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ALMOST SURE CENTRAL LIMIT THEOREMS FOR WEAKLY DEPENDENT RANDOM VARIABLES

Dedicated to Professor Kazimierz Urbanik on the occasion of his 70th birthday

Abstract. We present almost sure central limit theorems for weakly dependent random variables. The presented theorems generalize the results obtained by Peligrad and Shao (1995).

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with continuous symmetric distribution function and let $S_n = X_1 + \ldots + X_n$, $n \geq 1$. Then, for every $n \geq 1$, $P(S_n > 0) = 1/2$, but

$$\liminf_{n\to\infty} n^{-1} \sum_{i=1}^{n} I(S_i > 0) = 0 \text{ a.s.}$$

and

$$\limsup_{n\to\infty} n^{-1} \sum_{i=1}^{n} I(S_i > 0) = 1 \text{ a.s.},$$

where I denotes the indicator function. However, by the result of Erdős and Hunt (1953), we have

$$\lim_{n\to\infty} (\log n)^{-1} \sum_{i=1}^{n} \frac{1}{i} I(S_i > 0) = 1/2 \text{ a.s.} \tag{1.1}$$

Thus, it is not true that the random walk $\{S_n,\ n \geq 1\}$ spends half of the time on the positive and half of the time on the negative axis in the sense that the asymptotic density, in Cesàro mean,

$$\mu(A) = \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} I(S_i > 0)$$

of the set $A = \{i : S_i > 0\}$ equals a.s. 1/2. But, by (1.1), the logarithmic density

$$\mu_L(A) = \lim_{n\to\infty} (\log n)^{-1} \sum_{i=1}^{n} \frac{1}{i} I(S_i > 0) \tag{1.2}$$

of the set $A = \{i : S_i > 0\}$ exists a.s. and equals 1/2.

Recently many authors have investigated time averages with respect to a logarithmic scale rather than space averages and prove a.s. convergence for the resulting random measures. The obtained results are extensions of classical weak limit theorems in a generalized formulation involving logarithmic averages and logarithmic density. As a starting point of these investigations Brosamler (1988) and Schatte (1988) independently proved that if $\{X_n, n \geq 1\}$ is a sequence of independent and identically distributed random variables with $EX_1 = 0$, $EX_1^2 = 1$ and $E|X_1|^{2+\delta} < \infty$ for some $\delta > 0$ ($\delta = 1$ in Schatte's paper), then

$$\lim_{n\to\infty} (\log n)^{-1} \sum_{i=1}^{n} \frac{1}{i} I(S_i \leq x\sqrt{i}) = \Phi(x) \text{ a.s. for any } x, \tag{1.3}$$

where Φ denotes the standard normal distribution function. Lacey and Philipp (1990) showed that (1.3) remains valid assuming only $EX_1 = 0$ and $EX_1^2 = 1$. Conversely, Berkes and Dehling (1994) proved that if $\{X_n, n \geq 1\}$ is a sequence of independent and identically distributed random variables such that (1.3) holds, then $EX_1 = 0$ and $EX_1^2 = 1$. The limiting relation (1.3) is called the pointwise or almost sure central limit theorem. Thus in the special case $a_n = 0$, $b_n = \sqrt{n}$, $n \geq 1$, the almost sure central limit theorem (1.3) is equivalent to the classical central limit theorem

$$S_n/b_n - a_n \Rightarrow N(0,1) \text{ as } n \to \infty, \tag{1.4}$$

where $N(0,1)$ is the standard normal distribution on $\mathbb{R}$ and the convergence $\Rightarrow$ is weak converegence of the measures on $\mathbb{R}$. For general sequences $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ the situation is different and more interesting, cf. Berkes (1995 and 1998), Berkes and Dehling (1994).

In this paper we present almost sure central limit theorems for weakly dependent random variables. A general result of this kind is presented in Theorem 1. This result extends Theorem 1 of Peligrad and Shao (1995) to sequences without assuming finite variances. The next result, Theorem 3, gives the almost sure central limit theorem for associated sequences of random variables. Theorem 3 generalize Theorem 2 of Peligrad and Shao

(1995) and related results of Matuła (1996). The main results presented in this paper also extend, to the case of weakly dependent random variables, the related almost sure central limit theorems for independent nonidentically distributed random variables given by Atlagh (1993), Rodzik and Rychlik (1994, 1996). In the proofs we shall also follow the ideas of Lacey and Phillip (1990), Peligrad and Shao (1995), Rodzik and Rychlik (1994, 1996).

2. Results

Denote by $BL = BL(\mathbb{R}, \|\cdot\|_{BL})$ the class of functions $f:\ \mathbb{R} \to \mathbb{R}$ with $\|f\|_{BL} = \|f\|_L + \|f\|_\infty < \infty$, where

$$\|f\|_L = \sup\{|f(x) - f(y)|/|x-y| :\ x, y \in \mathbb{R},\ x \neq y\}$$

and

$$\|f\|_\infty = \sup\{|f(x)| :\ x \in \mathbb{R}\}.$$

THEOREM 1. *Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables. Let $\{a_n,\ n \geq 1\}$ be a sequence of positive numbers such that*

$$B_n^2 \to \infty \quad \text{and} \quad B_{n-1}^2/B_n^2 \to 1 \text{ as } n \to \infty, \tag{2.1}$$

where $B_n^2 = a_1 + \ldots + a_n$, $n \geq 1$. If $S_n = X_1 + \ldots + X_n$, $n \geq 1$,

$$S_n/B_n \Rightarrow N(0,1) \text{ as } n \to \infty \tag{2.2}$$

and for every $f \in BL$ there exists $\varepsilon = \varepsilon(f) > 0$ such that

$$\operatorname{Var}\Big(\sum_{k=1}^{n}(a_k/B_k^2)f(S_k/B_k)\Big) = O((\log B_n)^{2-\varepsilon}), \quad \text{as } n \to \infty, \tag{2.3}$$

then

$$\lim_{n\to\infty}(\log B_n^2)^{-1}\sum_{k=1}^{n}(a_k/B_k^2)I(S_k \leq xB_k) = \Phi(x) \text{ a.s. for any } x. \tag{2.4}$$

Let us observe that if for $x \in \mathbb{R}$ we denote by $\delta(x)$ the probability measure on $\mathbb{R}$ which assigns its total mass to x, then (2.4) can be restated in this way. Let the random variables X_n, $n \geq 1$, be defined on a probability space $(\Omega, \mathcal{A}, P)$. Then there is a P–null set $N \subset \Omega$ such that for all $w \in N^c$

$$(\log B_n^2)^{-1}\sum_{k=1}^{n}(a_k/B_k^2)\delta(S_k(w)/B_k) \Rightarrow N(0,1), \text{ as } n \to \infty.$$

A sequence $\{X_n,\ n \geq 1\}$ of random variables is called associated if for any $n \geq 2$ and any coordinatewise increasing functions $f, g :\ \mathbb{R}^n \to \mathbb{R}$ we have

$$\operatorname{Cov}(f(X_1, \ldots, X_n), g(X_1, \ldots, X_n)) \geq 0, \tag{2.5}$$

whenever the left hand side of (2.5) exists.

In many respect, associated sequences of random variables behave like sequences of independent random variables, cf. Esary, Proschan, Walkup (1967), Dabrowski and Dehling (1988) and the references therein.

Let, for every $n \geq 1$, $1 \leq m \leq n-1$,

$$u_n(m) = \sup_{1 \leq k \leq n} \sum_{1 \leq j \leq n: |j-k| \geq m} \operatorname{Cov}(X_j, X_k) \tag{2.6}$$

and

$$u(m) = \sup_{k \geq 1} \sum_{j: |j-k| \geq m} \operatorname{Cov}(X_j, X_k). \tag{2.7}$$

The function $u(m)$ introduced Cox and Grimmett (1984). Of course, for every $n \geq 1$ and $1 \leq m \leq n-1$, we have $u_n(m) \leq u(m)$. On the other hand, if $\{X_n,\ n \geq 1\}$ is associated, then

$$u_n(m) \leq u_{n+1}(m) \quad \text{for every } 1 \leq m < n$$

and, for every $n \geq 1$,

$$u_n(m) \geq u_n(m+1) \quad \text{for every } 1 \leq m \leq n-2.$$

THEOREM 2. *Let f be a bounded function that has a bounded Radon-Nikodym derivative $h(x)$. Let $\{X_n,\ n \geq 1\}$ be a sequence of associated zero mean random variables with finite second moments. If $V_n^2 = ES_n^2 \to \infty$, $V_n^2/V_{n+1}^2 \to 1$ as $n \to \infty$, then*

$$\begin{aligned} &\operatorname{Var}\Big(\sum_{k=1}^{n} (a_k/V_k^2) f(S_k/V_k)\Big) \\ &\quad \leq C\Big\{(\|f\|_\infty^2 + \|h\|_\infty^2)(\log V_n^2) + \|h\|_\infty^2 \sum_{i=1}^{n-1} u_n(i) \sum_{k=i}^{n-1} (a_k/V_k^4)\Big\}, \end{aligned} \tag{2.8}$$

where $a_k = V_k^2 - V_{k-1}^2$, $k \geq 2$, $a_1 = EX_1^2$ and C is an absolute constant.

THEOREM 3. *Let $\{X_n,\ n \geq 1\}$ be a sequence of associated zero mean random variables with finite second moments. If $V_n^2 = ES_n^2 \to \infty$, $V_n^2/V_{n+1}^2 \to 1$ as $n \to \infty$,*

$$S_n/V_n \Rightarrow N(0,1) \text{ as } n \to \infty \tag{2.9}$$

and, for some $\varepsilon > 0$,

$$\sum_{i=1}^{n-1} u_n(i) \sum_{k=i}^{n-1} (a_k/V_k^4) = O((\log V_n^2)^{2-\varepsilon}), \tag{2.10}$$

where $a_k = V_k^2 - V_{k-1}^2$, $k \geq 2$, $a_1 = EX_1^2$, *then*

$$\lim_{n\to\infty} (\log V_n^2)^{-1} \sum_{k=1}^{n} (a_k/V_k^2) I(S_k \leq xV_k) = \Phi(x) \text{ a.s. for any } x. \tag{2.11}$$

Let us observe that

$$\sum_{i=1}^{n-1} u_n(i) \sum_{k=i}^{n-1} (a_k/V_k^4) \leq \sum_{i=1}^{n-1} (u_n(i)/V_i^2) \sum_{k=1}^{n-1} (a_k/V_k^2).$$

Furthermore, we have (cf. (3.5) in the proof of Theorem 1)

$$\lim_{n\to\infty} (\log V_n^2)^{-1} \sum_{k=1}^{n} (a_k/V_k^2) = 1.$$

Thus if

$$\limsup_{n\to\infty} \sum_{k=1}^{n} (u_n(k)/V_k^2) < \infty, \tag{2.12}$$

then (2.10) holds with $\varepsilon = 1$.

3. Proofs

Proof of Theorem 1. We first note that (2.4) is equivalent to the following statement:

For each $f \in BL$

$$\lim_{n\to\infty} (\log B_n^2)^{-1} \sum_{k=1}^{n} (a_k/B_k^2) f(S_k/B_k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-x^2/2) dx \text{ a.s.} \tag{3.1}$$

This fact follows from Theorem 7.1 of Billingsley (1968), Theorem 11.3.3 (b⇒c) of Dudley (1989) and Section 2 of Lacey and Philipp (1990).

Let $f \in BL$ be a bounded Lipschitz function on $\mathbb{R}$. Let us put $f^+ = \max(0, f)$ and $f^- = \max(-f, 0)$. Clearly f^+ and f^- are nonnegative bounded Lipschitz functions on $\mathbb{R}$. By (2.2) and Theorem 2.1 of Billingsley (1968)

$$\lim_{n\to\infty} Ef^+(S_n/B_n) = Ef^+(N(0,1)), \tag{3.2}$$

where, here and in what follows,

$$Ef(N(0,1)) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x) \exp(-x^2/2) dx.$$

On the other hand, by (2.1), we have

$$\sum_{k=1}^{n}(a_k/B_k^2) \le 1 + \sum_{k=2}^{n} \int\limits_{B_{k-1}^2/B_n^2}^{B_k^2/B_n^2} x^{-1}dx \tag{3.3}$$
$$= 1 + \int\limits_{a_1/B_n^2}^{1} x^{-1}dx = 1 - \ln a_1 + \ln B_n^2$$

and for every $\varepsilon > 0$ there exists n_0 so that for every $n \ge n_0$

$$\sum_{k=1}^{n}(a_k/B_k^2) = 1 + \sum_{k=2}^{n}(B_{k-1}^2/B_k^2)(a_k/B_{k-1}^2) \tag{3.4}$$
$$\ge 1 + \sum_{k=2}^{n}(B_{k-1}^2/B_k^2) \int\limits_{B_{k-1}^2/B_n^2}^{B_k^2/B_n^2} x^{-1}dx$$
$$\ge \sum_{k=1}^{n_0}(a_k/B_k^2) + (1-\varepsilon)(\ln B_n^2 - \ln B_{n_0}^2).$$

Thus, by (3.3) and (3.4), we get

$$\lim_{n\to\infty}(\log B_n^2)^{-1}\sum_{k=1}^{n}(a_k/B_k^2) = 1. \tag{3.5}$$

By Toeplitz Lemma 7.1.2 of Ash (1972) and statements (3.2) and (3.5), proved above, we have

$$\lim_{n\to\infty}(\log B_n^2)^{-1}\sum_{k=1}^{n}(a_k/B_k^2)Ef^+(S_k/B_k) = Ef^+(N(0,1)). \tag{3.6}$$

Define an increasing sequence of integers $\{N_k,\ k \ge 1\}$ by

$$\exp(k^{2/\varepsilon}) \le B_{N_k}^2 < \exp((k+1)^{2/\varepsilon}),$$

where $\varepsilon = \varepsilon(f^+)$ is as in (2.3). Thus, by Chebyshev's inequality and (2.3), for every $\delta > 0$ we get

$$P\Big((\log B_{N_k}^2)^{-1}\Big|\sum_{i=1}^{N_k}(a_i/B_i^2)[f^+(S_i/B_i) - Ef^+(S_i/B_i)]\Big| \ge \delta\Big)$$
$$\le \delta^{-2}(\log B_{N_k}^2)^{-2}\,\mathrm{Var}\Big(\sum_{i=1}^{N_k}(a_i/B_i^2)f^+(S_i/B_i)\Big)$$
$$= O((\log B_{N_k}^2)^{-\varepsilon}) = O(k^{-2}) \text{ as } k \to \infty.$$

Hence, by Borel–Cantelli Lemma, we have

$$\lim_{k\to\infty}(\log B_{N_k}^2)^{-1}\sum_{i=1}^{N_k}(a_i/B_i^2)[f^+(S_i/B_i)-Ef^+(S_i/B_i)]=0 \text{ a.s.} \tag{3.7}$$

Now (3.6) and (3.7) yield

$$\lim_{k\to\infty}(\log B_{N_k}^2)^{-1}\sum_{i=1}^{N_k}(a_i/B_i^2)f^+(S_i/B_i)=Ef^+(N(0,1)) \text{ a.s.} \tag{3.8}$$

On the other hand, for $N_k < n < N_{k+1}$, we have

$$\begin{aligned}(\log B_{N_{k+1}}^2)^{-1}&\sum_{i=1}^{N_k}(a_i/B_i^2)f^+(S_i/B_i)\\ &\le(\log B_n^2)^{-1}\sum_{i=1}^{n}(a_i/B_i^2)f^+(S_i/B_i)\\ &\le(\log B_{N_k}^2)^{-1}\sum_{i=1}^{N_{k+1}}(a_i/B_i^2)f^+(S_i/B_i)\end{aligned} \tag{3.9}$$

and

$$\lim_{k\to\infty}(\log B_{N_k}^2)/(\log B_{N_{k+1}}^2)=1. \tag{3.10}$$

Consequently, by (3.8), (3.9) and (3.10), we get

$$\lim_{n\to\infty}(\log B_n^2)^{-1}\sum_{i=1}^{n}(a_i/B_i^2)f^+(S_i/B_i)=Ef^+(N(0,1)) \text{ a.s.} \tag{3.11}$$

The same proof works for f^-, so that we also have

$$\lim_{n\to\infty}(\log B_n^2)^{-1}\sum_{i=1}^{n}(a_i/B_i^2)f^-(S_i/B_i)=Ef^-(N(0,1)) \text{ a.s.} \tag{3.12}$$

Combining (3.11) with (3.12) we can assert that, for every $f\in BL$, (3.1) holds. Thus the proof of (2.4) is ended.

P r o o f of Theorem 2. We have

$$\begin{aligned}\operatorname{Var}&\Big(\sum_{i=1}^{n}(a_i/V_i^2)f(S_i/V_i)\Big)\\ &=\sum_{i=1}^{n}(a_i/V_i^2)^2\operatorname{Var}(f(S_i/V_i))\\ &\quad+2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(a_ia_j/V_i^2V_j^2)\operatorname{Cov}(f(S_i/V_i),f(Sj/V_j))\\ &=I_1(n)+2I_2(n).\end{aligned} \tag{3.13}$$

Clearly, $0 < a_i/V_i^2 \leq 1$, $i \geq 1$, so that (3.5) yields

$$(3.14) \quad I_1(n) \leq \|f\|_\infty^2 \sum_{i=1}^{n} (a_i/V_i^2)^2 \leq \|f\|_\infty^2 \sum_{i=1}^{n} (a_i/V_i^2) \leq C\|f\|_\infty^2 \log V_n^2,$$

where C is an absolute constatnt.

On the other hand, since $\{X_n,\ n \geq 1\}$ is a sequence of associated random variables, $\{S_n,\ n \geq 1\}$ is a sequence of associated random variables, too. Therefore the function

$$H(x,y) = P(S_i > xV_i, S_j > yV_j) - P(S_i > xV_i)P(S_j > yV_j)$$

is non–negative for every $x, y \in \mathbb{R}$, cf. Esary, Proschan and Walkup (1967). Thus, by Hoeffding–Lehman's types arguments, cf. Lehmann (1966), Peligrad and Shao (1995), we have

$$\begin{aligned} \operatorname{Cov}(f(S_i/V_i); f(S_j/V_j)) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x)h(y)H(x,y)dxdy \\ &\leq \|h\|_\infty^2 \operatorname{Cov}(S_i/V_i; S_j/V_j). \end{aligned}$$

This implies

$$\begin{aligned} (3.15) \quad I_2(n) &\leq \|h\|_\infty^2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (a_i a_j/V_i^2 V_j^2) \operatorname{Cov}(S_i/V_i; S_j/V_j) \\ &= \|h\|_\infty^2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (a_i a_j/V_i^3 V_j^3) [\operatorname{Cov}(S_i; S_i) + \operatorname{Cov}(S_i; S_j - S_i)] \\ &= \|h\|_\infty^2 \sum_{i=1}^{n-1} (a_i/V_i) \sum_{j=i+1}^{n} (a_j/V_j^3) \\ &\quad + \|h\|_\infty^2 \sum_{i=1}^{n-1} (a_i/V_i^3) \sum_{j=i+1}^{n} (a_j/V_j^3) \operatorname{Cov}(S_i, S_j - S_i). \end{aligned}$$

Moreover,

$$\begin{aligned} (3.16) \quad \sum_{j=i+1}^{n} (a_j/V_j^3) &\leq \sum_{j=i+1}^{n} V_n^{-1} \int_{V_{j-1}^2/V_n^2}^{V_j^2/V_n^2} x^{-3/2} dx \\ &= V_n^{-1} \int_{V_i^2/V_n^2}^{1} x^{-3/2} dx = 2(V_n - V_i)/(V_i V_n). \end{aligned}$$

Thus, by (3.5), we have

$$(3.17)\qquad \sum_{i=1}^{n-1}(a_i/V_i)\sum_{j=i+1}^{n}(a_j/V_j^3)\le 2\sum_{i=1}^{n}(a_i/V_i^2)(V_n-V_i)/V_n$$
$$\le 2\sum_{i=1}^{n}(a_i/V_i^2)\le C\log V_n^2.$$

On the other hand, since $\{X_n,\ n\ge 1\}$ is a sequence of associated random variables, therefore $\mathrm{Cov}(X_i,X_j)\ge 0$ for every $i,j\ge 1$. This proves that for every $i+1\le j\le n$

$$(3.18)\qquad \mathrm{Cov}(S_i,S_j-S_i)\le\sum_{k=1}^{i}EX_k(S_n-S_i)=\sum_{k=1}^{i}\sum_{j=i+1}^{n}EX_kX_j$$
$$\le\sum_{i=1}^{i}u_n(i+1-k)=\sum_{k=1}^{i}u_n(k).$$

Hence, by (3.18) and (3.16), we get

$$(3.19)\qquad \sum_{i=1}^{n-1}(a_i/V_i^3)\sum_{j=i+1}^{n}(a_j/V_j^3)\,\mathrm{Cov}(S_i;S_j-S_i)$$
$$\le\sum_{i=1}^{n-1}(a_i/V_i^3)\sum_{k=1}^{i}u_n(k)\sum_{j=i+1}^{n}(a_j/V_j^3)$$
$$\le 2\sum_{i=1}^{n-1}(a_i/V_i^4)(V_n-V_i)V_n^{-1}\sum_{k=1}^{i}u_n(k)\le 2\sum_{k=1}^{n-1}u_n(k)\sum_{i=k}^{n-1}(a_i/V_i^4).$$

Consequently, by (3.13), (3.14), (3.15), (3.17) and (3.19) we get (2.8), as desired.

Proof of Theorem 3. By Theorem 2 and (2.10) we get (2.3) with some $\varepsilon>0$. Of course (2.3) follows from (2.8) and (2.10). Thus Theorem 3 is now consequence of Theorem 1, so we omit details.

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