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ON THE ALMOST SURE CONVERGENCE OF ALL CONDITIONINGS OF POSITIVE RANDOM VARIABLES

The paper is dedicated to Professor Kazimierz Urbanik

Abstract. We prove that in a non-atomic probability space, for a sequence of positive r.v. (X_n), $\mathbb{E}(X_n|\mathfrak{A}) \to 0$ a.s. for any σ-field $\mathfrak{A}$ of events iff $X_n \to 0$ a.s. and $\mathbb{E}\sup_{n\geq 1}|X_n| < \infty$. We point out these classical theorems on orthogonal series and ergodic means in which the almost sure convergence of all conditionings of a discussed sequence can be obtained.

1. Introduction and main results

Let $\mathfrak{A}$ be any σ-field of events in any probability space. The operation of conditional expectation $\mathbb{E}(\cdot|\mathfrak{A})$ is a positive contraction in the space L_1 of integrable random variables. Thus the following theorem can be immediately obtained.

THEOREM 1.1. *For any sequence of random variables $(X_n) \subset L_1(\Omega, \mathcal{F}, P)$, the conditions $X_n \to 0$ a.s. and $\mathbb{E}(\sup_{n\geq 1}|X_n|) < \infty$ imply the convergence $\mathbb{E}(X_n|\mathfrak{A}) \to 0$ a.s. for any σ-field $\mathfrak{A} \subset \mathcal{F}$.*

On the other hand the conditions $X_n \to 0$ a.s., $\mathbb{E}|X_n| \to 0$ **do not** imply the convergence $\mathbb{E}(X_n|\mathfrak{A}) \to 0$ a.s.

EXAMPLE 1.2. Take the space $\Omega = [0, \infty)$ with probability

$$P(d\omega) = f(\omega)\lambda(d\omega)$$

given by the density

$$f(\omega) = \sum_{n\geq 0} \frac{1}{2^{n+1}} 1_{[n,n+1)}(x)$$

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with respect to the Lebesgue measure λ. For any index $i \geq 2$, let us define

$$X_i = 2^{n+1}\, 1_{[n+(k-1)4^{-n}3^{-1},\, n+k4^{-n}3^{-1})}$$

according to the unique representation $i = 4^n + k$, $n \geq 0$, $1 \leq k \leq 3 \cdot 4^n$. Then we have $\mathbb{E}X_i = 4^{-n}3^{-1} \to 0$ as $i \to \infty$, and $X_i \to 0$ a.s. But, for a σ-field $\mathfrak{A}$ generated by the function $Y(\omega) = \omega - [\omega]$ (being the fractional part of ω), the equality $\mathbb{E}(X_i|\mathfrak{A}) = 1$ is satisfied in particular on the set $[(k-1)4^{-n}3^{-1}, k4^{-n}3^{-1})$ for $i = 4^n + k$, $1 \leq k \leq 3 \cdot 4^n$.

Roughly speaking the σ-field $\mathfrak{A}$ is obtained by "glueing" points $x, 1+x, 2+x, \ldots$, for $x \in [0,1)$, satisfying

$$\sup_{i\geq 2} X_i(x), \sup_{i \geq 2} X_i(1+x), \ldots \to \infty.$$

Developping such an idea, we obtain the following general

THEOREM 1.3. *For any non-atomic probability space $(\Omega, \mathcal{F}, P)$ and any sequence (X_i) of positive random variables, the following conditions are equivalent:*

(i) $X_i \to 0$ *a.s.*, $\mathbb{E}\sup_{i\geq 1} X_i < \infty$;
(ii) $\mathbb{E}(X_i|\mathfrak{A}) \to 0$ *a.s. for any σ-field* $\mathfrak{A} \subset \mathcal{F}$.

The assumption that $(\Omega, \mathcal{F}, P)$ is non-atomic is essential. Namely,

$$|\mathbb{E}(X|\mathfrak{A})(\omega)| \leq (\mathbb{E}|X|)/p$$

for ω taken from any atom A of $\mathfrak{A}$ with probability $P(A) = p$, and we have

PROPOSITION 1.4. *If* $\mathbb{E}|X_i| \to 0$, *then* $\mathbb{E}(X_i|\mathfrak{A})1_A \to 0$ *a.s. for any σ-field $\mathfrak{A}$ and any atom A of $\mathcal{F}$.*

Some consequences of Theorem 1.3 and of (elementary) Theorem 1.1 are collected in Section 3. All these corollaries are obtained by analysing classical proves of specific theorems on almost sure convergence.

2. The proof of Theorem 1.3

We shall use the following, rather obvious

LEMMA 2.1. *Let $(A_s)_{s\geq 1}$ be a collection of disjoint sets in a non-atomic probability space $(\Omega, \mathcal{F}, P)$, and let $P(A_s) = \lambda^2(B_s)$, $s \geq 1$ for some disjoint Borel sets B_s in $[0,1) \times [0,1)$ with two-dimensional Lebesgue measure λ^2. Then there exists a transformation $T : \Omega \to [0,1) \times [0,1)$ satisfying*

$$T^{-1}(B_s) = A_s,$$
$$P(T^{-1}(C)) = \lambda^2(C) \quad \textit{for } C \in \text{Borel}[0,1) \times [0,1).$$

P r o o f. Let us fix $s \geq 1$. Assume at first that B_s is a compact set. One can define partitions $\mathcal{P}_1, \mathcal{P}_2, \ldots$ of A_s satisfying

$$\mathcal{P}_n = \{\Delta^n_{ij}; \quad i,j = 1,\ldots,2^n\},$$

$$P(\Delta^n_{ij}) = \lambda^2\Big(B_s \cap \Big(\Big[\frac{i-1}{2^n},\frac{i}{2^n}\Big] \times \Big[\frac{j-1}{2^n},\frac{j}{2^n}\Big]\Big)\Big),$$

$$\Delta^{n+1}_{i'j'} \subset \Delta^n_{ij} \text{ for } \Big[\frac{i'-1}{2^{n+1}},\frac{i'}{2^{n+1}}\Big] \times \Big[\frac{j'-1}{2^{n+1}},\frac{j'}{2^{n+1}}\Big] \subset \Big[\frac{i-1}{2^n},\frac{i}{2^n}\Big] \times \Big[\frac{j-1}{2^n},\frac{j}{2^n}\Big].$$

For a fixed $\omega \in A_s$, the indices $i(n)$, $j(n)$ satisfying $\omega \in \Delta^n_{i(n)j(n)}$, $n \geq 1$, can be defined. We put $T(\omega) = (x,y)$ with

$$(x,y) \in B_s \cap \Big(\Big[\frac{i(n)-1}{2^n},\frac{i(n)}{2^n}\Big] \times \Big[\frac{j(n)-1}{2^n},\frac{j(n)}{2^n}\Big]\Big), \quad n \geq 1.$$

If the set B_s is not compact, one can take disjoint compact sets B_{s1}, $B_{s2}, \ldots$ and disjoint events $A_{s1}, A_{s2}, \ldots$ satisfying $\lambda(\bigcup_{n\geq 1} B_{sn}) = \lambda(B_s)$, $P(A_{sn}) = \lambda^2(B_{sn})$, $\bigcup_{n\geq 1} A_{sn} = A_s$. Then the previous case can be used to define T on A_s. The proof is finished.

2.2. The proof of implication $\sim$ (i) $\Rightarrow\sim$ (ii)

Denote

$$Y = \sup_{n\geq 1} X_n - 1, \tag{1}$$
$$A^n_k = (X_1 < Y,\ldots,X_{n-1} < Y, X_n \geq Y, \quad k \leq X_n < k+1).$$

Obviously, we have

$$A^n_k \quad \text{are mutually disjoint for } n,k \geq 1, \tag{2}$$
$$\sum_{n\geq 1,k\geq 1} P(A^n_k)k = \infty.$$

By (2), we can find indices $n(1,1),\ldots,n(1,s(1))$ and $k(1,1),\ldots,k(1,s(1))$ satisfying

$$n(1,s) \geq 1, \quad k(1,s) \geq 2, \tag{3}$$
$$A^{n(1,s)}_{k(1,s)} \quad \text{are mutually disjoint for } 1 \leq s \leq s(1)$$

$$\sum_{1\leq s\leq s(1)} P(A^{n(1,s)}_{k(1,s)})k(1,s) > 1.$$

Thus, for some events $\overline{A}^{n(1,s)}_{k(1,s)} \subset A^{n(1,s)}_{k(1,s)}$, the equalities

$$\sum_{1\leq s\leq s'(1)} P(\overline{A}^{n(1,s)}_{k(1,s)})k(1,s) = 1 \tag{4}$$

can be obtained.

Assume now that we have defined indices $s(1),\dots,s(j)$ and $n(i,s),k(i,s)$, $1\le i\le j$, $1\le s\le s(i)$ in such a way that, for some events $\overline{A}^{n(i,s)}_{k(i,s)}\subset A^{n(i,s)}_{k(i,s)}$ we have

$$\begin{aligned}&n(i,s)\ge i\vee n(i-1,1)\vee\dots\vee n(i-1,s(i-1)),\\ &k(i,s)>2^i,\end{aligned}\tag{5}$$

and

$$\begin{aligned}&\overline{A}^{n(i,s)}_{k(i,s)},\quad \text{are mutually disjoint for } 1\le i\le j, 1\le s\le s(i),\\ &\sum_{1\le s\le s(i)} P(\overline{A}^{n(i,s)}_{k(i,s)})k(i,s)=1,\quad 1\le i\le j.\end{aligned}\tag{6}$$

By (2), it is now obvious that we can find indices $s(j+1), n(j+1,1),\dots,$ $n(j+1,s(j+1))$, $k(j+1,1),\dots,k(j+1,s(j+1))$, and events

$$\overline{A}^{n(j+1,s)}_{k(j+1,s)}\subset A^{n(j+1,s)}_{k(j+1,s)},\quad 1\le s\le s(j+1)$$

in such a way that conditions (5), (6) are satisfied for any i, $1\le i\le j+1$. Thus (5), (6) can be obtain for any $i\ge 1$.

Now we describe some properties of rectangles

$$B_{i,s}=[2^{-i},2^{-i}+\frac{1}{k(i,s)})\times[a^i_{s-1},a^i_s),\quad i\ge 1,\ 1\le s\le s(i),\tag{7}$$

with

$$\begin{aligned}&0=a^i_0<a^i_1<\dots<a^i_{s(i)}=1,\\ &a^i_s-a^i_{s-1}=P(\overline{A}^{n(i,s)}_{k(i,s)})k(i,s)\ \text{(cf. (5), (6))}.\end{aligned}\tag{8}$$

Obviously, $\lambda^2(B_{i,s})=P(\overline{A}^{n(i,s)}_{k(i,s)})$ and all the rectangles $B_{i,s}$ are mutually disjoint.

By Lemma 2.1, there exists a mapping $T:\Omega\to[0,1)\times[0,1)$, satisfying

$$\begin{aligned}&T^{-1}(B_{i,s})=\overline{A}^{n(i,s)}_{k(i,s)},\\ &P(T^{-1}(C))=\lambda^2(C)\quad\text{for}\quad C\in\ \text{Borel}\ [0,1)\times[0,1).\end{aligned}$$

Let $\mathfrak{A}$ be a σ-field generated by the second coordinate $T^2:\Omega\to\mathbb{R}$ of the transformation $T=(T^1,T^2)$. Then, according to (8) and to the definition (1), the inequalities $a^i_{s-1}\le\alpha<\beta<a^i_s$ imply that

$$\begin{aligned}\mathbb{E}X_{n(i,s)}1_{T^{-1}([0,1)\times[\alpha,\beta))}&\ge\mathbb{E}X_{n(i,s)}1_{T^{-1}(B_{i,s}\cap[0,1)\times[\alpha,\beta))}\\ &\ge k(i,s)\lambda^2(B_{i,s}\cap[0,1)\times[\alpha,\beta))\\ &=k(i,s)\frac{1}{k(i,s)}(\beta-\alpha).\end{aligned}$$

Thus the estimate $\mathbb{E}(X_{n(i,s)}|\mathfrak{A}) \geq 1$ holds almost surely on the set $(T^2)^{-1}([a^i_{s-1}, a^i_s))$. Obviously, $\Omega = \bigcup_{1\leq s\leq s(i)}(T^2)^{-1}([a^i_{s-1}, a^i_s))$ for any $i \geq 1$, and, by (5), $n(i,s) \geq i$ for any $s = 1, \dots, s(i)$. Thus, almost surely, $X_n(\omega)$ does not converges to 0.

3. Consequences of the equivalence of dominated convergence and convergence of all conditionings

By the well-known methods of estimation in the theory of orthogonal series, Theorem 1.1 gives a number of corollaries, see [4] and [2], [3]. All of them are "conditioned" versions of some classical results. The proves of "conditioned" versions need not any new ideas and can be left to the readers.

THEOREM 3.1 (conditioned version of Rademaher–Mienshov theorem). *Let* (Y_n) *be an othogonal sequence in* $L^2(\Omega, \mathcal{F}, P)$ *and*

$$\sum_{n\geq 1} ||Y_n||^2 \log^2 n < \infty.$$

Then $\mathbb{E}(\sum_{i\geq n} Y_i|\mathfrak{A}) \to 0$ *a.s. for any* σ*-field* $\mathfrak{A} \subset \mathcal{F}$.

THEOREM 3.2 (conditioned version of Tandori theorem). *Let* (Y_n) *be an orthogonal sequence in* $L^2(\Omega, \mathcal{F}, P)$ *and let*

$$\sum_{k\geq 1}\Big(\sum_{2^{2^k}<i\leq 2^{2^{k+1}}} ||Y_i||^2 \log^2 i\Big)^{1/2} < \infty.$$

Then $\mathbb{E}(\sum_{i\geq n} Y_{\sigma(i)}|\mathfrak{A}) \to 0$ *a.s. for any permutation* $\sigma(\cdot)$ *of* $\mathbb{N}^+ = \{1, 2, \dots\}$ *and any* σ*-field* $\mathfrak{A} \subset \mathcal{F}$.

THEOREM 3.3 (conditioned version of Gaposhkin theorem). *Let*

$$E : \text{Borel}[-\pi, \pi] \to L^2(\Omega, \mathcal{F}, P)$$

be an orthogonally scattered vector measure. Then, for any σ*-field* $\mathfrak{A} \subset \mathcal{F}$, *the following convergences are equivalent*

$$\mathbb{E}\bigg(\frac{1}{n}\sum_{1\leq k\leq n}\int_{[-\pi,\pi]} e^{ikt}E(dt)|\mathfrak{A}\bigg) \to \mathbb{E}(E\{0\}|\mathfrak{A}) \quad a.s.,$$

and

$$\mathbb{E}(E[-2^{-n}, 0) \cup (0, 2^{-n}]) \mid \mathfrak{A}) \to 0 \quad a.s.$$

Some connections of the Gaposhkin asymptotic with stationary processes are given in [2].

THEOREM 3.4 (conditioned version of Jajte's theorem). *Let*

$$E : \mathrm{Borel}[-\pi,\pi] \to L^2(\Omega,\mathcal{F},P)$$

be an orthogonally scattered vector measure. Then, for any σ-field $\mathfrak{A} \subset \mathcal{F}$, the convergence

$$\mathbb{E}\Big(\sum_{1\le|k|\le n} \frac{1}{k} \int_{[-\pi,\pi]} e^{ikt} E(dt)|\mathfrak{A}\Big) \to 0 \quad a.s.,$$

is equivalent to the condition

$$\mathbb{E}(E[-2^{-n},0) - E(0,2^{-n}]) \mid \mathfrak{A}) \to 0 \quad a.s.$$

THEOREM 3.5 (conditioned version of Banach theorem on arithmetic means of an orthonormal system (see [1])). *Let $X_1, X_2, \ldots$ be an orthonormal sequence in $L_2(\Omega,\mathcal{F},P)$. Then*

$$\mathbb{E}\bigg(\frac{1}{n}(X_1+\ldots+X_n)|\mathfrak{A}\bigg) \to 0 \quad a.s.$$

for any σ-field $\mathfrak{A} \subset \mathcal{F}$.

3.6. Problem. By 3.5, for any i.i.d. random variables $X_1, X_2, \ldots \in L_2(\Omega,\mathcal{F},P)$ we have

$$\mathbb{E}\bigg(\frac{1}{n}(X_1+\ldots+X_n)|\mathfrak{A}\bigg) \to \mathbb{E}X_1 \quad \text{a.s. for any } \sigma\text{-field } \mathfrak{A} \subset \mathcal{F}. \tag{9}$$

The author does not know for which $p \in (1,2)$ the assumption that $X_1, X_2, \ldots$ are i.i.d. in $L_p(\Omega,\mathcal{F},P)$ implies (9).

It is well-known that the convergences in Birkhoff ergodic theorem and in Kolmogorov S.L.L.N. for i.i.d. random variables are not necessarily dominated. We describe the typical example in the proof of Theorem 3.7. Thus, our Theorem 1.3 gives the following two theorems.

THEOREM 3.7. *For any non-atomic probability space $(\Omega,\mathcal{F},P)$, there exist i.i.d. random variables $X_n \in L^1(\Omega,\mathcal{F},P)$ satisfying $\mathbb{E}X_n = 0$ and*

$$\mathbb{E}\bigg(\frac{1}{n}(X_1+\ldots+X_n)|\mathfrak{A}\bigg) \not\to 0 \quad a.s.$$

for some σ-field $\mathfrak{A} \subset \mathcal{F}$.

Proof. For a given $\epsilon > 0$, take $n(\epsilon) \in \mathbb{N}$ satisfying $1 + \frac{1}{2} + \ldots + \frac{1}{n(\epsilon)} > \frac{2}{\epsilon}$, and then take $\eta(\epsilon) > 0$ satisfying $(1-\eta(\epsilon))^{n(\epsilon)} > \frac{1}{2}$ and $n(\epsilon)\eta(\epsilon) < \epsilon$. For any independent random variables $X_1^\epsilon, X_2^\epsilon, \ldots$ with distribution

$$p_\epsilon = \eta(\epsilon)\delta_{\epsilon/\eta(\epsilon)} + (1-\eta(\epsilon))\delta_0 \tag{10}$$

(as usually, δ_x denotes the Dirac distribution concentrated at x), and for a set $Z_\epsilon = (X_i^\epsilon > 0$ for exactly one index $i = 1, \ldots, n(\epsilon))$, we have $P(Z_\epsilon) < \epsilon$,

$$\mathbb{E}X_i^\epsilon = \epsilon, \quad X_i^\epsilon \geq 0, \tag{11}$$

and

$$\mathbb{E}\left(\max_{1\leq n\leq n(\epsilon)} \left| \frac{1}{n}(X_1^\epsilon + \ldots + X_n^\epsilon) \right| \right) \geq \mathbb{E}\left(\mathbf{1}_{Z_\epsilon} \max_{1\leq n\leq n(\epsilon)} \frac{1}{n}(X_1^\epsilon + \ldots + X_n^\epsilon) \right) = \eta(\epsilon)(1-\eta(\epsilon))^{n(\epsilon)-1} \cdot \frac{\epsilon}{\eta(\epsilon)} \left(1 + \ldots + \frac{1}{n(\epsilon)} \right) > 1.$$

It is obvious that, for any $\epsilon(1) > \epsilon(2) > \ldots > 0$, we can find random variables $X_1^{\epsilon(i)}, X_2^{\epsilon(i)}, \ldots$ with the same distribution $p_{\epsilon(i)}$ of the form (10) and such that the systems

$$(X_1^{\epsilon(1)}, X_1^{\epsilon(2)}, \ldots), (X_2^{\epsilon(1)}, X_2^{\epsilon(2)}, \ldots), \ldots$$

are independent. By (11), for $\epsilon(1), \epsilon(2), \ldots$ given by a suitable induction, random variables

$$X_1 = X_1^{\epsilon(1)} + X_1^{\epsilon(2)} + \ldots, \quad X_2 = X_2^{\epsilon(1)} + X_2^{\epsilon(2)} + \ldots, \ldots$$

satisfy $\mathbb{E}(\sup_{n\geq 1} |\frac{1}{n}(X_1+\ldots+X_n - \mathbb{E}(X_1+\ldots+X_n))|) = \infty$. Thus Theorem 1.3 can be used. ∎

THEOREM 3.8. *In any non-atomic probability space, there exist an endomorphism $T : \Omega \to \Omega$, a function $f \in L^2(\Omega, \mathcal{F}, P)$ and a σ-field $\mathfrak{A} \subset \mathcal{F}$ such that $\mathbb{E}(\frac{1}{n}\sum_{1\leq i\leq n} f \circ T^i | \mathfrak{A})$ does not converge almost surely.*

The proof of Theorem 3.8 can be obtained by the construction of T and f in such a way that random variables

$$X_1 = f \circ T, \qquad X_2 = f \circ T^2, \ldots$$

are independent and with the same distributions as $X_1, X_2, \ldots$ constructed in 3.7.

References

[1] S. Banach, *Sur la valuer moyenne des fonctions orthogonales*, Bull. Inst. Acad. Pol. Sci. Lettr. (1929), 66–72.

[2] V. F. Gaposhkin, *Criteria of the strong law of large numbers for some classes of stationary processes and homogeneous random fields*, Theory Probab. Appl. 22 (1977), 295–319 (in Russian).

[3] R. Jajte, *On the existence of the ergodic Hilbert transform*, Ann. Probab. 15 (1987), 831–825.

[4] B. S. Kashin and A. A. Saakyan, *Orthogonal Series*, Amer. Math. Soc. Press, 1989.

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