefficients D and B of (1).

The invariance problem in infinite dimensional case was studied by a number of authors. General invariant closed subsets of H are characterized by Jachimiak (1998), the set of nonnegative functions in L^2 by Kotelenez (1992), Goncharuk and Kotelenez (1996)and by Milian (1998). The set of positive functions in L^2 is characterized by Tessitore and Zabczyk (1998). Polyhedrons in Hilbert spaces are characterized by Milian (1998).The invariance problem with respect to weak solutions to (1), related to finite-dimensional submanifolds in H is studied by Filipović (1999). Compared to

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result by Jachimiak, we consider linear subspaces of H however our assumptions on coefficients are more general.

2. Statement of the main results

We are given a probability space $(\Omega, \mathcal{F}, P)$ together with a normal filtration $\mathcal{F} = \{\mathcal{F}_t, t \geq 0\}$. Let H and U be separable Hilbert spaces and let Q be a bounded, self-adjoint , strictly positive operator on U. Let W be a cylindrical $Q-$Wiener having values in U. Let U_0 denote a subspace $Q^{\frac{1}{2}}(U)$ of U, which, endowed with the inner product $\langle u, v\rangle_0 = \langle Q^{-\frac{1}{2}}u, Q^{-\frac{1}{2}}v\rangle$ is a Hilbert space. Further,let $L_2^0 = L_2(U_0, H)$ be the Hilbert space of Hilbert-Schmidt operators acting from U_0 to H, with the norm $\|.\|_{L_2^0}$. Let $L(U, H)$ be the Banach space of linear bounded operators from U into H with the norm $\|\cdot\|_L$.

We shall consider equation (1) on a time interval $[0, T]$, where $T > 0$ is fixed. We will make the following assumptions:

(2) A generates a strongly continuous semigroup $S(t), t \geq 0$ in H;
(3) $D : [0, T] \times H \to H$ is such that for some $C > 0, s, t \in [0, T]$ and

$$x, y \in H : |D(t, x) - D(s, y)| \leq C(|t - s| + \|x - y\|).$$

Moreover, we assume that either

$$(4) \quad \begin{cases} \text{(i) } TrQ < \infty, \\ \text{(ii) } B : [0, T] \times H \to L_2(U_0, H) \text{ is such that for some } C > 0 : \\ \qquad \|B(t, x) - B(s, y)\|_{L_2^0} \leq C(\mid t - s \mid + \|x - y\|), \\ \qquad s, t \in [0, T], \ x, y \in H, \end{cases}$$

or

$$(5) \quad \begin{cases} \text{(i) for an } \alpha \in \left(0, \frac{1}{2}\right) \text{ and for some } s > 0 : \\ \qquad \int_0^s t^{-2\alpha} \|S(t)\|^2_{L_2(H)} dt < \infty, \\ \text{(ii) } B : [0, T] \times H \to L(U, H) \text{ is such that for some } C > 0 : \\ \qquad \|B(t, x) - B(s, y)\|_L \leq C(\mid t - s \mid + \|x - y\|), \\ \qquad s, t \in [0, T], x, y \in H. \end{cases}$$

We can formulate two criterions for the invariance property of subspaces of H. The proofs are left to the next section.

THEOREM 1. *Assume that* (2), (3) *and either* (4) *or* (5) *hold. Let G be a subspace of H such that $G^\perp \subset D(A^*)$. If G is invariant by* (1) *then the following condition holds:*

(6) $\forall t \in [0, T], \forall x \in G, \forall f \in G^\perp$:
 (i) $\langle x, A^* f\rangle + \langle D(t, x), f\rangle = 0$,
 (ii) $(B(t, x))(U_0) \subset G$.

Inversely, suppose that (6) *holds, further* (2), (3) *and either* (4) *or* (5) *hold,* $G^{\perp} \subset D(A^*)$, *moreover assume that* $dim G^{\perp} < \infty$.

Then G *is invariant by* (1).

THEOREM 2. *Assume that* (2), (3) *and either* (4) *or* (5) *hold. Let* G *be a closed subspace of* H *such that* $S(t) : G \to G$, *for all* $t \geq 0$ *and* $G^{\perp} \subset D(A^*)$. *If* G *is invariant by* (1) *then the following condition holds:*

(7) $\forall t \in [0,T], \forall x \in G, \forall f \in G^{\perp}$:

(i) $\langle D(t,x), f \rangle = 0$,

(ii) $B(t,x)(U_0) \subset G$.

Inversely, assume that (7) *holds, moreover* (2), (3) *and either* (4) *or* (5) *hold,* G *is a closed subspace of* H *and* $S(t) : G \to G$, *for all* $t \geq 0$.

Then G *is invariant by* (1).

3. Proofs of the main results

We repeat the needed facts from [6] without proofs, thus making our exposition self-contained. The following lemma is a restatement of Lemma 2 from [6].

LEMMA 1. *If* $f \in D(A^*), x \in H$ *are such that* $\langle x, f \rangle = 0, X$ *is a mild solution to* (1) *on* $[t,T]$ *satisfying the conditions* $P\{X(t) = x\} = 1$ *and* $P\{\langle X(u), f \rangle \geq 0, u \in [t,T]\} = 1$, *then:*

(8) (i) $\langle x, A^* f \rangle + \langle D(t,x), f \rangle \geq 0$,

(ii) $B(t,x)(U_0) \perp f$.

We will use also the following lemma, proved in [6] as Lemma 4.

LEMMA 2. *Assume that* A, D *and* B *satisfy* (2), (3) *and either* (4) *or* (5), $f \in D(A^*)$ *and for* $t \in [0,T]$ *and* $x \in H$ *such that* $\langle x, f \rangle = 0$ *the condition* (8) *is satisfied.*

Then the half-space $\{x \in H : \langle x, f \rangle \geq 0\}$ *is invariant by* (1).

P r o o f of Theorem 1. Let us fix $t_0 \in [0,T]$, $x_0 \in G$ and $f \in G^{\perp}$. Let $X(t)$, $t \in [t_0, T]$ be the mild solution to (1) such that $X(t_0) = x_0$ a.s. If G is invariant by (1), then the assumptions of Lemma 1 are satisfied and we have (6)(ii) and moreover

$$\langle x_0, A^* f \rangle + \langle D(t_0, x_0), f \rangle \geq 0.$$

Taking $-f$ instead of f and repeating application of Lemma 1 we obtain (6)(i).

To prove that (6) is also sufficient for the invariance of G, denote by Π the orthogonal projector of H onto G and let us choose a complete orthonormal system $f_1, \dots f_n$ in $G^{\perp}$. Define

$$\tilde{D}(t,x) = D(t,\Pi x) + \sum_{i=1}^{n} \frac{1}{\|f_i\|^2} \langle \Pi x - x, A^* f_i \rangle f_i, \quad \tilde{B}(t,x) = B(t,\Pi x).$$

Consider the equation

$$dX = (AX + \tilde{D}(t,X))dt + \tilde{B}(t,X)dW(t), \quad X(t_0) = x_0. \tag{9}$$

Note that $\tilde{D}$ and $\tilde{B}$ are Lipschitz continuous and there exists a mild solution $\tilde{X}$ to (9). Let us fix $j \in \{1, \dots n\}$ and note that $\langle x_0, f_j \rangle \geq 0$ and $f_j \in D(A^*)$. By (6), for $x \in H$ such that $\langle x, f_j \rangle = 0$ and for $t \in [0,T]$ we have:

$$\langle x, A^* f_j \rangle + \langle \tilde{D}(t,x), f_j \rangle = \langle \Pi x, A^* f_j \rangle + \langle D(t,\Pi x), f_j \rangle \geq 0 \text{ and}$$
$$\langle \tilde{B}(t,x)g, f_j \rangle = \langle B(t,\Pi x)b, f_j \rangle = 0 \text{ for every } g \in U_0.$$

Hence the assumptions of Lemma 2 are satisfied and we obtain:

$$P\{\langle \tilde{X}(t), f_j \rangle \geq 0, t \in [t_0,T]\} = 1.$$

Similar considerations apply to the half-space $\{x \in H : \langle x, -f_j \rangle \geq 0\}$ and we conclude that

$$P\{\langle \tilde{X}(t), f_j \rangle = 0, t \in [t_0,T]\} = 1.$$

But f_j was fixed arbitrary which gives:

$$P\{\tilde{X}(t) \in G, t \in [t_0,T]\} = 1.$$

Since $\Pi\tilde{X}(t) = \tilde{X}(t), \tilde{D}(t,x) = D(t,x)$ and $\tilde{B}(t,x) = B(t,x)$ for $x \in G$, it follows that $\tilde{X} = X$ and the invariance of G is proved. ∎

Proof of Theorem 2. Let $t_0 \in [0,T], x_0 \in G$ and $f \in G^{\perp}$. As in the proof of Theorem 1, we get $(7)(ii)$ and the following condition:

$$\langle x_0, A^* f \rangle + \langle D(t_0, x_0), f \rangle = 0.$$

Since $S(t)x \in G$ for $t \geq 0$, we have

$$\begin{aligned}\langle x_0, A^* f \rangle &= \lim_{t \to 0} \left\langle x_0, \frac{S^*(t)f - f}{t} \right\rangle \\ &= \lim_{t \to 0} \frac{1}{t} \langle x_0, S^*(t) f \rangle = \lim_{t \to 0} \frac{1}{t} \langle S(t) x_0, f \rangle = 0\end{aligned}$$

and $(7)(i)$ follows. This finishes the proof of necessity of (7) for the invariance of G.

To prove sufficiency of the condition, let us fix $t_0 \in [0,T], x_0 \in G$ and let X be the mild solution to (1) starting at t_0 from x_0. Denote by A_k the Yosida approximation of A and let Π be the orthogonal projector of H onto G. Let X_k be the mild solution to the problem:

$$(10) \qquad dX = (A_k \Pi X + D(t, \Pi X))dt + B(t, \Pi X)dW(t), \quad X(t_0) = x_0.$$

We show that X_k stays in G. For this purpose, let us choose $f \in G^{\perp}$ and consider the half-space $K = \{x \in H : \langle x, f\rangle \geq 0\}$. Since $A_k \circ \Pi$ is bounded, we can treat (10) as the equation (1) with the operator $A = 0$ and with lipschitzean coefficients

$$\tilde{D}(t,x) = A_k \Pi x + D(t, \Pi x), \quad \tilde{B}(t,x) = B(t, \Pi x).$$

It is evident that $f \in D(A^*)$. Let $t \in [0,T]$ and $x \in H$ be such that $\langle x, f\rangle = 0$. By (7)(i) we have:

$$\langle D(t, \Pi x), f\rangle = 0 \text{ and } \langle B(t, \Pi x)u, f\rangle = 0 \text{ for every } u \in U_0.$$

Since G is invariant by $S(t), t \geq 0$ and G is closed,

$$R(k,A)x = \int_0^\infty e^{-kt} S(t)x dt \in G \text{ for } x \in G.$$

Hence $A_k x = (k^2 R(k,A) - kI)x \in G$ and consequently $\langle A_k \Pi x, f\rangle = 0$. The application of Lemma 2 enables us to assert that

$$P\{\langle X_k(t), f\rangle \geq 0, t \in [t_0, T]\} = 1.$$

Proceeding analogously to the proof of Theorem 1 we conclude that

$$P\{X_k(t) \in G, t \in [t_0, T]\} = 1.$$

Finally, the theorem is proved, letting k tend to infinity. ∎

COROLLARY 1. *Let* (2), (3) *and either* (4) *or* (5) *hold. Assume that*

(11) $\forall t \in [o,T], \forall x \in H, \forall f \in D(A^*)$ *such that* $\langle x, f\rangle = 0$ *we have :*
 (i) $\langle x, A^* f\rangle + \langle D(t,x), f\rangle = 0$,
 (ii) $\langle B(t,x)u, f\rangle = 0$ *for all* $u \in U_0$.

Then each subspace G *of* H *such that* $G^{\perp} \subset D(A^*)$ *is invariant by* (1).

Proof. Let $H = G \oplus U_n \oplus V_n$, where $\dim V_n < \infty, V_n \subset V_{n+1}$ and $\bigcup_{n=1}^{\infty} V_n = G^{\perp}$. By Theorem 1 it follows that for each $t_0 \in [0,T]$ and $x_0 \in G$ the solution X to (1) satisfies the condition

$$P\{X(t) \in G \oplus U_n, t \in [t_0, T]\} = 1.$$

But n was arbitrary so we conclude that

$$P\{X(t) \in G, t \in [t_0, T]\} = 1. \blacksquare$$

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