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A CLOSURE METHOD FOR RANDOMLY PERTURBED LINEAR SYSTEMS

Dedicated to Professor Kazimierz Urbanik

1. Introduction

Consider the following linear ordinary equation in R^d

$$\frac{dx}{dt} = A(t)x + \xi(t)C(t)x, \qquad 0 < t \leq T, \tag{1}$$

where $A(t)$, $C(t)$ are $d \times d$ deterministic matrices with bounded for $t \in [0, T]$ elements, $\xi(t)$ is a stochastic process with continuous trajectories.

It is known [1, 2, 3, 6, 7], that for the moments of the solution to (1) we have an infinite chain of equations. To close this chain, we consider the so called closure problem. Several methods of closure were proposed (see, for example, [6, 8]) but only few of them are mathematically justified.

The purpose of the paper is to present a closure method for special case of process $\xi(t)$ and to provide a mathematical justification. Asymptotic expansions for the mean value of the solution of equation (1) are given in the case of large and fast random perturbations. Similar results can be applied also to higher-order moments.

2. Infinite chain for mean value

Let the process $\xi(t)$ in equation (1) be a trigonometric polynomial with respect to Wiener process $w(t)$. For notational simplicity, consider the case $\xi(t) = \beta \sin(\alpha w(t))$, where β, α are nonrandom parameters.

The solution $x(t)$ of equation (1) in this case is a functional with respect to the process $w(t)$, and therefore we shall write $x(t) = x(t; w(z))$. It follows

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from Cameron-Martin formula for the density of Wiener measure under translation [4], that

$$(2) \qquad E\left[\exp\{i\gamma w(t)\}x(t;w(z))\right] = \exp\{-\frac{\gamma^2 t}{2}\}Ex(t;w(z)+i\gamma z).$$

Since

$$\sin(\alpha w(t)) = \frac{\exp\{i\alpha w(t)\} - \exp\{-i\alpha w(t)\}}{2i}$$

using (2) we obtain

$$(3) \qquad \begin{aligned} \frac{dEx(t)}{dt} &= A(t)Ex(t) \\ &+\frac{\beta}{2i}\exp\left\{\frac{-\alpha^2 t}{2}\right\}C(t)[Ex(t;w(z)+i\alpha z) - Ex(t;w(z)-i\alpha z)], \end{aligned}$$

$$\begin{aligned} &\frac{dEx(t;w(z)\pm ik\alpha z)}{dt} \\ &= A(t)Ex(t;w(z)\pm ik\alpha z) \\ &\quad + \frac{\beta}{2i}C(t)[\exp\{\frac{-\alpha^2 t}{2} \mp k\alpha^2 t\}Ex(t;w(z)\pm ik\alpha z + i\alpha z) \\ &\quad - \exp\{\frac{-\alpha^2 t}{2} \pm k\alpha^2 t\}Ex(t;w(z)\pm ik\alpha z - i\alpha z)], \quad k = 1,2,3,\ldots \end{aligned}$$

Let

$$\begin{aligned} v_0(t) &:= Ex(t), \\ v_k(t) &:= \frac{1}{(2i)^k}\exp\{\frac{-k^2\alpha^2 t}{2}\}[Ex(t;w(z)+ik\alpha z) \\ &\qquad + (-1)^k Ex(t;w(z)-ik\alpha z)], \quad k = 1,2,3,\ldots. \end{aligned}$$

Then from (3) we obtain for the mean value of the solution of equation (1) the following infinite chain of differential equation:

$$(4) \qquad \begin{aligned} \frac{dv_0}{dt} &= A(t)v_0 + \beta C(t)v_1, \\ \frac{dv_1}{dt} &= -\frac{\alpha^2}{2}v_1 + A(t)v_1 + \beta C(t)v_2 + \frac{1}{2}\beta C(t)v_0, \\ \frac{dv_k}{dt} &= -\frac{k^2\alpha^2}{2}v_k + A(t)v_k + \beta C(t)v_{k+1} + \frac{1}{4}\beta C(t)v_{k-1}, \quad k = 2,\ldots, \\ v_0(0) &= x(0), \\ v_k(0) &= \frac{1}{(2i)^k}[1+(-1)^k]x(0), \quad k = 1,2,3,\ldots \end{aligned}$$

We prove first that the above chain has a unique solution.

Let $X(t,s)$ be a Cauchy matrix for the equation $\dot{x} = A(t)x$. Since the matrix $A(t)$ is bounded, there exist $a \in R$, $b > 0$ such that for all $t, s \in [0, T]$

$$|X(t,s)| \le b \exp\{-a(t-s)\}. \tag{5}$$

Let $\lambda > 0$ and G be a Banach space of infinite sequences of measurable functions $\varphi = \{\varphi_k(t)\}$ with the norm

$$\|\varphi\| := \sup_k \sup_t [e^{-\lambda t} |\varphi_k(t)|], \quad k = 0, 1, \dots, \quad t \in [0, T].$$

One can easily verify that $\{\varphi_k(t)\} \in G$.

Transforming (4) into the integral form we have

$$\begin{aligned}
v_0(t) &= X(t,0)x(0) + \beta \int_0^t X(t,s)C(s)v_1(s)ds,\\
v_1(t) &= 0 + \beta \int_0^t \exp\{-\frac{\alpha^2}{2}(t-s)\} X(t,s)C(s)[v_2(s) + \frac{v_0(s)}{2}]ds,\\
v_k(t) &= \frac{1}{(2i)^k}[1 + (-1)^k] X(t,0)x(0)\\
&\quad + \beta \int_0^t \exp\left\{ - \frac{k^2\alpha^2}{2}(t-s)\right\} X(t,s)C(s)\left[v_{k+1}(s) + \frac{v_{k-1}(s)}{4}\right] ds.
\end{aligned} \tag{6}$$

The chain (6) can be written in the space G in the form of the following linear equation

$$v = g + Fv,$$

where the operator F and g are determined by the right hand side of (6). Let

$$c := \sup_{t \in [0,T]} |C(t)| \quad \text{and} \quad \lambda > -a.$$

Using (5) we obtain the following estimation for the norm $\|F\|$:

$$\|F\| \le \max_{k=0,1,\dots} \left[\frac{2bc\beta}{2\lambda + 2a + k^2\alpha^2} \right].$$

Consequently there is a $\lambda > 0$ for which $\|F\| < 1$ and therefore the chain (4) has a unique solution.

3. Main result

To close the chain (4) let us restrict our attention to the first n equations letting $v_{n+1}(t) \equiv 0$. Then the functions $v_k^{(n)}(t)$, $k = 0, 1, 2, \dots$, form the following closed chain:

$$\frac{dv_0^{(n)}}{dt} = A(t)v_0^{(n)} + \beta C(t)v_1^{(n)}, \tag{7}$$

$$\frac{dv_1^{(n)}}{dt} = -\frac{\alpha^2}{2}v_1^{(n)} + A(t)v_1^{(n)} + \beta C(t)v_2^{(n)} + \frac{1}{2}\beta C(t)v_0^{(n)},$$

$$\frac{dv_k^{(n)}}{dt} = -\frac{k^2\alpha^2}{2}v_k^{(n)} + A(t)v_k^{(n)} + \beta C(t)v_{k+1}^{(n)} + \frac{1}{4}\beta C(t)v_{k-1}^{(n)}, \quad k = 2,\ldots,n-1,$$

$$\frac{dv_n^{(n)}}{dt} = -\frac{n^2\alpha^2}{2}v_n^{(n)} + A(t)v_n^{(n)} + \frac{1}{4}\beta C(t)v_{n-1}^{(n)},$$

$$v_0^{(n)}(0) = x(0), \quad v_k^{(n)}(0) = \frac{1}{(2i)^k}[1 + (-1)^k]x(0), \quad k = 1,\ldots,n.$$

Let

$$\mu_k := \frac{2bc\beta}{2\sigma + 2a + k^2\alpha^2}, \quad k = 0,1,\ldots, \quad \eta_1 := \frac{bc\beta}{2\sigma + 2a + \alpha^2}, \tag{8}$$

$$\eta_k := \frac{bc\beta}{4\sigma + 4a + 2k^2\alpha^2}, \quad k = 2,3,\ldots .$$

We shall assume that the parameter σ is such that

$$\sigma > -a, \quad \max_{k=0,1,\ldots} (\mu_k \eta_{k+1}) \le \frac{1}{4}. \tag{9}$$

We have

THEOREM. *The solution $v_0^{(n)}(t)$ of the closed chain (7) converges uniformly to the mean value $Ex(t)$ of the solution of equation (1) and*

$$\sup_{t\in[0,T]} \left[e^{-\sigma t}|Ex(t) - v_0^{(n)}(t)|\right] \le \frac{b(4bc\beta)^{n+1}\exp\{4bc\beta\alpha^{-2}\}|x(0)|}{(\sigma + a)(2\sigma + 2a + \alpha^2)\cdots(2\sigma + 2a + n^2\alpha^2)}. \tag{10}$$

Proof. Let

$$\delta_{nm}^{(k)} = \sup_{t\in[0,T]} \left[e^{-\sigma t}|v_k^{(n)}(t) - v_k^{(m)}(t)|\right], \quad k = 0,1,\ldots,n,$$

$$\gamma_m^{(k)} = \sup_{t\in[0,T]} \left[e^{-\sigma t}|v_k^{(m)}(t)|\right], \quad k = 0,1,\ldots,m, \quad n < m.$$

Transforming (7) into integral form we obtain the following inequalities for $\delta_{nm}^{(j)}$:

$$\delta_{nm}^{(0)} \le \mu_0\delta_{nm}^{(1)}, \quad \delta_{nm}^{(k)} \le \mu_k\delta_{nm}^{(k+1)} + \eta_k\delta_{nm}^{(k-1)}, \quad k = 1,\ldots,n-1, \tag{11}$$

$$\delta_{nm}^{(n)} \le \mu_n \gamma_m^{(n+1)} + \eta_n \delta_{nm}^{(n-1)}, \quad n < m,$$

and the inequalities for $\gamma_m^{(j)}$:

$$\gamma_m^{(0)} \le \mu_0 \gamma_m^{(1)} + b|x(0)|, \quad \gamma_m^{(2k+1)} \le \mu_{2k+1}\gamma_m^{(2k+2)} + \eta_{2k+1}\gamma_m^{(2k)}, \quad 2k+1 < m,$$

$$\text{(12)} \qquad \gamma_m^{(2k)} \le \mu_{2k}\gamma_m^{(2k+1)} + \eta_{2k}\gamma_m^{(2k-1)} + 2^{-2k+1} b|x(0)|, \quad 2k < m,$$

$$m = 2p, \quad \gamma_{2p}^{(2p)} \le \eta_{2p}\gamma_{2p}^{(2p-1)} + 2^{-2p+1} b|x(0)|,$$

$$m = 2p+1, \quad \gamma_{2p+1}^{(2p+1)} \le \eta_{2p+1}\gamma_{2p+1}^{(2p)}.$$

From theorem of Worpitzky for continued fractions [10, p. 42] it follows that under conditions (9) we have

$$\text{(13)} \qquad D_j^{(k)} := 1 - \cfrac{\mu_j \eta_{j+1}}{1 - \cfrac{\mu_{j+1}\eta_{j+2}}{1 - \cfrac{\vdots}{\vdots - \mu_{k-1}\eta_k}}} \ge \frac{1}{2}.$$

Using the positiveness of $D_j^{(k)}$ from (11) we obtain the estimates:

$$\text{(14)} \qquad \begin{aligned} \delta_{nm}^{(k)} &\le \frac{\gamma_m^{(n+1)} \prod_{j=k}^{n} \mu_j}{\prod_{j=k}^{n-1} D_j^{(n)}} + \frac{\eta_k \delta_{nm}^{(k-1)}}{D_k^{(n)}}, \quad k = 1, 2, \ldots, n-1, \\ \delta_{nm}^{(0)} &\le \frac{\gamma_m^{(n+1)} \prod_{j=0}^{n} \mu_j}{\prod_{j=0}^{n-1} D_j^{(n)}}. \end{aligned}$$

Let m be even, $m = 2p$, and

$$S_{2p}^{(0)} := \frac{b|x(0)|}{D_0^{(2p)}} + b|x(0)| \sum_{k=1}^{p} 2^{-2k+1} \frac{\prod_{l=0}^{2k-1} \mu_l}{\prod_{l=0}^{2k} D_l^{(2p)}},$$

$$S_{2p}^{(2j-1)} := b|x(0)| \sum_{k=j}^{p} 2^{-2k+1} \frac{\prod_{l=2j-1}^{2k-1} \mu_l}{\prod_{l=2j-1}^{2k} D_l^{(2p)}},$$

$$S_{2p}^{(2j)} := \frac{D_{2j-1}^{(2p)}}{\mu_{2j-1}} S_{2p}^{(2j-1)}, \quad j = 1, 2, 3, \ldots, p.$$

By positivity of $D_j^{(k)}$ again from (12) we obtain

$$\gamma_{2p}^{(j)} \le \frac{\eta_j \gamma_{2p}^{(j-1)}}{D_j^{(2p)}} + S_{2p}^{(j)}, \quad j = 1, \dots, 2p-1, \quad \gamma_{2p}^{(0)} \le S_{2p}^{(0)}. \tag{15}$$

Now, from (15) we have

$$\gamma_{2p}^{(n+1)} \le S_{2p}^{(n+1)} + \sum_{k=0}^{n} S_{2p}^{(k)} \prod_{l=0}^{n-k} \frac{\eta_{n-l+1}}{D_{n-l+1}^{(2p)}}. \tag{16}$$

Taking into account (13) we obtain the estimates for $S_{2p}^{(2j-1)}$ and $S_{2p}^{(2j)}$:

$$\begin{aligned}
S_{2p}^{(2j-1)} &\le b|x(0)| \sum_{k=j}^{p} \frac{2^{2k-4j+2}(bc\beta)^{2k-2j+1}}{\prod_{l=2j-1}^{2k-1} (2\sigma + 2a + l^2\alpha^2)} \\
&\le 4b|x(0)| \Big[\frac{2^{-2j+1}(2bc\beta)}{(2j-1)^2\alpha^2} + \frac{2^{-2j+1}(2bc\beta)^3}{(2j-1)^2\alpha^2(2j)^2\alpha^2(2j+1)^2\alpha^2} + \cdots \\
&\quad + \frac{2^{-2j+1}(2bc\beta)^{2p-2j+1}}{(2j-1)^2\alpha^2 \cdots (2p-1)^2\alpha^2} \Big] \le 4b|x(0)| \exp\{2bc\beta\alpha^{-2}\}, \\
S_{2p}^{(2j)} &\le b|x(0)| \\
&\quad \times \sum_{k=j}^{p} \frac{2^{-2j+2}(2bc\beta)^{2k-2j}}{(2\sigma+2a+(2j)^2\alpha^2)(2\sigma+2a+(2j+1)^2\alpha^2) \cdots (2\sigma+2a+(2k-1)^2\alpha^2)} \\
&\le 4b|x(0)| \exp\{2bc\beta\alpha^{-2}\}.
\end{aligned}$$

Combining these estimations with (13), (16) we get

$$\begin{aligned}
\gamma_{2p}^{(n+1)} &\le 4b|x(0)| \Big(1 + \sum_{k=0}^{n} 2^{n-k+1} \prod_{l=0}^{n-k} 2^l \eta_{n-l+1}\Big) \exp\{2bc\beta\alpha^{-2}\} \\
&\le 4b|x(0)| \exp\{2bc\beta\alpha^{-2}\} \Big[1 + \frac{2bc\beta}{4\sigma + 4a + 2(n+1)^2\alpha^2} + \cdots \\
&\quad + \frac{2^{n+1}(bc\beta)^{n+1}}{(2\sigma + 2a + \alpha^2)(4\sigma + 4a + 2(2\alpha)^2) \cdots (4\sigma + 4a + 2(n+1)^2\alpha^2)} \Big] \\
&\le 4b|x(0)| \exp\{4bc\beta\alpha^{-2}\}.
\end{aligned}$$

Notice, that in the same way one can obtain similar estimates for odd m.

Finally from (13), the last estimate in (14) and the uniqueness of the solution of the chain (4) we conclude that $v_0^{(n)}(t)$ converges uniformly to $v_0(t)$ and the estimate (10) holds.

4. Asymptotic expansions

Let ε be a small parameter, $\beta = \varepsilon^{-1}$, $\alpha = \varepsilon^{-1}$, i.e. the random disturbances are large and fast. According to the estimate (10) we have

$$|Ex(t) - v_0^{(n)}(t)| = O(\varepsilon^{n-1}), \quad \varepsilon \to 0. \tag{17}$$

In order to get an asymptotic expansion for $Ex(t)$ in powers of ε it is sufficient to obtain it for the solution of the chain (7). Let $u_0(t) =: v_0^{(n)}(t)$, $u_1(t) := \varepsilon^{-1} v_1^{(n)}(t)$, $u_k(t) := v_k^{(n)}(t)$, $k = 2, 3, \ldots, n$.

We have the following equations

$$\begin{aligned}
\frac{du_0}{dt} &= A(t)u_0 + C(t)u_1, \\
\varepsilon^2 \frac{du_1}{dt} &= -\frac{1}{2}u_1 + \varepsilon^2 A(t)u_1 + C(t)u_2 + \frac{1}{2}C(t)u_0, \\
\varepsilon^2 \frac{du_2}{dt} &= -2u_2 + \varepsilon^2 A(t)u_2 + \varepsilon C(t)u_3 + \frac{1}{4}\varepsilon^2 C(t)u_1, \\
\varepsilon^2 \frac{du_k}{dt} &= -\frac{k^2}{2}u_k + \varepsilon^2 A(t)u_k + \varepsilon C(t)u_{k+1} + \frac{1}{4}\varepsilon C(t)u_{k-1}, \quad k = 3, \ldots, n-1, \\
\varepsilon^2 \frac{du_n}{dt} &= -\frac{n^2}{2}u_n + \varepsilon^2 A(t)u_n + \frac{1}{4}\varepsilon C(t)u_{n-1}, \\
u_0(0) = x(0), \quad u_k(0) &= \frac{1}{(2i)^k}[1 + (-1)^k]x(0), \quad k = 1, \ldots, n.
\end{aligned} \tag{18}$$

Consequently we obtain for $u_0(t)$ the chain of differential equations with a small parameter in the derivatives. The construction of asymptotic expansions in powers of ε for these equations is well known (see [5, 9]). If the matrices $A(t)$, $C(t)$ are smooth enough, for this purpose we can use the boundary function method since all conditions required by this method are fulfiled in our case.

According to the boundary function method, we seek the asymptotic expansions as the sum of regular parts and boundary layer parts

$$u_k(t) \sim \sum_{m=0}^{\infty} \varepsilon^m u_{km}(t) + \sum_{m=0}^{\infty} \varepsilon^m g_{km}(\tau), \quad \tau = t\varepsilon^{-2}.$$

Substituting the regular parts of these expansions to (18) we obtain the equations:

$$\frac{du_{00}}{dt} = A(t)u_{00} + C(t)u_{10}, \quad -\frac{1}{2}u_{10} + C(t)u_{20} + \frac{1}{2}C(t)u_{00} = 0,$$
$$-2u_{20} = 0, \quad -\frac{9}{2}u_{30} = 0, \quad -\frac{16}{2}u_{40} = 0,$$

$$\frac{du_{01}}{dt} = A(t)u_{01} + C(t)u_{11}, \quad -\frac{1}{2}u_{11} + C(t)u_{21} + \frac{1}{2}C(t)u_{01} = 0,$$
$$-2u_{21} + C(t)u_{30} = 0, \quad -\frac{9}{2}u_{31} + C(t)u_{40} + \frac{1}{4}C(t)u_{20} = 0,$$
$$\frac{du_{02}}{dt} = A(t)u_{02} + C(t)u_{12},$$
$$\frac{du_{10}}{dt} = -\frac{1}{2}u_{12} + A(t)u_{10} + C(t)u_{22} + \frac{1}{2}C(t)u_{02},$$
$$\frac{du_{20}}{dt} = -2u_{22} + A(t)u_{20} + C(t)u_{31} + \frac{1}{4}C(t)u_{10}.$$

Therefore $u_{20} \equiv 0$, $u_{30} \equiv 0$, $u_{40} \equiv 0$, $u_{21} \equiv 0$, $u_{31} \equiv 0$, $u_{10} = C(t)u_{00}$ and u_{00}, u_{01}, u_{02} are solutions to the equations:

$$\begin{aligned} \frac{du_{00}}{dt} &= A(t)u_{00} + C^2(t)u_{00}, \\ \frac{du_{01}}{dt} &= A(t)u_{01} + C^2(t)u_{01}, \\ \frac{du_{02}}{dt} &= A(t)u_{02} + C^2(t)u_{02} \\ &\quad + 2C(t)[A(t)C(t) - C(t)A(t) - C'(t) - \frac{7}{8}C^3(t)]u_{00}. \end{aligned} \tag{19}$$

In order to find the initial values to the equations (19) we consider the boundary layer parts of the asymptotic expansions.

Assume that $A(\tau\varepsilon^2) = A(0) + O(\varepsilon)$, $C(\tau\varepsilon^2) = C(0) + O(\varepsilon)$. Considering in the equations (18) the stretched time $\tau = t\varepsilon^{-2}$ and substituting the boundary layer parts of asymptotic expansions, we obtain

$$\begin{aligned} &\frac{dg_{00}}{d\tau} = 0, \quad \frac{dg_{01}}{d\tau} = 0, \quad \frac{dg_{02}}{d\tau} = A(0)g_{00} + C(0)g_{10}, \\ &\frac{dg_{10}}{d\tau} = -\frac{1}{2}g_{10} + C(0)g_{20} + \frac{1}{2}C(0)g_{00}, \quad \frac{dg_{20}}{d\tau} = -2g_{20}. \end{aligned} \tag{20}$$

Taking into account the initial values to (18) and conditions

$$\lim_{\tau\to\infty} g_{km}(\tau) = 0,$$

from (20) we obtain that

$$g_{00}(\tau) \equiv 0, \quad g_{01}(\tau) \equiv 0, \quad u_{00} = x(0), \quad u_{01} = 0, \quad g_{20}(\tau) = \frac{-e^{-2\tau}x(0)}{2},$$
$$g_{10}(\tau) = \frac{e^{-\tau/2}}{3}[e^{-3\tau/2} - 4]C(0)x(0), \quad g_{02}(0) = \frac{5}{2}C^2(0)x(0),$$

$$u_{02}(0) = -\frac{5}{2}C^2(0)x(0), \quad g_{02}(\tau) = \frac{1}{6}[16e^{-\tau/2} - e^{-2\tau}]C^2(0)x(0).$$

Therefore by (17) and [5, 9] we have the following result.

PROPOSITION. *If the matrices $A(t)$ and $C(t)$ are twice differentiable for $t \in [0,T]$, the mean value $Ex(t)$ of the solution of equation* (1) *has the following asymptotic expansion*

$$Ex(t) = u_{00}(t) + \varepsilon^2 u_{02}(t) + \varepsilon^2 g_{02}(\tau) + O(\varepsilon^3), \quad \tau = t\varepsilon^{-2}, \quad \varepsilon \to 0,$$

where $u_{00}(t)$, $u_{02}(t)$, $g_{02}(\tau)$ are determined by equations:

$$\frac{du_{00}}{dt} = A(t)u_{00} + C^2(t)u_{00}, \quad u_{00}(0) = x(0),$$

$$\frac{du_{02}}{dt} = A(t)u_{02} + C^2(t)u_{02}$$

$$+ 2C(t)[A(t)C(t) - C(t)A(t) - C'(t) - \frac{7}{8}C^3(t)]u_{00},$$

$$u_{02}(0) = -\frac{5}{2}C^2(0)x(0),$$

$$g_{02}(\tau) = \frac{1}{6}[16e^{-\tau/2} - e^{-2\tau}]C^2(0)x(0).$$

Notice that the above approach permits also to construct asymptotic expansions in the case of fast random disturbances.

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