DEMONSTRATIO MATHEMATICA
Vol. XXXIV No 2 2001

Agnieszka Plucińska, Monika Bisińska

POLYNOMIAL-GAUSSIAN VECTORS AND POLYNOMIAL-GAUSSIAN PROCESSES

Dedicated to Professor Kazimierz Urbanik

Abstract. The density of a d-dimensional polynomial-Gaussian distribution (PGD_d) is the product of a non-negative polynomial and a Gaussian density. The density of a (PGD_d) has many properties similar to a d-dimensional Gaussian distribution (GD_d), but one-dimensional marginal distributions of (PGD_d) are (PGD_1). Analogously one-dimensional densities of a polynomial-Gaussian process (PGP) are (PGD_1). We investigate the differences and similarities between the Gaussian and non-Gaussian cases.

1. Introduction

Let $\mathbf{X}_d = (X_1, \ldots, X_d)$ denote a d-dimensional random variable with a non-degenerate distribution. We suppose that X_1 has a polynomial-Gaussian distribution (PGD_1), i.e. the density of X_1 is the product of a non-negative polynomial and a Gaussian density; see Evans and Swartz (1994).

Thus the density of X_1 has the following form

$$f(x) = \frac{1}{\alpha\sqrt{2\pi}} p_{2l}(x) \exp\left(-\frac{x^2}{2\alpha^2}\right), \tag{1.1}$$

where $\alpha > 0$ and $p_{2l}(x)$ is a non-negative polynomial of degree $2l$.

We will construct a d-dimensional polynomial-Gaussian distribution (PGD_d) in such a way that various properties of Gaussian vectors are preserved.

Let the density of $\mathbf{X}_d$ be the product of a non-negative polynomial in x_1 and a d-dimensional Gaussian density

$$f(\mathbf{x}_d) = \frac{\sqrt{\det A}}{(2\pi)^{\frac{d}{2}}} p_{2l}(x_1) \exp\left\{-\frac{1}{2}\left(A\mathbf{x}_d, \mathbf{x}_d\right)\right\} = p_{2l}(x_1)\widetilde{f}(\mathbf{x}_d), \tag{1.2}$$

where $\mathcal{K} = [k_{rs}]_{r,s=1}^{d}$ is a symmetric, positive definite $d \times d$ matrix, $A = \mathcal{K}^{-1}$, $p_{2l}(x_1)$ is a non-negative polynomial in x_1, $\mathbf{x}_d = (x_1, \ldots, x_d) \in R^d$, and $\tilde{f}(\mathbf{x}_d)$ denotes a d-dimensional Gaussian density.

In Section 2 we find the characteristic function of $\mathbf{X}_d$ and the moments of the first and second order of $\mathbf{X}_d$. We show that all the one-dimensional marginal distributions of (PGD_d) are (PGD_1). We show that there exist linear transformations

$$\begin{aligned} Y_1 &= X_1, \\ Y_2 &= X_2 + a_{12}X_1, \\ &\dots \\ Y_d &= X_d + a_{1d}X_1 + \ldots + a_{d-1,d}X_{d-1}, \end{aligned}$$

such that $Y_1, Y_2, \ldots, Y_d$ are independent. We show that for (PGD_d) vectors the random variables $X_1, X_2, \ldots, X_d$ are independent iff they are uncorrelated.

Section 3 is devoted to the properties of sums of independent (PGD_d) vectors.

In Section 4 we give a characterization of (PGD_d) .

In Section 5 we give some necessary and sufficient conditions for characteristic functions.

In Section 6 we construct a stochastic process such that the one-dimensional distributions of this process are (PGD_1).

There are various generalizations of Gaussian distributions and Gaussian processes, see for example Johnson and Kotz' monograph (1972). The general idea of these generalizations is to introduce new forms, but, on the other hand, to preserve as far as possible the properties which hold in the Gaussian case such as the properties of linear transformations, the properties of marginal distributions, conditional distributions, the equivalence between the independence of random variables and the vanishing of the correlation coefficients.

The distribution given by (1.2) belongs to the class named "conditionally Gaussian", considered for example by Liptser and Shiryaev (1978). Some properties of (PGD_1) were considered by Plucińska (1999).

The characteristic function corresponding to (1.2) is given by (2.5) and is the product of a polynomial and an exponential function. Characteristic functions of such a form were considered by Lukacs (1970), who gave a sufficient condition for the product of a polynomial and an exponential function to be a characteristic function. In Section 5 we give a necessary and sufficient condition.

2. Properties of (PGD_d) vectors

We will consider some special form of the polynomial $p_{2l}(x)$. It is known that every polynomial can be represented as a linear combination of Hermite polynomials. Thus we will put

$$p_{2l}(x_1) = \sum_{r=0}^{2l} \frac{c_r}{\alpha^{r+1}} H_r\left(\frac{x_1}{\alpha}\right) = \sum_{r=0}^{2l} \frac{c_r}{k_{11}^{\frac{r}{2}}} H_r\left(\frac{x_1}{\sqrt{k_{11}}}\right). \tag{2.1}$$

Formula (2.1) indicates some assumptions we make on the coefficients of the polynomial p_{2l}. Of course the coefficients c_r are such that $p_{2l} \geq 0$ and $\int_{R^d} f(\mathbf{x}_d) d\mathbf{x}_d = 1$. Throughout the paper we will consider the density (1.2), where the polynomial p_{2l} is given by (2.1). More exactly the density of $\mathbf{X}_d$ will be

$$f(\mathbf{x}_d) = \frac{\sqrt{\det A}}{(2\pi)^{\frac{d}{2}}} \sum_{r=0}^{2l} \frac{c_r}{k_{11}^{\frac{r}{2}}} H_r\left(\frac{x_1}{\sqrt{k_{11}}}\right) \exp\left\{-\frac{1}{2}(A\mathbf{x}_d, \mathbf{x}_d)\right\}. \tag{2.2}$$

Let us denote the cofactor of k_{rs} in the matrix $\mathcal{K}^{(n)} = [k_{rs}]_{r,s=1}^{n}$ by $\mathcal{K}_{rs}^{(n)}$ and let

$$a_{rn} = \frac{1}{\mathcal{K}_{nn}^{(n)}} \mathcal{K}_{rn}^{(n)}, \ \mu_n = \sum_{r=1}^{n-1} a_{rn} X_r, \ \sigma_n^2 = \frac{\det \mathcal{K}^{(n)}}{\det \mathcal{K}^{(n-1)}}, \ n \leq d.$$

We are going to show that there exist independent linear forms of X_1, $X_2, \ldots, X_d$.

PROPOSITION 1. *Let the density of $\mathbf{X}_d$ be given by (2.2). Then the random variables*

$$\begin{cases} Y_1 = X_1, \\ Y_2 = X_2 + \mu_2 = X_2 + a_{12}X_1, \\ \ldots \\ Y_d = X_d + \mu_d = X_d + a_{1d}X_1 + \ldots + a_{d-1,d}X_{d-1} \end{cases} \tag{2.3}$$

are independent and every Y_r $(r \geq 2)$ has a Gaussian distribution with parameters $EY_r = 0$, $EY_r^2 = \sigma_r^2$. Moreover the conditional distribution of $X_r \,|X_1, \ldots, X_{r-1}$ where $r \geq 2$ is Gaussian with parameters

$$E(X_r \,|X_1, \ldots, X_{r-1}) = -\mu_r,$$
$$Var(X_r \,|X_1, \ldots, X_{r-1}) = \sigma_r^2.$$

Proof. It follows from the properties of Gaussian distributions that the function (2.2) can be written in the following form

(2.4) $$f(\mathbf{x}_d) = \frac{\sqrt{\det A}}{(2\pi)^{\frac{d}{2}}} p_{2l}(x_1) \exp\left\{-\frac{x_1^2}{2\sigma_1^2} - \frac{(x_2+\mu_2)^2}{2\sigma_2^2} - \ldots - \frac{(x_d+\mu_d)^2}{2\sigma_d^2}\right\}.$$

The function (2.4) is the product of d functions: the first depends only on y_1, the second on $y_2, \ldots$, the last one depends only on y_d, where

$$y_1 = x_1,$$
$$y_2 = x_2 + \mu_2,$$
$$\ldots$$
$$y_d = x_d + \mu_d.$$

Thus the random variables $Y_1, Y_2, \ldots, Y_d$ are independent and every Y_r $(r \geq 2)$ has a Gaussian distribution. The last statement of proposition 1 follows immediately from (2.4). ∎

PROPOSITION 2. *Let the density of $\mathbf{X}_d$ be given by (2.2). Then the characteristic function of $\mathbf{X}_d$ has the following form*

(2.5) $$\varphi(\boldsymbol{\xi}_d) = E\exp\left[i(\boldsymbol{\xi}_d, \mathbf{X}_d)\right] = \sum_{r=0}^{2l} c_r (i\eta)^r \exp\left[-\frac{1}{2}\sum_{r,s=1}^{d} k_{rs}\xi_r\xi_s\right] = \sum_{r=0}^{2l} c_r (i\eta)^r \exp\left[-\frac{1}{2}(\mathcal{K}\boldsymbol{\xi}_d, \boldsymbol{\xi}_d)\right] = \Psi_{2l}(\eta)\widetilde{\varphi}(\boldsymbol{\xi}_d),$$

where $\eta = \xi_1 + \frac{k_{12}}{k_{11}}\xi_2 + \ldots + \frac{k_{1d}}{k_{11}}\xi_d = \frac{1}{k_{11}}\left[\xi_1 k_{11} + \ldots + \xi_d k_{1d}\right]$. $\Psi_{2l}(\eta)$ *is a polynomial of degree 2l and* $\widetilde{\varphi}(\boldsymbol{\xi}_d)$ *denotes the characteristic function of a Gaussian distribution.*

Proof. It is evident that the characteristic function $\varphi(\boldsymbol{\xi}_d)$ is a product of a polynomial and an exponential function. We must only show that the parameters of the characteristic function have the form stated in (2.5). If the density is given by (2.2), then for d=1 , the characteristic function (see Plucińska 2001) has the form

(2.5′) $$\varphi(\xi_1) = \sum_{r=0}^{2l} c_r (i\xi_1)^r \exp\left[-\frac{1}{2}k_{11}\xi_1^2\right].$$

We shall use the properties of characteristic functions for linear transformations. In order to calculate

$$\varphi(\boldsymbol{\xi}_d) = E\left[\exp\left(i(\mathbf{X}_d, \boldsymbol{\xi}_d)\right)\right] = \int_{R^d} \exp\left[i(\mathbf{x}_d, \boldsymbol{\xi}_d)\right] f(\mathbf{x}_d) d\mathbf{x}_d$$

we solve (2.3) with respect to $X_1, X_2, \ldots, X_d$. This system is a Cramer system and thus there exists a unique solution

$$X_r = \sum_{j=1}^{r} b_{jr} Y_j \ , \ r = 1, \ldots, d, \tag{2.3'}$$

where b_{jr} can be found by the Cramer formulas; evidently $b_{rr} = 1$ ($r = 1, \ldots, d$). It follows from (2.4) that $Y_1, Y_2, \ldots, Y_d$ are independent, Y_1 has (PGD_1) and Y_r ($r > 1$) have Gaussian distributions. Therefore the characteristic function φ can be written in the form

$$\begin{aligned} \varphi(\boldsymbol{\xi}_d) &= E\Big\{ \exp\Big[i\Big(\xi_1 Y_1 + \xi_2 \sum_{j=1}^{2} b_{j2} Y_j + \ldots + \xi_d \sum_{j=1}^{d} b_{jd} Y_j\Big)\Big]\Big\} \\ &= E\left\{\exp\left[iY_1\left(b_{11}\xi_1 + b_{12}\xi_2 + \ldots + b_{1d}\xi_d\right)\right]\right\} \\ &\quad \times E\left\{\exp\left[iY_2\left(b_{22}\xi_2 + \ldots + b_{2d}\xi_d\right)\right]\right\} \cdot \ldots \cdot E\left\{\exp\left[iY_d b_{dd}\xi_d\right]\right\}. \end{aligned} \tag{2.6}$$

The first factor on the right-hand side in (2.6) is the characteristic function of (PGD_1), and each remaining factor is the characteristic function of a Gaussian distribution.

Now we are going to find the coefficients b_{1r} ($r > 1$). By the Cramer formula

$$X_r = \begin{vmatrix} 1 & 0 & 0 & \ldots & 0 & Y_1 \\ a_{12} & 1 & 0 & \ldots & 0 & Y_2 \\ a_{13} & a_{23} & 1 & \ldots & 0 & Y_3 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{1r} & a_{2r} & a_{3r} & \ldots & a_{r-1,r} & Y_r \end{vmatrix}.$$

Thus

$$b_{1r} = (-1)^{r-1} \begin{vmatrix} a_{12} & 1 & 0 & \ldots & 0 \\ a_{13} & a_{23} & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{1r} & a_{2r} & a_{3r} & \ldots & a_{r-1,r} \end{vmatrix}. \tag{2.7}$$

By induction we now prove that

$$b_{1r} = \frac{k_{1r}}{k_{11}}, \ r = 1, \ldots, d. \tag{2.8}$$

In view of (2.7),

$$b_{12} = -a_{12} = \frac{k_{12}}{k_{11}}.$$

Suppose that (2.8) holds for $r \le n$. We are going to prove that

$$b_{1,n+1} = \frac{k_{1,n+1}}{k_{11}}.$$

By (2.7) and the induction assumption we have

$$\begin{aligned} b_{1,n+1} &= (-1)^n\{a_{n,n+1}(-1)^{n-1}b_{1n} - a_{n-1,n+1}(-1)^{n-2}b_{1,n-1} \\ &\quad + \ldots + (-1)^{n-1}a_{1,n+1}\} \\ &= (-1)^{2n-1}\{a_{n,n+1}\frac{k_{1n}}{k_{11}} + \ldots + a_{1,n+1}\} \\ &= \frac{-1}{\det \mathcal{K}^{(n)}k_{11}}[k_{1n}\mathcal{K}^{(n+1)}_{n,n+1} + \ldots + k_{11}\mathcal{K}^{(n+1)}_{1,n+1} \\ &\quad + k_{1,n+1}\mathcal{K}^{(n+1)}_{n+1,n+1} - k_{1,n+1}\mathcal{K}^{(n+1)}_{n+1,n+1}] \\ &= \frac{k_{1,n+1}}{k_{11}}. \end{aligned}$$

Formula (2.8) follows from (2.5'), (2.6) and (2.7). ■

PROPOSITION 3. *Every δ-dimensional ($\delta < d$) marginal distribution of (PGD_d) is (PGD_δ). The characteristic function of the one-dimensional marginal distribution for every $s \le d$ has the form*

$$\varphi(\xi_s) = E\exp(i\xi_s X_s) = \sum_{r=0}^{2l} c_r\left(\frac{k_{1s}}{k_{11}}i\xi_s\right)^r \exp\left[-\frac{1}{2}k_{ss}\xi_s^2\right]. \tag{2.9}$$

P r o o f. Proposition 3 follows immediately from proposition 2. In particular case we get (2.9) by putting: $\xi_1 = \ldots = \xi_{s-1} = \xi_{s+1} = \xi_d = 0$ in (2.5). ■

PROPOSITION 4. *The moments of the first and second order of $\mathbf{X}_d$ are given by*

$$m_r = EX_r = c_1\frac{k_{1r}}{k_{11}}, \tag{2.10}$$

$$q_{rs} = EX_rX_s = k_{rs} + 2c_2\frac{k_{1s}k_{1r}}{k_{11}^2} = k_{rs} + (q_{11} - k_{11})\frac{k_{1s}k_{1r}}{k_{11}^2}. \tag{2.11}$$

Moreover,

$$k_{rs} = q_{rs} - 2c_2\frac{q_{1s}q_{1r}}{q_{11}^2} = q_{rs} - (q_{11} - k_{11})\frac{q_{1s}q_{1r}}{q_{11}^2}. \tag{2.11'}$$

P r o o f. After some simple calculations we find that the derivatives of the function φ given by (2.5) for $\boldsymbol{\xi}_d = \mathbf{0}$ have the following forms

$$\frac{\partial}{\partial \xi_r}\varphi(\xi_1, .., \xi_d)\,|_{\xi_1=\ldots=\xi_d=0} = ic_1\frac{k_{1r}}{k_{11}}, \tag{2.12}$$

$$\frac{\partial^2}{\partial \xi_r \partial \xi_s}\varphi(\xi_1, \ldots, \xi_d)\,|_{\xi_1=\ldots=\xi_d=0} = i^2\left(k_{rs} + 2c_2\frac{k_{1r}k_{1s}}{k_{11}^2}\right). \tag{2.13}$$

Formula (2.10) is thus proved. Moreover by (2.13) for r=s=1 we have $q_{11} = k_{11} + 2c_2$. Thus (2.11) follows from (2.13). And conversely solving equation (2.11) we get (2.11'). ∎

Now we are going to show that the coefficients a_{rs} of the linear forms given by (2.3) can be expressed by the moments of $\mathbf{X}_d$. Let $Q^{(n)} = [q_{rs}]_{r,s=1}^{n}$, $n \le d$ be the matrix of the second order moments of $\mathbf{X}_d$. Let $Q_{rs}^{(n)}$ be the cofactor of q_{rs} in the matrix $Q^{(n)}, n \le d$.

PROPOSITION 5. *Let the density of* $\mathbf{X}_d$ *be given by* (2.2). *Then*

$$\det Q^{(d)} = \left(1 + \frac{2c_2}{k_{11}}\right) \det \mathcal{K}^{(d)} \tag{2.14}$$

and moreover the coefficients a_{rs} *of the linear forms given by* (2.3) *are*

$$a_{rs} = \frac{1}{\mathcal{K}_{ss}^{(s)}} \mathcal{K}_{rs}^{(s)} = \frac{1}{Q_{ss}^{(s)}} Q_{rs}^{(s)}. \tag{2.15}$$

Proof. The matrix Q has the following form:

$$Q^{(d)} = \begin{bmatrix} k_{11} + 2c_2 & k_{12} + 2c_2\frac{k_{12}}{k_{11}} & \cdots & k_{1d} + 2c_2\frac{k_{1d}}{k_{11}} \\ k_{12} + 2c_2\frac{k_{12}}{k_{11}} & k_{22} + 2c_2\frac{k_{12}^2}{k_{11}^2} & \cdots & k_{2d} + 2c_2\frac{k_{1d}k_{12}}{k_{11}^2} \\ \cdots & \cdots & \cdots & \cdots \\ k_{1d} + 2c_2\frac{k_{1d}}{k_{11}} & k_{2d} + 2c_2\frac{k_{1d}k_{12}}{k_{11}^2} & \cdots & k_{dd} + 2c_2\frac{k_{1d}^2}{k_{11}^2} \end{bmatrix} =$$

$$= \left(1 + \frac{2c_2}{k_{11}}\right) \begin{bmatrix} k_{11} & k_{12} + 2c_2\frac{k_{12}}{k_{11}} & \cdots & k_{1d} + 2c_2\frac{k_{1d}}{k_{11}} \\ k_{12} & k_{22} + 2c_2\frac{k_{12}^2}{k_{11}^2} & \cdots & k_{2d} + 2c_2\frac{k_{1d}k_{12}}{k_{11}^2} \\ \cdots & \cdots & \cdots & \cdots \\ k_{1d} & k_{2d} + 2c_2\frac{k_{1d}k_{12}}{k_{11}^2} & \cdots & k_{dd} + 2c_2\frac{k_{1d}^2}{k_{11}^2} \end{bmatrix}.$$

The elements of the r-th column form a vector which can be written as the sum of two vectors

$$\left(k_{1r} + 2c_2\frac{k_{1r}}{k_{11}}, k_{2r} + 2c_2\frac{k_{1r}k_{12}}{k_{11}^2}, \ldots, k_{rd} + 2c_2\frac{k_{1d}k_{1r}}{k_{11}^2}\right)$$

$$= (k_{1r}, \ldots, k_{rd}) + 2c_2\frac{k_{1r}}{k_{11}^2}(k_{11}, \ldots, k_{1d}).$$

The second term is proportional to the vector in the first column. Thus formula (2.14) is proved.

The cofactors of $Q^{(d)}$ can be computed analogously to $Q^{(d)}$. Thus

$$Q_{rs}^{(s)} = \left(1 + \frac{2c_2}{k_{11}}\right) \mathcal{K}_{rs}^{(s)}.$$

Therefore (15) is proved. ∎

PROPOSITION 6. *Let the density of $\mathbf{X}_d$ be (2.2). The random variables $X_1, X_2, \ldots, X_d$ are independent iff they are uncorrelated.*

Proof. We need only show that if $X_1, X_2, \ldots, X_d$ are uncorrelated then they are independent. It is evident that

$$cov\,(X_1 X_r) = k_{1r} + \left(2c_2 - c_1^2\right)\frac{k_{1r}}{k_{11}} = \frac{k_{1r}}{k_{11}}\left(k_{11} + 2c_2 - c_1^2\right). \tag{2.16}$$

The r.v. X_1 is non-degenerate when

$$Var X_1 = k_{11} + 2c_2 - c_1^2 > 0. \tag{2.17}$$

We suppose that $cov\,(X_1 X_r) = 0$ for $r \neq 1$. Then by (2.16) and (2.17) we get

$$k_{1r} = 0 \; for \; r \neq 1. \tag{2.18}$$

It is easy to compute that

$$cov\,(X_s X_r) = k_{sr} + \left(2c_2 - c_1^2\right)\frac{k_{1r}k_{1s}}{k_{11}^2}.$$

The condition $cov\,(X_s X_r) = 0$ and (2.18) imply that

$$k_{sr} = 0 \; for \; r \neq s. \tag{2.19}$$

By (2.19) we have

$$a_{rs} = 0 \; for \; r \neq s. \tag{2.20}$$

Condition (2.20) implies that $X_1, X_2, \ldots, X_d$ are independent. ∎

3. Properties of sums

We consider the characteristic function of the form (2.5).

Let $z_1, z_2, \ldots, z_{2l}$ be all the complex zeros of the polynomial $\Psi_{2l}(z)$. We set $\mathcal{Z}_{2l} = (z_1, z_2, \ldots, z_{2l})$. The characteristic function (2.5) is determined by the matrix $\mathcal{K}$ and the parameters $c_1, \ldots, c_{2l}$. It is evident that the knowledge of the parameters $c_1, \ldots, c_{2l}$ is equivalent to the knowledge of the set of zeros $z_1, \ldots, z_{2l}$ and to the knowledge of EX_1^r $(r \leq 2l)$. Sometimes it is more convenient to use the parameters $c_1, \ldots, c_{2l}$, sometimes $z_1, \ldots, z_{2l}$ and sometimes EX_1^r $(r \leq 2l)$. We use the abbreviated denotation

$$PGD_d\,(2l, \mathcal{K}, \mathcal{Z}_{2l})$$

in order to show the dependence on parameters.

PROPOSITION 7. *Let $\mathbf{X}_d^{(n)} = (X_1^{(n)}, \ldots, X_d^{(n)})$, $n = 1, 2$, be two independent vectors and suppose $\mathbf{X}_d^{(n)}$, $n = 1, 2$, has the distribution $PGD_d(2l_n, \mathcal{K}_n, \mathcal{Z}_{2l_n})$, where $\mathcal{K}_n = [k_{rs}^{(n)}]_{r,s=1}^d$ and the set of zeros is $\mathcal{Z}_{2l_n} = (z_1^{(n)}, \ldots, z_{2l_n}^{(n)})$. Let moreover*

$$k_{1r}^{(n)} = k_{11}^{(n)} \; for \; r \leq d, \quad n = 1, 2. \tag{3.1}$$

Then $\mathbf{X}_d^{(1)} + \mathbf{X}_d^{(2)}$ *has the distribution*

$$PGD_d\left(2l_1 + 2l_2, \mathcal{K}_1 + \mathcal{K}_2, \mathcal{Z}_{2l_1} \cup \mathcal{Z}_{2l_2}\right). \tag{3.2}$$

Proof. The product of the characteristic functions of $\mathbf{X}_d^{(s)}$ has the form

$$\varphi_1(\boldsymbol{\xi}_d)\,\varphi_2(\boldsymbol{\xi}_d) = \prod_{r=1}^{2l_1}\left(1 - \frac{\eta}{z_r^{(1)}}\right)\prod_{s=1}^{2l_2}\left(1 - \frac{\eta}{z_s^{(2)}}\right)\cdot \exp\left[-\frac{1}{2}\left(\mathcal{K}_1\boldsymbol{\xi}_d, \boldsymbol{\xi}_d\right) - \frac{1}{2}\left(\mathcal{K}_2\boldsymbol{\xi}_d, \boldsymbol{\xi}_d\right)\right], \tag{3.3}$$

where $\eta = \sum_{i=1}^{d} \xi_i$ and the set of zeros of (3.3) can be written in the form

$$\mathcal{Z}_{2l_1} \cup \mathcal{Z}_{2l_2} = \left(z_1^{(1)}, \ldots, z_{2l_1}^{(1)}, z_1^{(2)}, \ldots, z_{2l_2}^{(2)}\right).$$

Moreover, taking into account the well known properties of the sums of the matrices, we see that (3.3) corresponds to the distribution (3.2). ∎

REMARK 8. *Assumption* (3.1) *has a special interpretation. Let us observe that assumption (3.1) implies* $a_{12} = -1$. *That means that the random variables* $X_1, X_2 - X_1$ *are independent. We can also find the two-dimensional marginal distribution of* (X_1, X_r). *Next, taking into account assumption (3.1), we see that* $X_1, X_r - X_1$ *are independent.*

REMARK 9. *If* $\mathbf{X}_d^{(n)}$ *(n=1,2) are Gaussian vectors then assumption* (3.1) *is needless.*

4. The characterization of (PGD_d)

Now we are going to give a characterization of (PGD_d) by the independence of linear forms in a triangular system.

We shall use the following lemma (see Plucińska 2001):

LEMMA 10. *The characteristic function* φ *of* $PGD(2l, \mathcal{K}, \mathcal{Z}_{2l})$ *has only* PGD_1 *factors. Moreover if* $\varphi = \varphi_1\varphi_2$ *then* $l = l_1 + l_2$, $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$, $\mathcal{Z}_{2l} = \mathcal{Z}_{2l_1} \cup \mathcal{Z}_{2l_2}$.

PROPOSITION 11. *Let* $\mathcal{Q} = [q_{rs}]_{r,s=1}^{d}$ *be the positive definite matrix of the second order moments of the vector* $\mathbf{X}_d$, *let* k_{11} *be a given number such that* $k_{11} \in (0, q_{11})$ *and let* k_{dd} *be given by* (2.11′). *Moreover let*

$$\begin{cases} X_1 \sim PGD_1(2l, k_{11}, \mathcal{Z}_{2l}), \\ X_d \sim PGD_1(2l, k_{dd}, \frac{q_{11}}{q_{1d}}\mathcal{Z}_{2l}). \end{cases} \tag{4.1}$$

Suppose that there exist coefficients a_{rs} *such that* $Y_1, \ldots, Y_d$ *given by* (2.3) *are independent and* $EY_r = 0$ *for* $r = 2, \ldots, d$.

Then $\mathbf{X}_d$ *has the distribution* PGD_d $(2l, \mathcal{K}, \mathcal{Z}_{2l})$, *where* $\mathcal{K} = [k_{rs}]_{r,s=1}^{d}$ *and* k_{rs} *are given by* (2.11′).

Proof. The system (2.3) is a Cramer system. We solve it with respect to $X_1, \dots, X_d$. The solution is given by (2.3'). For r=d we have

$$X_d = Y_d + b_{1d}Y_1 + \dots + b_{d-1,d}Y_{d-1}. \tag{4.2}$$

According to (2.8), $b_{1d} = \frac{E(X_1 X_d)}{EX_1^2} = \frac{q_{1d}}{q_{11}} = 1$. By the decomposition lemma 10 every component of the sum (4.2) must have PGD_1. The zeros of the characteristic function of the left side of (4.2) must be the same as the zeros of the right side. The set of zeros of the characteristic function of X_1 is $\mathcal{Z}_{2l}$, and the set of zeros of the characteristic function of X_d is $\frac{q_{11}}{q_{1d}}\mathcal{Z}_{2l}$. Thus the sets of zeros of the left and right sides are really the same. Thus every Y_r, $r \geq 2$, has normal distribution. Therefore

$$\begin{aligned}
\varphi(\boldsymbol{\xi}_d) &= E\exp(i\boldsymbol{\xi}_d, \mathbf{X}_d) \\
&= E\exp[i\xi_1 Y_1 b_{11} + i\xi_2(b_{22}Y_2 + b_{12}Y_1) + \dots + i\xi_d(b_{1d}Y_1 + \dots + b_{dd}Y_d)] \\
&= E\exp[iY_1(b_{11}\xi_1 + \dots + b_{1d}\xi_d)]E\exp[iY_2(b_{22}\xi_2 + \dots + b_{2d}\xi_d)]\cdot \dots \\
&\quad \cdot E\exp[i\xi_d b_{dd}Y_d],
\end{aligned}$$

where Y_1 has (PGD_1), while Y_r, $r \geq 2$, has a Gaussian distribution. Thus we get the characteristic function of the form (2.5).

Then the density of $(X_1, \dots, X_d)$ is

$$f(\mathbf{x}_d) = p_{2l}(x_1)\exp\left(-\frac{x_1^2}{2\sigma_1^2}\right)\exp\left[-\frac{(x_2 + a_{12}x_1)^2}{2\sigma_2^2}\right]\cdot \dots$$

$$\dots \cdot \exp\left[-\frac{(x_d + a_{1d}x_1 + \dots + a_{d-1,d}x_{d-1})^2}{2\sigma_d^2}\right]. \ \blacksquare$$

5. Conditions for characteristic functions

In this Section we give a necessary and sufficient condition for a function which is the product of a polynomial and an exponential function to be a characteristic function.

Let $L_k^w(x)$ denote Laguerre's polynomials defined by

$$L_k^w(x) = \sum_{i=0}^{k} \frac{(-1)^i}{i!}\binom{k+w}{k-i}x^i.$$

PROPOSITION 12. *A function of the form*

$$\varphi(\xi) = \sum_{r=0}^{2l} c_r (i\xi)^r \exp\left(-\frac{\xi^2}{2}\right)$$

is a characteristic function iff

$$\varphi(\xi) = \exp\left(-\frac{\xi^2}{2}\right) \cdot \left[\sum_{k=0}^{l} \left(a_k^2 + b_k^2\right) k! L_k^0 \left(\xi^2\right)\right] + \exp\left(-\frac{\xi^2}{2}\right) \cdot \left[2 \sum_{k<w}^{l} \left(a_k a_w + b_k b_w\right) k! \left(i\xi\right)^{w-k} L_k^{w-k} \left(\xi^2\right)\right] \tag{5.1}$$

where a_k *and* b_k *are real parameters satisfying the condition*

$$\sum_{k=0}^{l} \left(a_k^2 + b_k^2\right) k! = 1.$$

LEMMA 13. *The relationship between Laguerre's and Hermit'e polynomials is given by*

$$\frac{1}{\sqrt{2\pi}} \int_R H_l(x) H_m(x) \exp\left(-\frac{1}{2}(x-b)^2\right) dx = m! b^{l-m} L_m^{l-m}(-b^2). \tag{5.2}$$

It is easy to derive (5.2) using the formulas given by Prudnikov et al. (1983).

Proof of Proposition 12. It is well known that a non-negative function has the form

$$p_{2l}(x) = \left(\widetilde{a_0} + \widetilde{a_1} x + \ldots + \widetilde{a_l} x^l\right)^2 + \left(\widetilde{b_0} + \widetilde{b_1} x + \ldots + \widetilde{b_l} x^l\right)^2.$$

Of course this function is a polynomial of degree 2l. It is very well known that every polynomial can be written as a linear combination of Hermite polynomials. So we can write the above equation in the form

$$p_{2l}(x) = \left(a_0 H_0(x) + a_1 H_1(x) + \ldots + a_l H_l(x)\right)^2 + \left(b_0 H_0(x) + b_1 H_1(x) + \ldots + b_l H_l(x)\right)^2. \tag{5.3}$$

The function

$$f(x) = \frac{1}{\sqrt{2\pi}} p_{2l}(x) \exp\left(-\frac{x^2}{2}\right)$$

is a density function when $\int_R f(x)dx = 1$. It is well known that there is one-to-one correspondence between the characteristic function and the density function. Thus we can start from the density function. Moreover, from the orthogonality condition for Hermite polynomials:

$$\frac{1}{\sqrt{2\pi}} \int_R \exp\left(-\frac{x^2}{2}\right) H_l(x) H_m(x) dx = \begin{cases} 0\ , & l \neq m, \\ l!, & l = m, \end{cases}$$

we have

$$\sum_{k=0}^{l}\left(a_k^2+b_k^2\right)k!=1.$$

In order to calculate

$$\varphi(\xi)=\int_R f(x)\exp(i\xi x)dx$$

we use the equation (5.3). We have

$$\varphi(\xi)=\frac{1}{\sqrt{2\pi}}\int_R[(a_0H_0(x)+a_1H_1(x)+\ldots+a_lH_l(x))^2]\exp(i\xi x)\exp\left(-\frac{x^2}{2}\right)$$

$$+\frac{1}{\sqrt{2\pi}}\int_R[(b_0H_0(x)+b_1H_1(x)+\ldots+b_lH_l(x))^2]\exp(i\xi x)\exp\left(-\frac{x^2}{2}\right)$$

$$=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\xi^2}{2}\right)\int_R\Big[\sum_{r=0}^{l}\left(a_k^2+b_k^2\right)H_k^2(x)$$

$$+2\sum_{k\neq l}a_ka_wH_k(x)H_w(x)+2\sum_{k\neq w}b_kb_wH_k(x)H_w(x)\Big]$$

$$\times\exp\left[-\frac{1}{2}(x-i\xi)^2\right]dx.$$

Then using (5.2), after some simple calculations, we get (5.1). ∎

6. Polynomial-Gaussian processes

We will construct a stochastic process $\mathfrak{X}=(X_t,t_1\geq 0)$ such that the one-dimensional distributions of $\mathfrak{X}$ are (PGD_1). We shall consider two cases.

6.1. Case A

Let $K:(0,\infty)\times(0,\infty)\to R^1$ be a positive definite function and let $t_1>0$ be a given number. For brevity we will write $k_{rs}=K(t_r,t_s)$. Let X_{t_1} be a random variable with distribution $PGD_1(2l,K(t_1,t_1),\underset{\sim}{Z}_{2l})$ and $\widetilde{\mathfrak{X}}=(\widetilde{X}_t,\ t\geq 0)$ be a Gaussian process such that $E\widetilde{X}_t=0$, $E\widetilde{X}_{t_r}\widetilde{X}_{t_s}=k_{rs}-\frac{k_{1r}k_{1s}}{k_{11}}$ for $t_r,t_s\geq t_1$. We take $t_1\leq t_2\leq\ldots\leq t_d$, $\mathbf{t}_d=(t_1,\ldots,t_d)$. We suppose that $\widetilde{\mathfrak{X}}$ and X_{t_1} are independent.

We define a stochastic process $\mathfrak{X}$ in the following way:

$$\mathfrak{X}=(X_t,\ t\geq t_1)=\left(\widetilde{X}_t+\frac{K(t_1,t)}{k_{11}}X_{t_1},\ t\geq t_1\right).$$

PROPOSITION 14. *For every $d \geq 1$ the distribution of $(X_{t_1}, \ldots, X_{t_d})$ is given by* (2.2).

Proof. First observe that

$$q_{rs} = EX_{t_r}X_{t_s} = k_{rs} + \frac{k_{1r}k_{1s}}{k_{11}^2}\left(EX_{t_1}^2 - k_{11}\right) = k_{rs} + \frac{k_{1r}k_{1s}}{k_{11}^2}\left(q_{11} - k_{11}\right).$$

Thus we get formula (2.11). Moreover, by formula (2.5), the characteristic function is

$$\begin{aligned}
\varphi(\boldsymbol{\xi}_d, \mathbf{t}_d) &= \exp\left[i\left(\xi_1 X_{t_1} + \ldots + \xi_d X_{t_d}\right)\right] \\
&= \exp\left\{i\left[\xi_1 + \frac{k_{12}}{k_{11}}\xi_2 + \ldots + \frac{k_{1d}}{k_{11}}\xi_d\right] X_{t_1} + \xi_1 \widetilde{X}_{t_1} + \ldots + \xi_d \widetilde{X}_{t_d}\right\} \\
&= \varphi_{X_{t_1}}\left(\xi_1 + \ldots + \frac{k_{1d}}{k_{11}}\xi_d\right) \varphi_{\left(\widetilde{X}_{t_1}, \ldots, \widetilde{X}_{t_d}\right)}(\boldsymbol{\xi}_d, \mathbf{t}_d) \\
&= \Psi_{2l}(\eta) \exp\left\{-\frac{1}{2}\sum_{r,s=1}^{d} \frac{k_{1r}k_{1s}}{k_{11}}\xi_r\xi_s - \frac{1}{2}\sum_{r,s=1}^{d} \xi_r\xi_s E\left(\widetilde{X}_{t_r}\widetilde{X}_{t_s}\right)\right\} \\
&= \Psi_{2l}(\eta) \exp\left\{-\frac{1}{2}\sum_{r,s=1}^{d} \xi_r\xi_s k_{rs}\right\},
\end{aligned}$$

where: $\eta = \xi_1 + \frac{k_{12}}{k_{11}}\xi_2 + \ldots + \frac{k_{1d}}{k_{11}}\xi_d$. Thus φ is given by (2.5). ∎

6.2. Case B

Now we shall consider a special case of the above. Namely let $W = (W_t,\ t \geq 0)$ be a Wiener process independent of X_{t_1}. Let $\mathfrak{X} = (W_{t-t_1} + X_{t_1},\ t \geq t_1)$. It follows from the previous considerutions that the process $\mathfrak{X}$ has the following properties:

1) $\mathfrak{X}$ is conditionally Gaussian, i.e. the conditional distribution of $X_{t_d} \mid X_{t_1}, \ldots, X_{t_{d-1}}$ is Gaussian for $t_1 < t_2 < \ldots < t_d$. Conditionally Gaussian processes are considered for example by Liptser and Shiryaev (1978).

2) The coefficients a_{rs} have the following form

$$a_{rs} = 0 \; for \; r = 1, \ldots, s-2; \; s \geq 3;$$
$$a_{s-2,s-1} = 1.$$

Thus $\mathfrak{X}$ is a process with independent increments.

3) $\mathfrak{X}$ is a martingale.

4) The moments of $\mathfrak{X}$ have the form:

$$E(X_t) = c_1;$$
$$E\left(X_t^2\right) = 2c_2 + t;$$
$$E(X_{t_r}, X_{t_s}) = 2c_2 + t_r \quad \text{for } t_r \le t_s.$$

5) The characteristic function of $(X_{t_1}, \dots, X_{t_d})$ is

$$\varphi_d(\boldsymbol{\xi}_d, \mathbf{t}_d) = \sum_{r=0}^{2l} c_r i^r (\xi_1 + \dots + \xi_d)^r \exp\left[-\frac{1}{2}(\mathcal{K}\boldsymbol{\xi}_d, \boldsymbol{\xi}_d)\right],$$

where

$$\mathcal{K} = \begin{bmatrix} t_1 & \cdots & t_1 \\ \cdots & \cdots & \cdots \\ t_1 & \cdots & t_d \end{bmatrix}.$$

6) The characteristic function of X_{t_s} is

$$\varphi(\xi_s, t_s) = \sum_{r=0}^{2l} c_r (i\xi_s)^r \exp\left(-\frac{1}{2}\xi_s^2 t_s\right).$$

7. Example

Consider case B of Section 6. Take l=2. Then

$$p_2(x,t) = 1 + \frac{c_1 x}{t} + \frac{c_2}{t}\left(\frac{x^2}{t} - 1\right). \tag{7.1}$$

This polynomial is non-negative iff $t \ge \frac{4c_2^2}{4c_2 - c_1^2}$, $4c_2 - c_1^2 > 0$. Thus we have $\frac{4c_2^2}{4c_2 - c_1^2} \le t_1 < t_2 < \dots < t_d$. For $c_1 = c_2 = 0$ we take $t_1 \ge 0$. The characteristic function has the form

$$\begin{aligned} \varphi(\xi_1, \dots, \xi_d, t_1, \dots, t_d) &= [1 + ic_1(\xi_1 + \dots + \xi_d) + i^2 c_2 (\xi_1 + \dots + \xi_d)^2] \\ &\quad \times \exp\left\{-\frac{1}{2}\sum_{i=1}^{d} \xi_i^2 t_i - \sum_{i<j} \xi_i \xi_j t_i\right\}. \end{aligned} \tag{7.2}$$

By formula (7.2) it is easy to observe that the consistency conditions are satisfied.

The polynomial (7.1) has quite a general form. But evidently not every second degree polynomial can be represented in the form (7.1). For example we cannot take the Maxwell distribution, i.e. we cannot put

$$p_2(x,t) = \frac{x^2}{t}.$$

The consistency conditions are not satisfied, since

$$\frac{1}{2\pi\sqrt{t_1(t_2-t_1)}}\int\frac{x_1^2}{t_1}\exp\left\{-\frac{x_1^2}{2t_1}+\frac{(x_2-x_1)^2}{2(t_2-t_1)}\right\}dx_1$$
$$=\frac{1}{\sqrt{2\pi}t_2\sqrt{t_2}}\left[t_2-t_1+\frac{t_1}{t_2}x_2^2\right]\exp\left\{-\frac{x_2^2}{2t_2}\right\}.$$

The two-dimensional Fourier transformation has the form

$$\begin{aligned}\varphi(\xi_1,\xi_2,t_1,t_2) &= E\exp(i\xi_1X_{t_1}+i\xi_2X_{t_2})\\ &= \frac{1}{2\pi\sqrt{(t_2-t_1)t_1}}\int_{R^2}\frac{x_1^2}{t_1}\exp(i\xi_1x_1+i\xi_2x_2)\\ &\quad\times\exp\left\{-\frac{x_1^2}{2t_1}+\frac{(x_2-x_1)^2}{2(t_2-t_1)}\right\}dx_1dx_2\\ &= \left[1-t_1(\xi_1+\xi_2)^2\right]\exp\left[-\frac{1}{2}\left(\xi_1^2t_1+\xi_2^2t_2-2\xi_1\xi_2t_1\right)\right].\end{aligned}$$

It is also evident that the consistency conditions are not satisfied: $\varphi_2(0,\xi_2,t_1,t_2)$ depends on t_1.

Another explanation that we cannot take the Maxwell distribution is the following. Let us represent X_{t_2} as a sum of two independent components:

$$X_{t_2}=(X_{t_2}-X_{t_1})+X_{t_1}.$$

It is known that the characteristic function of the Maxwell distribution is indecomposable. Thus we get a contradiction.

References

[1] M. Evans and T. Swartz (1994), *Distribution theory and inference for polynomial-normal densities*, Comm. Stat., Theory Methods 23 (4), 1123–1148.

[2] N. Johnson and S. Kotz (1972), *Distributions in Statistics: Continuous Multivariate Distributions*, John Wiley, New York.

[3] R. S. Liptser and A. N. Shiryaev (1978), *Statistics of Random Processes*, Springer, New York.

[4] E. Lukacs (1970), *Characteristic Functions*, Griffin, London.

[5] A. Plucińska (2001), *Composition and Decomposition of Polynomial-Normal Distributions*, Janos Bolyai Math. Society, Proc of "Fourth Hungarian Colloquium on Limit Theorems of Probability and Statistics", (to appear in 2001).

[6] A. Plucińska (1999), *Some properties of polynomial-normal distributions associated with Hermite polynomials*, Demonstratio Math. 32, No 1, 195–206.

[7] A. P. Prudnikov, Y. A. Brychkov and O. I. Marichev (1983), *Integrals and Special Functions*, Nauka, Moscow (in Russian).
[8] A. N. Shiryaev (1996), *Probability*, Springer, New York.

FACULTY OF MATHEMATICS AND INFORMATION SCIENCE
WARSAW UNIVERSITY OF TECHNOLOGY
Pl. Politechniki 1
00-661 WARSAW, POLAND

Received July 13, 2000.