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ASYMPTOTICS AND HIGH DIMENSIONAL APPROXIMATIONS FOR NONLINEAR PSEUDODIFFERENTIAL EQUATIONS INVOLVING LÉVY GENERATORS

Dedicated to Professor Kazimierz Urbanik
on the occasion of His 70th birthday

Abstract. Nonlinear pseudodifferential equations involving Lévy semigroup generators are used in physical models where the diffusive behavior is affected by hopping and trapping phenomena. In this paper we present several results concerning asymptotics and high dimensional Monte Carlo-type approximations via interacting particle systems for two classes of such equations.

1. Introduction

In this paper we present several asymptotic and approximation results for the Cauchy problem for nonlinear pseudodifferential equations of the form

$$u_t + \mathcal{L}u + \nabla \mathcal{N} u = 0, \quad u(x,0) = u_0(x), \tag{1}$$

where $u = u(x,t)$, $x \in \mathbb{R}^d$, $t \geq 0$, $u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$, $-\mathcal{L}$ is a (linear) generator of a symmetric positive semigroup $e^{-t\mathcal{L}}$ on $L^1(\mathbb{R}^d)$, with the symbol defined by the Lévy–Khintchine formula

$$a(\xi) = ib\xi + q(\xi) + \int_{\mathbb{R}^d} (1 - e^{-i\eta\xi} - i\eta\xi \mathbb{1}_{\{|\eta|<1\}}(\eta))\, \Pi(d\eta), \tag{2}$$

see [3, Ch.1, Th. 1], and $\mathcal{N}$ is a nonlinear operator to be specified later. We will assume that $b = 0$. The function $q(\xi) = \sum_{j,k=1}^{d} q_{jk}\xi_j\xi_k$ is a positive definite quadratic function on $\mathbb{R}^d_\xi$ and the Lévy measure Π satisfies the usual integrability condition $\Pi(\{0\}) = 0$, $\int_{\mathbb{R}^d} \min(1, |\eta|^2)\, \Pi(d\eta) < \infty$.

The solutions to the Cauchy problem (1) have to be understood in some weak sense and several options are here available and have been studied in the papers quoted in the references. For the sake of this presentation let us just say that as the *mild* solution to (1) we mean a solution of the integral equation

$$u(t) = e^{-t\mathcal{L}}u_0 - \int_0^t \nabla \cdot e^{-(t-\tau)\mathcal{L}}(\mathcal{N}u)(\tau)\, d\tau, \tag{3}$$

motivated by the classical Duhamel formula.

Such equations are used in physical models where the diffusive behavior is affected by hopping, trapping and other nonlocal, but possibly self-similar, phenomena (see, e.g., [1], [2], [13], [17], [24], [26], [27], [29], [30], [32], [33]).

Recently, we have studied the questions of existence, uniqueness, regularity, temporal asymptotics, and interacting particle approximations (*propagation of chaos*) for certain special cases of equation (1), in particular, the *fractal Burgers equation* (see, [4], [16]),

$$u_t + (-\Delta)^{\alpha/2}u + c\cdot\nabla(u|u|^{r-1}) = 0, \quad c \in \mathbb{R}^d, \tag{4}$$

and the one-dimensional *multifractal conservation laws* (see [6]),

$$u_t + \mathcal{L}u + f(u)_x = 0, \tag{5}$$

with the *multifractal operator*

$$\mathcal{L} = -a_0\Delta + \sum_{j=1}^{k} a_j(-\Delta)^{\alpha_j/2}, \tag{6}$$

$0 < \alpha_j < 2$, $a_j > 0, j = 0, 1, \ldots, k$, where $(-\Delta)^{\alpha/2}$, $0 < \alpha < 2$, is the fractional Laplacian defined as the Fourier multiplier operator

$$((-\Delta)^{\alpha/2}v) = \mathcal{F}^{-1}(|\xi|^\alpha(\mathcal{F}v)(\xi)). \tag{7}$$

All these equations are generalizations of the classical Burgers equation

$$u_t - u_{xx} + (u^2)_x = 0, \tag{8}$$

and the results provided below extend our work quoted above. The detailed proofs of these results will appear in [8], [9], see also [7].

2. Asymptotics for conservation laws

Intuitively speaking, our results from [6] have shown that the first order asymptotics (as $t \to \infty$) for solutions of the Cauchy problem for the multifractal conservation laws (5-6) is essentially linear. More precisely, if $1 \le \alpha < 2$, with

$$\alpha = \min\{\alpha_1, ..., \alpha_k\},$$

and if the nonlinearity f has a polynomial growth then, for

$$u_0(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}),$$

the Cauchy problem has a global-in-time mild solution such that the first term of its asymptotic expansion is given by the solution of the Cauchy problem of the linear equation $u_t + \mathcal{L}u = 0$ with the same initial data, or quantitatively, in any L^p-norm $\|.\|_p$, $1 \le p \le \infty$,

$$t^{(1-1/p)/\alpha}\|u(t) - e^{t\mathcal{L}} * u_0\|_p \to 0$$

as $t \to \infty$. For the linear equation the asymptotics is clear: there exists a nonnegative function $\eta \in L^\infty(0,\infty)$ satisfying $\lim_{t\to\infty}\eta(t) = 0$, and such that

$$\Big\|e^{t\mathcal{L}} * u_0 - \int_{\mathbb{R}} u_0(x)\,dx \cdot p_{\mathcal{L}}(t)\Big\|_p \le t^{-(1-1/p)/\alpha}\eta(t),$$

where $p_{\mathcal{L}}(t)$ is the kernel of the operator $\mathcal{L}$ in (6).

The results summarized below extend the above first order asymptotics results to a more general class of nonlocal diffusion operators $\mathcal{L}$ and also give information about the second order asymptotics. They will be formulated under various sets of assumptions on either the semigroup $e^{-t\mathcal{L}}$ or the symbol a of $\mathcal{L}$ which will include the following conditions satisfied for all $t > 0$, $1 \le p \le \infty$ and some $0 < \alpha,\ \widetilde{\alpha} < 2$

$$\|e^{-t\mathcal{L}}\|_{1,p} \le \min(c_1 t^{-n(1-1/p)/2}, c_2 t^{-n(1-1/p)/\alpha}), \tag{9}$$

$$\|\nabla e^{-t\mathcal{L}}\|_{1,p} \le \min(c_1 t^{-n(1-1/p)/2-1/2}, c_2 t^{-n(1-1/p)/\alpha)-1/\alpha}), \tag{10}$$

$$0 < \liminf_{\xi\to 0}\frac{a(\xi)}{|\xi|^\alpha} \le \limsup_{\xi\to 0}\frac{a(\xi)}{|\xi|^\alpha} < \infty, \quad 0 < \inf_{\xi}\frac{a(\xi)}{|\xi|^2}, \tag{11}$$

$$\limsup_{|\xi|\to\infty}\frac{a(\xi) - a_0|\xi|^2}{|\xi|^{\widetilde{\alpha}}} < \infty \quad \text{for some } a_0 > 0. \tag{12}$$

Here and later on, $\|.\|_{k,p}$ stands for the usual Sobolev space $W^{k,p}$-norm. The conditions (11)–(12), together with a smoothness assumption on a for $\xi \ne 0$, imply decay rates (9)–(10). All these assumptions are verified by, e.g., *multifractal diffusion operators* (6) with $a_0 > 0$.

THEOREM 1. *Assume that* $f \in C^1(\mathbb{R},\mathbb{R}^d)$ *and* $\mathcal{L}$ *is of the form* (2) *and satisfies* (12). *Given* $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, *there exists a unique solution* $u \in C([0,\infty);\ L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ *of the problem*

$$u_t + \mathcal{L}u + \nabla \cdot f(u) = 0, \qquad u(x,0) = u_0(x). \tag{13}$$

This solution is regular, $u \in C((0,\infty); W^{2,2}(\mathbb{R}^d)) \cap C^1((0,\infty); L^2(\mathbb{R}^d))$, *satisfies the conservation of integral property* $\int u(x,t)\,dx = \int u_0(x)\,dx$, *and the contraction property*

$$\|u(t)\|_p \le \|u_0\|_p, \tag{14}$$

for each $p \in [1,\infty]$ *and all* $t > 0$. *Moreover, the maximum and minimum principles hold:* $\operatorname{ess\,inf} u_0 \le u(x,t) \le \operatorname{ess\,sup} u_0$, *a.e.* x, t, *as well as the* *comparison principle for* $u_0 \le v_0 \in L^1(\mathbb{R}^d)$*:*

$$u(x,t) \le v(x,t) \ \ \text{a.e.} \ \ x,\ t, \ \ \text{and} \ \ \|u(t) - v(t)\|_1 \le \|u_0 - v_0\|_1. \tag{15}$$

The estimates of solutions of the nonlinear equation (13), which turn out to be the same as for the linear semigroup, can be proved under quite general assumptions on the decay of the semigroup, much weaker than (9).

THEOREM 2. (i) *If the semigroup* $e^{-t\mathcal{L}}$ *verifies the estimate* $\|e^{-t\mathcal{L}}\|_{1,\infty} \le m(t)$ *for some decreasing* C^1 *function* m*, then positive solutions of the Cauchy problem (13) satisfy the bound*

$$\|u(t)\|_2 \le m(t)^{1/2}\|u_0\|_1. \tag{16}$$

Moreover, if $m(t) = ct^{-\varepsilon}$ *(as it is whenever (9) holds), then the same estimate is valid for solutions of arbitrary sign.*

(ii) *If* $\|e^{-t\mathcal{L}}\|_{2,\infty} \le M(t)$*, then* $\|u(t)\|_\infty \le M(t)\|u_0\|_2$*, for* u_0 *of arbitrary sign.*

(iii) *Under assumption (9) on* $e^{-t\mathcal{L}}$ *the bound*

$$\|u(t)\|_p \le C_p \min(t^{-n(1-1/p)/2}, t^{-n(1-1/p)/\alpha})\|u_0\|_1$$

holds for all $1 \le p \le \infty$*. Moreover, if* $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$*, then*

$$\|u(t)\|_p \le C(1+t)^{-n(1-1/p)/\alpha} \tag{17}$$

with a constant C *which depends on* $\|u_0\|_1$ *and* $\|u_0\|_p$*.*

Two consecutive terms of asymptotics of solutions of (13) are described in the next two theorems.

THEOREM 3. *Assume that* u *is a solution of the Cauchy problem (13) with* $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ *and* $e^{-t\mathcal{L}}$ *satisfies (9)–(10) with some* $0 < \alpha < 2$*. Furthermore, suppose that* $f \in C^1$*,* $\limsup_{s\to 0} |f(s)|/|s|^r < \infty$ *for some*

$$r > \max((\alpha - 1)/n + 1, 1)). \tag{18}$$

Then, for every $1 \le p \le \infty$*, the relation*

$$\lim_{t\to\infty} t^{n(1-1/p)/\alpha}\|u(t) - e^{-t\mathcal{L}}u_0\|_p = 0 \tag{19}$$

holds.

THEOREM 4. *Let the symbol* a *of* $\mathcal{L}$ *satisfy the assumptions* (11)*, the semigroup satisfy* (9)–(10)*, and* $f \in C^2$*,* $f'(0) = 0$*. If* $n = 1$ *and* $\alpha \ge 1$*, suppose moreover that* $f \in C^3$*,* $f''(0) = 0$*. Then for each* $1 < p \le \infty$*, the solution of* (13) *satisfies the limit relation*

$$\lim_{t\to\infty} t^{n(1-1/p)/\alpha+1/\alpha}\|u(t) - e^{-t\mathcal{L}}u_0 + F\cdot(\nabla e^{-t\mathcal{L}}\delta_0)\|_p = 0 \tag{20}$$

with $F = \int_0^\infty \int_{\mathbb{R}^d} f(u(y,\tau))\,dy\,d\tau$*.*

3. Critical nonlinearity exponents

By contrast with the results of the previous section, let us note that the first order asymptotics of solutions to the Cauchy problem for the Burgers equation (8) is described by the relation

$$t^{(1-1/p)/2}\|u(t)-U_M(t)\|_p \to 0, \qquad \text{as} \qquad t\to\infty,$$

where

$$U_M(x,t)=t^{-1/2}\exp(-x^2/4t)\Big(K(M)+\frac{1}{2}\int_0^{x/\sqrt{t}}\exp(-\xi^2/4)\,d\xi\Big)^{-1}$$

is the, so-called, source solution such that $u(x,0)=M\delta_0$. It is easy to verify that this solution is self-similar, i.e., $U_M(x,t)=t^{-1/2}U(xt^{-1/2},1)$. Thus, the long time behavior of solutions to the classical Burgers equation is genuinely nonlinear, i.e., it is not determined by the asymptotics of the linear heat equation.

As it turns out that genuinely nonlinear behavior of the Burgers equation is due to the precisely matched balancing influence of the regularizing Laplacian diffusion operator and the gradient-steepening quadratic inertial nonlinearity.

The next result finds such a matching critical nonlinearity exponent for the nonlocal multifractal Burgers equation so that the solutions of (13) with a multifractal operator $\mathcal{L}$ (see (6)) behave asymptotically like self-similar source solutions U of (4) with singular initial data $M\delta_0$. Note that here u_0 is not necessarily positive, while positivity of U is a subtle consequence of (4) and $M>0$.

THEOREM 5. *Let u be a solution of the Cauchy problem* (13) *with the operator $\mathcal{L}=(-\Delta)^{\alpha/2}+\mathcal{K}$ for some $1<\alpha<2$, and another Lévy operator $\mathcal{K}$ whose symbol k fulfills $\lim_{\xi\to 0}k(\xi)/|\xi|^\alpha=0$ (in particular, $\mathcal{L}$ can be a multifractal operator of the form* (6) *with $a_0\geq 0$, $1<\alpha_j<2$, $\alpha=\min\{\alpha_1,\dots,\alpha_k\}$), and $u_0\in L^1(\mathbb{R}^d)$, $\int_{\mathbb{R}^d}u_0(x)\,dx=M>0$. Assume that f satisfies the condition*

$$\lim_{s\to 0}\frac{f(s)}{s|s|^{(\alpha-1)/n}}\in\mathbb{R}. \tag{21}$$

Then, for each $1\leq p\leq\infty$,

$$\lim_{t\to\infty}t^{n(1-1/p)/\alpha}\|u(t)-U(t)\|_p=0, \tag{22}$$

where $U=U_M$ is the unique solution of the problem (4) *with $r=\max((\alpha-1)/n+1,1))$ and the initial data $M\delta_0$. Moreover, U is of self-similar form $U(x,t)=t^{-n/\alpha}U(xt^{-1/\alpha},1)$, $\int_{\mathbb{R}^d}U(x,1)\,dx=M$, and $U\geq 0$.*

4. Interacting particle approximations

The results of this section can be viewed as extension of our work from [16] where we established the existence of McKean's nonlinear diffusions, and related interacting particle approximation schemes (propagation of chaos in a wide sense) for the fractal Burgers equation and of [10], where we studied global and exploding solutions for equations of the form

$$u_t + (-\Delta)^{\alpha/2}u - \nabla \cdot (uB(u)) = 0. \tag{23}$$

Here $u : \Omega \times (0,T) \subset \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$, $(-\Delta)^{\alpha/2}$ is a fractional power of the minus Laplacian in $\mathbb{R}^d$, $0 < \alpha \le 2$, and

$$B(u)(x) = \int_{\mathbb{R}^d} b(x,y)u(y)\,dy$$

is a linear $\mathbb{R}^d$-valued integral operator with the kernel $b : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$. The dimension is restricted to the physically interesting cases $d = 1, 2$, or 3. The proofs of the statements provided below appear in [5].

Equations (23) describe various physical phenomena involving diffusion and interaction of pairs of particles when suitable assumptions are made on the possibly singular integral operator B. Since our main interest is in u as a description of the density of particles in $\mathbb{R}^d$, we will only consider nonnegative solutions to (23). For instance, if $b(x,y) = c(x-y)|x-y|^{-d}$ then the equation (23) models the diffusion of charge carriers ($c < 0$) in electrolytes, semiconductors or plasmas interacting via Coulomb forces. If $c > 0$, it describes gravitational interaction of particles in a cloud, or galaxies in a nebula, or the Biot–Savart kernel $b(x,y) = (2\pi)(x_2 - y_2, y_1 - x_1)|x - y|^{-2}$ in $\mathbb{R}^2$, the equation (23) with $\alpha = 2$ is equivalent to the vorticity formulation of the Navier–Stokes equations. Its solutions are global in time. Also, formally, the singular kernel $b(x,y) = c\delta(x-y)$ leads to the classical Burgers equation (8).

We restrict ourselves to the most important in the applications case of convolution operators B in (23), so that from now on $b(x,y) = b(x-y)$. Moreover, we assume that b satisfies potential estimates like either

$$|b(x)| \le C|x|^{\beta-d} \tag{24}$$

or

$$|Db(x)| \le C|x|^{\gamma-d} \tag{25}$$

for some $0 < \beta < d$, $0 < \gamma < d$, which is motivated by the above mentioned physical examples. Formally, the case of the Burgers equation (8) corresponds to the limit case $\beta = 0$ but, of course, the operator $B(u) = cu$, $0 \ne c \in \mathbb{R}^d$, is not an integral one. In fact, assumptions (24), (25) can be weakened as, e.g., in [10], but we prefer to keep the potential character and smoothing properties of B clear.

We begin with the construction of a nonlinear Markov process for which the equation (23) serves as the Fokker–Planck–Kolmogorov equation. The assumption $\alpha \in (1,2)$ permits us to freely use the expectations of the α-stable processes involved in the construction.

Let $u \geq 0$ be a (local in time) solution of (23). Without loss of generality we can assume that u is bounded, i.e.

$$\sup_{x \in \mathbb{R}^d, t \in [0,T]} |u(x,t)| < \infty. \tag{26}$$

Moreover, since we are working with $(L^1 \cap L^\infty)$-solutions

$$\sup_{x \in \mathbb{R}^d, t \in [0,T]} |B(u(t))(x)| < \infty$$

follows from the potential estimate (24), Sobolev embedding theorem and (26).

Consider a solution $X(t)$ of the stochastic differential equation

$$dX(t) = dS(t) - B(u(t))(X(t))\,dt, \tag{27}$$

where u is a given (bounded) solution of (23), $X(0) \sim u(x,0)\,dx$ in law, and $S(t)$ is a standard α-stable spherically symmetric process with its values in $\mathbb{R}^d$. Since the coefficient $B(u)$ in (27) is bounded, based on the work [20], we infer that the stochastic differential equation (27) has a unique solution X. The measure-valued function

$$v(dx,t) \equiv P(X(t) \in dx) \tag{28}$$

satisfies the weak forward equation

$$\frac{d}{dt}\langle v(t), \eta\rangle = \langle v(t), \mathcal{L}_{u(t)}\eta\rangle, \tag{29}$$

for all $\eta \in \mathcal{S}(\mathbb{R}^d)$, the Schwartz class of functions on $\mathbb{R}^d$, with the initial condition $v(0) = u(x,0)\,dx$ and the operator

$$\mathcal{L}_u = -(-\Delta)^{\alpha/2} - B(u) \cdot \nabla, \quad u = u(x,t).$$

THEOREM 6. *Let* $1 < \alpha < 2$ *and* u *be a solution of* (23) *satisfying* (24). *The process* $X(t)$ *in* (27) *is the McKean process* (*nonlinear Markov process*) *corresponding to* (23), *that is, it satisfies the relation*

$$P(X(t) \in dx) = u(x,t)\,dx.$$

The proof of the above result can be sketched as follows. From the results of [14] (see [15]), the following two statements are equivalent:

- The martingale problem for the operator $\mathcal{L}_{u(t)}$ is well posed, and
- The existence and uniqueness theorem holds for the corresponding *linear* weak forward equation (29).

Here, the martingale problem associated with (27) is well posed. However, $u(dx,t) \equiv u(x,t)\,dx$ is also a solution of (29) since

$$\frac{d}{dt}\langle u(t),\eta\rangle = \langle -(-\Delta)^{\alpha/2}u + \nabla\cdot(uB(u)),\eta\rangle = \langle u,\mathcal{L}_u\eta\rangle.$$

Since the coefficients of the *linear* equation (29) are regular ($B(u)\in L^\infty$), the problem $w_t = \mathcal{L}_u w$, $w(0)=0$, has the unique solution $w\equiv 0$. This can be easily seen from the energy estimates used in [10]. Now, the uniqueness for (29) implies that $v(dx,t) = u(dx,t)$, which yields Theorem 6.

Now we can describe our high dimensional interacting particle approximation for the equation (13). Results in this spirit, when $\mathcal{L}$ is replaced by the usual Laplacian, have been proved in various situations after the pioneering work [22]. The following references contain reformulations, extensions and generalizations of the McKean's scheme (also often called "propagation of chaos results") for various evolution problems of physical origin: [19], [12], [21], [25], [31], [34], [11]. Besides a purely mathematical interest, they also give reasonably well working tools for the numerical approximation of solutions, especially when convergence rates can be found. In the case of the Biot-Savart kernel mentioned at the beginning of this section, they coincide with the "random vortex method" for the two-dimensional Navier-Stokes equations (see, e.g., [18]).

Let us consider a standard smoothing kernel

(30) $$\delta_\epsilon(x) = (2\pi\epsilon)^{-d/2}\exp(-|x|^2/(2\epsilon)),\quad \epsilon>0,$$

and the system of regularized equations (27)

(31) $$dX^{i,n,\epsilon}(t) = dS^i(t) - \frac{1}{n}\sum_{j\neq i} b_\epsilon\left(X^{i,n,\epsilon}(t) - X^{j,n,\epsilon}(t)\right)dt,$$

where $b(x,y)=b(x-y)$, $b_\epsilon = b*\delta_\epsilon$. Then define random empirical measures

(32) $$Y^{n,\epsilon}(t) = \frac{1}{n}\sum_{i=1}^{n}\delta(X^{i,n,\epsilon}(t)).$$

THEOREM 7. *Let the conditions ensuring the local in time existence of solutions of (23) on* $\mathbb{R}^d\times(0,T)$ *be satisfied. Moreover, assume that*

$$|(\mathcal{F}b)(\xi)| \le C(1+|\xi|^{-\beta})$$

(which is, of course, compatible with the potential estimate (24)), *and that the initial conditions* $\{X^{i,n,\epsilon}(0)\}_{i=1,\dots,n}$ *satisfy*

$$\sup_n \sup_{\lambda\in\mathbb{R}^d} n^{1-1/\alpha}(1+|\lambda|^a)^{-1}E\,|\langle Y^{n,\epsilon}(0) - u^\epsilon(x,0),\chi_\lambda\rangle| < \infty$$

for some $a\ge 0$ *and all the characters* $\chi_\lambda(x)=e^{i\lambda x}$. *Then:*

(i) *For each $\epsilon > 0$ the empirical process is weakly convergent*

$$Y^{n,\epsilon}(t) \Rightarrow u^\epsilon(x,t)\,dx, \text{ in probability, as } n \to \infty.$$

The limit density $u^\epsilon = u^\epsilon(x,t)$, $x \in \mathbb{R}^d$, $t \in (0,T)$, solves the regularized equation (13)

$$u_t^\epsilon + (-\Delta)^{\alpha/2} u^\epsilon - \nabla \cdot (u^\epsilon B_\epsilon(u^\epsilon)) = 0 \tag{33}$$

with $B_\epsilon = \delta_\epsilon * B$ defined by the kernel $b_\epsilon = \delta_\epsilon * b$.

(ii) *For each $\epsilon > 0$, there exists a constant C_ϵ such that for any $\phi \in \mathcal{S}(\mathbb{R}^d)$*

$$E\,|\langle Y^{n,\epsilon}(t) - u^\epsilon(t), \phi\rangle| \le C_\epsilon n^{1/\alpha - 1} \int_{\mathbb{R}^d} (1 + |\lambda|^a)|(\mathcal{F}\phi)(\lambda)|\,d\lambda.$$

(iii) *Under the assumptions guaranteeing the global in time existence of solutions of* (23), *the conclusions* (i), (ii) *are valid for all $t \in (0,\infty)$.*

A crucial tool in the proof of Theorem 7 were the estimates for α-stable stochastic Itô integrals from [23]. It is also possible to prove the "propagation of chaos in a wide sense" for equation (23), by which we mean that given any sequence of regularizations (33) with $\epsilon \to 0$, the family of empirical distributions $\{Y^{n,\epsilon}(t)\}$ contains a subsequence weakly convergent to a solution $u(t)$ of (23).

THEOREM 8. *Let the general conditions of Theorem 7 be satisfied. Assume that $u^\epsilon(t)$ are solutions of the regularized equation* (33) *such that their initial conditions satisfy $|u^\epsilon(0) - u(0)|_2 \to 0$ as $\epsilon \to 0$ for some $u(0) \in L^2(\mathbb{R}^d)$. Then given any sequence $\epsilon_k \to 0$ as $k \to \infty$, there exists a sequence $n_k \to \infty$ and a weak solution $u(t)$ of (23) such that for each $\phi \in C_0^\infty(\mathbb{R}^d)$*

$$E\,|\langle Y^{n_k,\epsilon_k}(t) - u(t), \phi\rangle| \to 0.$$

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