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RECURRENCE RELATIONS FOR SINGLE AND PRODUCT MOMENTS OF LOWER GENERALIZED ORDER STATISTICS FROM THE INVERSE WEIBULL DISTRIBUTION

Dedicated to Professor Kazimierz Urbanik

Abstract. We present a concept of lower generalized order statistics. With this definition we give recurrence relations for single and product moments of lower generalized order statistics from the inverse Weibull distribution.

1. Introduction

The concept of generalized order statistics was introduced by Kamps (1995). A variety of order models of random variables is contained in this concept.

DEFINITION 1.1 [1]. Let $n \in \mathbb{N}$, $k \geq 1$, $m \in \mathbb{R}$, be parameters such that

$$\gamma_r = k + (n-r)(m+1) \geq 1, \quad \text{for all } 1 \leq r \leq n.$$

By generalized order statistics from an absolutely continuous distribution function F with the probability density function (pdf) f we mean random variables $X(1,n,m,k), \ldots, X(n,n,m,k)$ having joint density function of the form

$$f^{X(1,n,m,k),\ldots,X(n,n,m,k)}(x_1,\ldots,x_n)$$
$$= k\Big(\prod_{j=1}^{n-1}\gamma_j\Big)\Big(\prod_{i=1}^{n-1}(1-F(x_i))^m f(x_i)\Big)(1-F(x_n))^{k-1}f(x_n)$$

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$$F^{-1}(0) < x_1 \le \ldots \le x_n < F^{-1}(1).$$

The joint density function of $X(r,n,m,k)$ and $X(s,n,m,k)$ is given by

$$f^{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{c_{s-1}}{(r-1)!(s-r-1)!}(1-F(x))^m f(x) g^{r-1}(F(x)) \times [h(F(y)) - h(F(x))]^{s-r-1} (1-F(y))^{\gamma_s - 1} f(y)$$

for $x < y$, $r < s$, where

$$c_{s-1} = \prod_{i=1}^{s} \gamma_i,$$

$$h(x) = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1}, & m \ne -1, \\ -\log(1-x), & m = -1, \end{cases}$$

$$g(x) = h(x) - h(0), \ x \in [0,1).$$

The marginal density of the r–th generalized order statistic is given by

$$f^{X(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!}(1-F(x))^{\gamma_r - 1} g^{r-1}(F(x)) f(x).$$

The model of generalized order statistics contains as special cases, order statistics, sequential order statistics, Stigler's order statistics, record values. They can be easily applicable in practice problems except when F is so–called the inverse distribution function (inverse exponential, inverse Weibull, inverse Pareto, etc.) In this case when F is an inverse distribution function, we need a concept of lower generalized order statistics.

DEFINITION 1.2. Let $n \in \mathbb{N}$, $k \ge 1$, $m \in \mathbb{R}$, be parameters such that

$$\gamma_r = k + (n-r)(m+1) \ge 1, \quad \text{for all } 1 \le r \le n.$$

By the lower generalized order statistics from an absolutely continuous distribution function F with the density function f we mean random variables $X'(1,n,m,k), \ldots, X'(n,n,m,k)$ having joint density function of the form

$$(1.1) \quad f^{X'(1,n,m,k),\ldots,X'(n,n,m,k)}(x_1,\ldots,x_n) = k\Big(\prod_{r=1}^{n-1} \gamma_r\Big)\Big(\prod_{j=1}^{n-1} (F(x_j))^m f(x_j)\Big)(F(x_n))^{k-1} f(x_n)$$

for

$$F^{-1}(1) > x_1 \ge x_2 \ge \ldots \ge x_n > F^{-1}(0).$$

Note that for $m = 0$, $k = 1$ we obtain the joint pdf for the ordinary order statistics, and when $m = -1$ we get the joint pdf of the k–th lower record values.

In the sequel we shall need the pdf of $(X'(r,n,m,k), X'(s,n,m,k))$, $r < s$, namely

$$(1.2)\qquad f^{X'(r,n,m,k),X'(s,n,m,k)}(x,y) = \frac{c_{s-1}}{(r-1)!(s-r-1)!}(F(x))^m f(x) g^{r-1}(F(x)) \times [h(F(y)) - h(F(x))]^{s-r-1}(F(y))^{\gamma_s - 1} f(y),\ x > y,$$

where

$$h(x) = \begin{cases} -\dfrac{1}{m+1}x^{m+1}, & m \neq -1, \\ -\log x, & m = -1, \end{cases}$$
$$g(x) = h(x) - h(1),\ x \in [0,1).$$

We shall also take $X'(0,n,m,k) = 0$. From (1.1) we see that the pdf of the r–th lower generalized order statistic is given by

$$(1.3)\qquad f^{X'(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!}(F(x))^{\gamma_r - 1} g^{r-1}(F(x)) f(x).$$

A random variable X is said to have the inverse Weibull distribution if the pdf is of the form

$$(1.4)\qquad f(x) = \frac{\tau\left(\frac{\theta}{x}\right)^{\tau} e^{-\left(\frac{\theta}{x}\right)^{\tau}}}{x},\ x > 0;\ \tau,\ \theta > 0.$$

Note that for the inverse Weibull distribution we have

$$(1.5)\qquad f(x)x^{\tau+1} = \theta^{\tau}\tau F(x).$$

For $\tau = 1$ in (1.4) we obtain the pdf of the inverse exponential distribution of the form

$$(1.6)\qquad f(x) = \frac{\theta e^{-\frac{\theta}{x}}}{x^2},\ x > 0;\ \theta > 0.$$

More details on inverse distributions and their applications can be found in [2].

2. Relations for single moments

Using (1.3) and (1.5) we obtain the following recurrence relations for single moments of lower generalized order statistics from the inverse Weibull distribution (1.4).

THEOREM 2.1. *Fix a positive integer k. For $n \in \mathbb{N}$, $m \in \mathbb{R}$, $1 \le r \le n$, and $j = 0, 1, 2, \ldots$*

$$E[X'(r,n,m,k)]^{j+\tau+1} = \frac{\tau\theta^\tau\gamma_r}{j+1}\left[E(X'(r-1,n,m,k))^{j+1} - E(X'(r,n,m,k))^{j+1}\right].$$

Proof. By (1.3) and (1.5)

$$E[X'(r,n,m,k)]^{j+\tau+1} = \frac{\tau\theta^\tau c_{r-1}}{(r-1)!}\int x^j [F(x)]^{\gamma_r} g^{r-1}(F(x))dx.$$

Integrating by parts, treating x^j as the part for integration, we get

$$\begin{aligned} E[X'(r,n,m,k)]^{j+\tau+1} &= \frac{\tau\theta^\tau c_{r-1}}{(r-1)!}\left[-\frac{1}{j+1}\int x^{j+1}\gamma_r [F(x)]^{\gamma_r-1}[g_m(F(x))]^{r-1} f(x)dx \right. \\ &\quad \left. + \frac{r-1}{j+1}\int x^{j+1}(g(F(x)))^{r-2}[F(x)]^{\gamma_{r-1}-1} f(x)dx\right] \\ &= \frac{\tau\theta^\tau\gamma_r}{j+1}\left[E(X'(r-1,n,m,k))^{j+1} - E(X'(r,n,m,k))^{j+1}\right]. \end{aligned}$$

COROLLARY 2.1. *The recurrence relations for single moments of order statistics from the inverse Weibull distribution have the form*

$$E(X_{n-r+1:n})^{j+\tau+1} = \frac{\tau\theta^\tau(n-r+1)}{j+1}\left[E(X_{n-r+2:n})^{j+1} - E(X_{n-r+1:n})^{j+1}\right].$$

COROLLARY 2.2. *The recurrence relations for single moments of k–th lower record values from the inverse Weibull distribution have the form*

$$E(Z_n^{(k)})^{j+\tau+1} = \frac{\tau\theta^\tau k}{j+1}\left[E(Z_{n-1}^{(k)})^{j+1} - E(Z_n^{(k)})^{j+1}\right].$$

COROLLARY 2.3. *For $m = 0$ and $k = \alpha - n + 1$, $\alpha \in \mathbb{R}_+$, we obtain the recurrence relations for single moments of order statistics with non–integral sample size*

$$E(X_{\alpha-r+1:\alpha})^{j+\tau+1} = \frac{\tau\theta^\tau(\alpha-r+1)}{j+1}\left[E(X_{\alpha-r+2:\alpha})^{j+1} - E(X_{\alpha-r+1:\alpha})^{j+1}\right].$$

COROLLARY 2.4. *For $m = \alpha - 1$, $k = \alpha$, we obtain the recurrence relations for sequential order statistics*

$$E[X'(r,n,\alpha-1,\alpha)]^{j+\tau+1} = \frac{\tau\theta^{\tau}\alpha(n-r+1)}{j+1}[E(X'(r-1,m,\alpha-1,\alpha))^{j+1} - E(X'(r,n,\alpha-1,\alpha))^{j+1}].$$

COROLLARY 2.5. *Under the assumptions of Theorem* 2.1 *with* $\tau = 1$ *we have the corresponding recurrence relations for the inverse exponential distribution*

$$E[X'(r,n,m,k)]^{j+\tau+1} = \frac{\theta\gamma_r}{j+1}\left[E[X'(r-1,n,m,k)]^{j+1} - E[X'(r,n,m,k)]^{j+1}\right].$$

3. Relations for product moments

Using (1.2) and (1.5) we get the following recurrence relations for product moments of generalized order statistics from the inverse Weibull distribution.

THEOREM 3.1. *Fix a positive integer* k. *For* $n \geq 1$, $m \in Z$, $1 \leq r \leq n$, *and* $i, j = 0, 1, 2, \ldots,$

$$\begin{aligned}(3.1)\quad & E[(X'(r,n,m,k))^i(X'(s,n,m,k))^{j+\tau+1}] \\ &= \frac{\theta^{\tau}\tau\gamma_s}{j+1}\left[E[(X'(r,n,m,k))^i(X'(s-1,n,m,k))^{j+1}] \right. \\ &\quad \left. -E[(X'(r,n,m,k))^i(X'(s,n,m,k))^{j+1}]\right]\end{aligned}$$

and for $r + 1 = s$

$$\begin{aligned}(3.2)\quad & E[(X'(r,n,m,k))^i(X'(r+1,n,m,k))^{j+\tau+1}] \\ &= \frac{\theta^{\tau}\tau\gamma_s}{j+1}[E(X'(r,n,m,k))^{i+j+1} - E[(X'(r,n,m,k))^i(X'(r+1,n,m,k))^{j+1}]].\end{aligned}$$

Proof. We note that for $1 \leq r \leq s-2$ and $i, j = 0, 1, 2, \ldots,$

$$\begin{aligned}(3.3)\quad & E[(X'(r,n,m,k))^i(X'(s,n,m,k))^{j+\tau+1}] \\ &= \frac{c_{s-1}}{(r-1)!(s-r-1)!}\int x^i[F(x)]^m f(x) g^{r-1}(F(x)) I(x)\,dx\end{aligned}$$

where

$$I(x) = \int y^{j+\tau+1}\left[h_m(F(y)) - h_m(F(x))\right]^{s-r-1}(F(y))^{\gamma_s}\,dy.$$

Integrating $I(x)$ by parts and substituting this expression into (3.3) we obtain (3.1). For $s = r + 1$ we have (3.2).

COROLLARY 3.1. *Under the assumptions of Theorem* 3.1 *for* $m = 0$, $k = 1$, *we have*

$$E\left[X^i_{n-r+1:n}X^{j+\tau+1}_{n-s+1:n}\right]$$
$$= \frac{\tau\theta^\tau(n-s+1)}{j+1}\left\{E[X^i_{n-r+1:n}X^{j+1}_{n-s+2:n}] - E[X^i_{n-r+1:n}X^{j+1}_{n-s+1:n}]\right\}.$$

COROLLARY 3.2. *For $m=-1$ from Theorem 3.1 we have*

$$E[(Z^{(k)}_m)^i(Z^{(k)}_n)^{j+\tau+1}]$$
$$= \frac{\tau\theta^\tau k}{j+1}\left\{E(Z^{(k)}_m)^i(Z^{(k)}_{n-1})^{j+1} - E(Z^{(k)}_m)^i(Z^{(k)}_n)^{j+1}\right\}.$$

By analogy with previous expressions for $\tau = 1$ we have recurrence relations for the inverse exponential distribution, when $m=0$, $k=\alpha-n-1$ we get Stigler statistics and for $m=\alpha-1$, $k=\alpha$ the sequential order statistics.

References

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