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SOME REMARKS ON QUADRATIC FORMS IN STABLE RANDOM VARIABLES

Dedicated to Professor Kazimierz Urbanik

Abstract. We consider quadratic forms that appear in the least squares estimator for the unknown parameter in the AR(1) model with stable innovations. In three cases we obtain different limit distributions.

1. Introduction

Consider the following AR(1) model

$$X_0 = 0 \qquad \text{a.s.}$$

and

$$X_k = \gamma X_{k-1} + \epsilon_k, \qquad k = 1, 2, \ldots,$$

where $\epsilon_1, \epsilon_2, \ldots$ are independent identically distributed (i.i.d.) with a stable distribution function $(F(\cdot; \alpha, \beta))$ and satisfy the stability property

$$\epsilon_1 + \cdots + \epsilon_k \stackrel{\mathrm{d}}{=} k^{\frac{1}{\alpha}} \epsilon_1 \tag{1.1}$$

for $k = 1, 2, \ldots$. $\stackrel{\mathrm{d}}{=}$ stands for: has the same distribution as. This implies that we restrict ourselves to strictly stable random variables. Thus, for $\alpha = 1$ we restrict ourselves to the case $\beta = 0$ (Cauchy distribution).

The least squares estimator $\hat{\gamma}_n$ for γ is given by

$$\hat{\gamma}_n = \Big(\sum_{k=1}^{n} X_{k-1} X_k\Big)\Big(\sum_{k=1}^{n} X_{k-1}^2\Big)^{-1}$$

and satisfies

$$\hat{\gamma}_n - \gamma = \Big(\sum_{k=1}^{n} X_{k-1} \epsilon_k\Big)\Big(\sum_{k=1}^{n} X_{k-1}^2\Big)^{-1}. \tag{1.2}$$

In this paper we consider only the two quadratic forms that appear on the right hand side (r.h.s.) of (1.2). We restrict ourselves to random variables with a strictly stable distribution and do not consider random variables in the domain of attraction. We do not make the assumption that ϵ_1 is symmetrically distributed. We also consider $\beta \neq 0$. Only for $\alpha = 1$ we assume symmetry. In papers on quadratic forms or double stochastic integrals one often makes this assumption. See Kwapien and Woyczynski (1992) and Hu and Woyczynski (1995).

We make use of the notation as used in Mijnheer (1998a). The random variables also satisfy the following stability property. For $s, t > 0$ we have

$$s^{\frac{1}{\alpha}}\epsilon_1 + t^{\frac{1}{\alpha}}\epsilon_2 \stackrel{\mathrm{d}}{=} (s+t)^{\frac{1}{\alpha}}\epsilon_1. \tag{1.3}$$

From the theory of time series with innovations with a finite variance we know that we have to distinguish the cases $|\gamma| < 1$, $|\gamma| = 1$ and $|\gamma| > 1$. In the (non-normal) stable case we also have to distinguish $\gamma > 0$ and $\gamma < 0$. See Mijnheer (1998a).

Let $e_n' = (\epsilon_1, \ldots, \epsilon_n)$ and Γ_n a symmetric $n \times n$-matrix. We write the quadratic forms as $e_n'\Gamma_n e_n$.

2. The case $0 < \gamma < 1$

We first consider the case where $\Gamma_n = (\gamma_{i,j})$ is given by

$$\begin{cases} \gamma_{i,j} = 0 & \text{for } i = j \\ \gamma^{j-i-1} & \text{for } 1 \le i < j \le n. \end{cases} \tag{2.1}$$

Then we have

$$\Gamma_{2n} = \begin{pmatrix} \Gamma_n & A_n \\ A_n^{\mathrm{T}} & \Gamma_n \end{pmatrix} \tag{2.2}$$

where $A_n = (a_{i,j})$ is given by

$$a_{i,j} = \gamma^{n+j-i-1} \qquad \text{for} \quad 1 \le i < j \le n.$$

A_n is not symmetric.

Define $\widetilde{e}_n' = (\epsilon_{n+1}, \ldots, \epsilon_{2n})$. Then we have

$$e_{2n}'\Gamma_{2n}e_{2n} = e_n'\Gamma_n e_n + \widetilde{e}_n'\Gamma_n\widetilde{e}_n + 2e_n'A_n\widetilde{e}_n \tag{2.3}$$

and

$$\begin{aligned} e_n'A_n\widetilde{e}_n &= (\epsilon_{n+1} + \cdots + \gamma^{n-1}\epsilon_{2n})(\gamma^{n-1}\epsilon_1 + \cdots + \epsilon_n) \\ &\stackrel{\mathrm{d}}{=} (1-\gamma^{n\alpha})^{\frac{2}{\alpha}}(1-\gamma^{\alpha})^{-\frac{2}{\alpha}}\epsilon_1\epsilon_2 \end{aligned} \tag{2.4}$$

by using stability property (1.3). In Mijnheer (1998b) we have derived the tail behavior of $\epsilon_1\epsilon_2$.

PROPOSITION 2.1.

$$P(\epsilon_1\epsilon_2 > x) = c_1 x^{-\alpha}(1+o(1))\log x \qquad \textit{for} \quad x \to \infty.$$

Making use of this proposition we have

$$e_n' A_n \widetilde{e}_n \cdot n^{-\frac{1}{\alpha}}(\log n)^{-\frac{1}{\alpha}} \xrightarrow{\mathrm{P}} 0 \qquad \text{for} \quad n \to \infty. \tag{2.5}$$

$\xrightarrow{\mathrm{P}}$ means: convergence in probability. Let $Y_1, Y_2, \ldots$ be independent copies with the limit distribution of $(n \log n)^{-\frac{1}{\alpha}} e_n' \Gamma_n e_n$ for $n \to \infty$. From (2.3) and (2.5) we obtain

$$Y_1 + Y_2 \stackrel{\mathrm{d}}{=} 2^{\frac{1}{\alpha}} Y_1.$$

We may also consider $e_{3n}' \Gamma_{3n} e_{3n}$. Similarly we obtain

$$Y_1 + Y_2 + Y_3 \stackrel{\mathrm{d}}{=} 3^{\frac{1}{\alpha}} Y_1.$$

Now one may use criterion 3 on p. 14 of Zolotarev (1986). But one has to prove that Y_1 is non-degenerated or, in other words, that $(n \log n)^{\frac{1}{\alpha}}$ is the proper choice for the norming constants. In Mijnheer (1998b) we proved

PROPOSITION 2.2.

$$(n \log n)^{-\frac{1}{\alpha}} e_n' \Gamma_n e_n \quad \xrightarrow{\mathrm{d}} \quad \textit{stable law}$$

for $n \to \infty$.

Next we consider the quadratic form $\sum_{k=1}^{n} X_{k-1}^2$. From the $AR(1)$ model, as described in section 1, we obtain

$$X_n^2 + (1-\gamma^2)\sum_{k=1}^{n} X_{k-1}^2 = 2\gamma \sum_{k=1}^{n} X_{k-1}\epsilon_k + \sum_{k=1}^{n} \epsilon_k^2. \tag{2.6}$$

The random variables $\epsilon_1^2, \epsilon_2^2, \ldots$ are i.i.d. and in the domain of normal attraction of the stable law $F(\cdot; \frac{\alpha}{2}, 1)$. The norming constants are $c_2^{-1} n^{\frac{2}{\alpha}}$, $n = 1, 2, \ldots$, for some constant c_2. From the foregoing results in this section we derive that $n^{-\frac{2}{\alpha}} \sum_{k=1}^{n} X_{k-1}\epsilon_k \xrightarrow{\mathrm{P}} 0$ for $n \to \infty$. We also have, using the stability property (1.3), $X_n \stackrel{\mathrm{d}}{=} (1-\gamma^{n\alpha})^{\frac{1}{\alpha}}(1-\gamma^{\alpha})^{-\frac{1}{\alpha}}\epsilon_1$. Since $0 < \gamma < 1$, we have $n^{-\frac{2}{\alpha}} X_n^2 \xrightarrow{\mathrm{P}} 0$ for $n \to \infty$. In theorem 2.3 in Mijnheer (1998b) we proved the following proposition.

PROPOSITION 2.3. *For* $n \to \infty$

$$\Big(c_1 (n \log n)^{-\frac{1}{\alpha}} \sum_{k=1}^{n} X_{k-1}\epsilon_k, c_2 n^{-\frac{2}{\alpha}} \sum_{k=1}^{n} \epsilon_k^2\Big) \xrightarrow{\mathrm{d}} (S_1, S_0)$$

where S_0 *and* S_1, *are independent with stable distributions* $F(\cdot;\frac{\alpha}{2},1)$ *and* $F(\cdot;\alpha,\widetilde{\beta})$.

The parameter $\widetilde{\beta}$ is given in Proposition 2.1. in that paper. If we write $\beta = p^\alpha - q^\alpha$ with $p^\alpha + q^\alpha = 1$. Then $\widetilde{\beta} = (p^{2\alpha} + q^{2\alpha})^{\frac{1}{\alpha}} - 2^{\frac{1}{\alpha}}pq$.

3. The case $\gamma = 1$

In this section we first consider the matrix $\Gamma_n = (\gamma_{i,j})$ where $\gamma_{i,j} = 1-\delta_{i,j}$ and $\delta_{i,j}$ is Kronecker's delta function. The matrix has the property as given in (2.2) with $A_n = (a_{i,j})$ and $a_{i,j} \equiv 1$, $1 \le i,j \le n$. Again relation (2.3) is true. But now we have

$$\begin{aligned} e_n' A_n \widetilde{e}_n &= (\epsilon_1 + \cdots + \epsilon_n)(\epsilon_{n+1} + \cdots + \epsilon_{2n}) \\ &\stackrel{\mathrm{d}}{=} n^{\frac{2}{\alpha}} \epsilon_1 \epsilon_2. \end{aligned} \tag{3.1}$$

Let $\{S(t) : 0 \le t \le 1\}$ be a stable process such that $S(1) \stackrel{\mathrm{d}}{=} \epsilon_1$. One easily proves, for $n \to \infty$,

$$n^{-\frac{2}{\alpha}} e_n' \Gamma_n e_n \stackrel{\mathrm{d}}{\longrightarrow} 2\int_0^1 S(t-)dS(t). \tag{3.2}$$

The tail behavior of the double stable integral on the r.h.s. of (3.2) is, in the case $0 < \alpha < 1$ and $\beta = 1$, given in Mijnheer (1991). For the general case, see remark 2 in Mijnheer (1997). More information about the norming constant $n^{\frac{2}{\alpha}}$ in the left hand side (l.h.s.) of (3.2) will be given in appendix 1.

Consider relation (2.3). Making use of (3.1) and (3.2), let $Y_1, Y_2, \ldots$ stand for independent copies of the r.v. on the r.h.s. of (3.2), dividing (2.3) by $n^{\frac{2}{\alpha}}$ gives, for $n \to \infty$,

$$2^{\frac{2}{\alpha}} Y_1 \stackrel{\mathrm{d}}{=} Y_1 + Y_2 + 2\epsilon_1\epsilon_2.$$

Remark that $2\epsilon_1\epsilon_2 = e_2'\Gamma_2 e_2$. Next we consider $e_{3n}'\Gamma_{3n}e_{3n}$. Similarly we obtain

$$3^{\frac{2}{\alpha}} Y_1 \stackrel{\mathrm{d}}{=} Y_1 + Y_2 + Y_3 + e_3'\Gamma_3 e_3.$$

The random variable Y_1 belongs to the domain of (non-normal) attraction of a stable law with characteristic exponent α.

Now we consider $\sum_{k=1}^{n} X_k^2$. Then

$$\begin{aligned} n^{-1-\frac{2}{\alpha}} \sum_{k=1}^{n} X_k^2 = n^{-1-\frac{2}{\alpha}} \sum_{k=1}^{n} (\epsilon_1 + \cdots + \epsilon_k)^2 \stackrel{\mathrm{d}}{=} n^{-1} \sum_{k=1}^{n} S^2\left(\frac{k}{n}\right) \\ \stackrel{\mathrm{d}}{\longrightarrow} \int_0^1 S^2(t)dt \qquad \text{for} \quad n \to \infty. \end{aligned}$$

In matrix notation $\sum_{k=1}^{n} X_k^2 = e_n' \Gamma_n e_n$, where $\Gamma_n = (\gamma_{i,j})$ with $\gamma_{i,j} = n-j+1$ for $1 \le i \le j \le n$. We write $\Gamma_n = B_n + C_n$, where $B_n = (b_{i,j})$ with

$$b_{i,j} = \begin{cases} 0 & \text{for } i = j \\ n-j+1 & \text{for } 1 \le i < j \le n. \end{cases}$$

Thus B_n is equal to Γ_n except there are zeros on the diagonal. Then $e_n' C_n e_n = \sum_{k=1}^{n} (n-k+1)\epsilon_k^2$. The random variables $\epsilon_1^2, \epsilon_2^2, \ldots$ are i.i.d. and in the domain of normal attraction of the law $F(\cdot; \frac{\alpha}{2}, 1)$. It is easy to prove that, for some constant c, $cn^{-1-\frac{2}{\alpha}} \sum_{k=1}^{n} (n-k+1)\epsilon_k^2$ converges in distribution to a r.v. S_0 with distribution function $F(\cdot; \frac{\alpha}{2}, 1)$. Using the tail expansion of ϵ_1 one obtains the behavior of the characteristic function of ϵ_1^2 near the origin. Then we obtain that S_0 is non-degenerated. Next we continue as in section 2 and make use of criterion 3 mentioned in Zolotarev (1986 p. 14) in order to conclude that S_0 has a stable distribution.

REMARK. In the case $\alpha = 2$, i.e. the innovations have a standard normal distribution, one easily shows $n^{-2} e_n' C_n e_n \xrightarrow{\text{P}} \frac{1}{2}$ for $n \to \infty$.

Next we consider $e_n' B_n e_n$. First we assume that $0 < \alpha < 1$ and $\beta = 1$, i.e. the random variable ϵ_1 is positive. Then we have

$$n \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} \epsilon_i \epsilon_j < e_{2n}' B_{2n} e_{2n} < 2n \sum_{\substack{i=1 \\ i \neq j}}^{2n} \sum_{j=1}^{2n} \epsilon_i \epsilon_j.$$

Divide by $(2n)^{1+\frac{2}{\alpha}}$ and let $n \to \infty$. We obtain the following result.

PROPOSITION 3.1. *In the case* $0 < \alpha < 1$ *and* $\beta = 1$

$$\int_0^1 S^2(t)dt = cS_0 + B$$

where S_0 *has distribution function* $F(\cdot; \frac{\alpha}{2}, 1)$ *and the random variable* B *is non-degenerated. The tail of* S_0 *dominates the tail of* B.

In the general case we may use Remark 2 from Mijnheer (1997), in order to obtain the assertion as above.

4. The case $\gamma > 1$

As in the previous sections we first consider $\sum_{k=1}^{n} X_{k-1}\epsilon_k = \frac{1}{2} e_n' \Gamma_n e_n$, where Γ_n is given by (2.1). Making use of (2.3) and $e_n' A_n \widetilde{e}_n \stackrel{\text{d}}{=}$

$(\gamma^{n\alpha}-1)^{\frac{2}{\alpha}}(\gamma^{\alpha}-1)^{-\frac{2}{\alpha}}\epsilon_1\epsilon_2$ we obtain, after dividing by γ^{2n} and taking the limit for $n\to\infty$, that for $Y=\lim_{n\to\infty}\gamma^{-2n}e'_{-2n}\Gamma_{2n}e_{2n}$ that, for some $c>0$,

$$Y \stackrel{\mathrm{d}}{=} 2c\epsilon_1\epsilon_2. \tag{4.1}$$

This is the analogue of Theorem 2.2 in Anderson (1959). Thus $e'_n A_n \widetilde{e}_n$ dominates in (2.3).

Next we rewrite (2.6)

$$X_n^2 - 2\gamma\sum_{k=1}^{n} X_{k-1}\epsilon_k = (\gamma^2-1)\sum_{k=1}^{n} X_{k-1}^2 + \sum_{k=1}^{n}\epsilon_k^2. \tag{4.2}$$

From the model it follows that $X_n \stackrel{\mathrm{d}}{=} (\gamma^{n\alpha}-1)^{\frac{1}{\alpha}}(\gamma^{\alpha}-1)^{-\frac{1}{\alpha}}\epsilon_1$. The random variables $\epsilon_1^2,\epsilon_2^2,\ldots$ belong to the domain of attraction of the stable law $F(\cdot;\frac{\alpha}{2},1)$. Therefore $\gamma^{-2n}\sum_{k=1}^{n}\epsilon_k^2 \to 0$ a.s. for $n\to\infty$. We also have, for $n\to\infty$, $\gamma^{-2n}\sum_{k=1}^{n}X_{k-1}\epsilon_k \xrightarrow{\mathrm{P}} 0$. Thus we have proved that, for $n\to\infty$,

$$\gamma^{-2n}\Big\{X_n^2-(\gamma^2-1)\sum_{k=1}^{n}X_{k-1}^2\Big\} \xrightarrow{\mathrm{P}} 0. \tag{4.3}$$

In the finite variance case this result is proved in Theorem 2.1 in Anderson (1959). He also asserts a.s. convergence in (4.3) when the innovations have a finite variance.

Now we can formulate and prove a similar assertion as in the case $\gamma=1$.

THEOREM 4.1. *The limit distribution of* $\gamma^{-2n}\sum_{k=1}^{n}X_{k-1}^2$, *for* $n\to\infty$, *is given by* cS_0+B, *where* S_0 *has distribution function* $F(\cdot;\frac{\alpha}{2},1)$ *and the tail of* B *is dominated by the tail of* S_0.

Proof. In view of (4.3) we may also consider $\gamma^{-2n}X_n^2$. We have

$$\gamma^{-2n}X_n^2 = \sum_{k=1}^{n}\gamma^{-2k}\epsilon_k^2 + \sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=1}^{n}\gamma^{-i-j}\epsilon_i\epsilon_j. \tag{4.4}$$

As described in section 3 we show that $\sum_{k=1}^{n}\gamma^{-2k}\epsilon_k^2 \xrightarrow{\mathrm{d}} cS_0$ for $n\to\infty$. In appendix 2 we shall prove that the second sum on the r.h.s. of (4.4) converges in distribution to a r.v. B and $P(|B|>x)<x^{-\alpha}\log x$ for $x\to\infty$. ∎

5. Appendix 1. On the norming constants

Let ϵ_1 have a stable distribution function $F(\cdot;\alpha,1)$ with $0<\alpha<1$. Let $\Gamma_n=(\gamma_{i,j})$ with $\gamma_{i,j}=1-\delta_{i,j}$ as in section 3. In Mijnheer (1995) we proved that

$$\mathrm{E}\left\{(e_n'\Gamma_n e_n)(\epsilon_{(n)}\epsilon_{(n-1)})^{-1}\right\} \to \text{ constant } \neq 0$$

for $n \to \infty$, where $\epsilon_{(1)} < \cdots < \epsilon_{(n-1)} < \epsilon_{(n)}$ are the ordered random variables $\epsilon_1, \dots, \epsilon_n$. Thus we see a dominant influence of $\epsilon_{(n)}\epsilon_{(n-1)}$ in $e_n'\Gamma_n e_n$. If we apply arguments – well-known in the theory of order statistics and U-statistics we may write

$$\frac{1}{2}e_n'\Gamma_n e_n = \epsilon_{(n)}\epsilon_{(n-1)} + (\epsilon_{(n)} + \epsilon_{(n-1)})\sum_{i=1}^{n-2} Y_i + \frac{1}{2}\sum_{\substack{i=1\\ i\neq j}}^{n-2}\sum_{j=1}^{n-2} Y_iY_j \tag{5.1}$$

where $Y_1, \dots, Y_{n-2}$ are the unordered random variables $\epsilon_{(1)}, \dots, \epsilon_{(n-2)}$. If we condition on $\epsilon_{(n-1)}$ one easily shows that the last two sums on the r.h.s. of (5.1) are small w.r.t. $\epsilon_{(n)}\epsilon_{(n-1)}$. In order to see that $n^{\frac{2}{\alpha}}$ is the right norming constant, we consider a r.v. in the domain of attraction of $F(\cdot;\alpha,1)$. Let $U_1, U_2, \dots, U_n$ be i.i.d. with a uniform distribution on $(0,1)$. Let $U_{(1)} < \cdots < U_{(n)}$ be the order statistics.

PROPOSITION 5.1. *For x fixed and large*

$$\lim_{n\to\infty} P(U_{(1)}^{-\frac{1}{\alpha}}U_{(2)}^{-\frac{1}{\alpha}} > n^{\frac{2}{\alpha}}x) = cx^{-\alpha}\log x.$$

Proof. One easily checks that $U_1^{-\frac{1}{\alpha}}$ belongs to the domain of normal attraction of $F(\cdot;\alpha,1)$. Let $y = n^{-2}x^{-\alpha}$. Then

$$\begin{aligned} P(U_{(1)}^{-\frac{1}{\alpha}}U_{(2)}^{-\frac{1}{\alpha}} > n^{\frac{2}{\alpha}}x) &= P(U_{(1)}U_{(2)} < y) \\ &= \int_y^{y^{\frac{1}{2}}}\int_u^{u^{-1}y} n(n-1)(1-v)^{n-2}\,dv\,du + \text{error}. \end{aligned} \tag{5.2}$$

The error is $O(n^{-1})$ for $n \to \infty$ and x fixed. The assertion follows by computing the double integral in the r.h.s. of (5.2). ∎

6. Appendix 2. Tail behavior of B

In this appendix we shall complete the proof of Theorem 4.1.

First we consider (4.4) in the case $\alpha = 2$ (i.e. ϵ_1 has a standard normal distribution). Martingale theory gives the existence of $Y = \sum_{k=1}^{\infty} \gamma^{-2k}(\epsilon_k^2 - 1)$.

Define the sequence Z_n by $Z_n = \sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=1}^{n} \gamma^{-i}\gamma^{-j}\epsilon_i\epsilon_j$. Then we have

$$Z_{n+1} - Z_n = 2(\gamma^{-n-1}\epsilon_{n+1})\Big(\sum_{i=1}^{n}\gamma^{-i}\epsilon_i\Big)$$
$$\stackrel{\mathrm{d}}{=} 2\gamma^{-n-1}(1-\gamma^{-n\alpha})^{\frac{1}{\alpha}}(\gamma^{\alpha}-1)^{-\frac{1}{\alpha}}\epsilon_1\epsilon_{n+1}.$$

For $\alpha = 2$

$$\sum_{n=1}^{\infty}\mathrm{E}(Z_{n+1}-Z_n)^2 = 4\sum_{n=1}^{\infty}\gamma^{-2n-2}(1-\gamma^{-2n})(\gamma^2-1)^{-1} < \infty.$$

Martingale theory gives us that, for $n \to \infty$, $Z_n \to Z$ a.s. and in $\mathcal{L}^2$. See, for example, Williams (1990) Theorem 12.1. Thus for $\alpha = 2$ we have, for $n \to \infty$,

$$\gamma^{-2n}X_n^2 \xrightarrow{\mathrm{d}} (\gamma^2-1)^{-1} + Y + Z,$$

where Y and Z are as above; random variables with zero expectations and finite variances. We can prove the same assertion if we assume $\epsilon_1, \epsilon_2, \dots$ i.i.d., $\mathrm{E}\epsilon_1 = 0$ and $\sigma^2(\epsilon_1) < \infty$. See Varberg (1968).

The case $\alpha \neq 2$. Again we define $Z_n = \sum\limits_{\substack{i=1\\ i\neq j}}^{n}\sum\limits_{j=1}^{n}\gamma^{-i}\gamma^{-j}\epsilon_i\epsilon_j$. Then we have

$$Z_{2n} = Z_n + 2\gamma^{-n}\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma^{-i}\gamma^{-j}\epsilon_{n+i}\epsilon_j + \gamma^{-2n}\sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=1}^{n}\gamma^{-i}\gamma^{-j}\epsilon_{n+i}\epsilon_{n+j}.$$

For each n we have $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}\gamma^{-i}\gamma^{-j}\epsilon_{n+i}\epsilon_j \stackrel{\mathrm{d}}{=} c_n\epsilon_1\epsilon_2$ where c_n converges to some constant c for $n \to \infty$. The sum $\sum\limits_{\substack{i=1\\ i\neq j}}^{n}\sum\limits_{j=1}^{n}\gamma^{-i}\gamma^{-j}\epsilon_{n+i}\epsilon_{n+j}$ has the same distribution as Z_n and is independent of Z_n.

We follow the proof of the inequality in Theorem 1 in Mijnheer (1995). Define $A_n = \max\limits_{1\le i\le n}\gamma^{-i}|\epsilon_i|$ and A_{n-1} the second largest

$$\begin{aligned} P(|Z_n| \ge x) &= P(|Z_n| \ge x \wedge Z_n \le x^{\frac{1}{2}}) \\ &\quad + P(|Z_n| \ge x \wedge x^{\frac{1}{2}} \le A_n \le x(\log x)^{-\frac{1}{\alpha}} \wedge A_nA_{n-1} \le x) \\ &\quad + P(|Z_n| \ge x \wedge x^{\frac{1}{2}} \le A_n \le x(\log x)^{-\frac{1}{\alpha}} \wedge A_nA_{n-1} > x) \\ &\quad + P(|Z_n| \ge x \wedge A_n > x(\log x)^{-\frac{1}{\alpha}}) \\ &= P_1 + P_2 + P_3 + P_4. \end{aligned}$$

Now we have

$$P_4 \le P(\max_{1\le i\le n} \gamma^{-i}|\epsilon_i| > x(\log x)^{-\frac{1}{\alpha}})$$
$$\le \sum_{i=1}^{n} P(|\epsilon_i| > \gamma^i x(\log x)^{-\frac{1}{\alpha}})$$
$$\le \sum_{i=1}^{n} \frac{c\log x}{\gamma^{i\alpha}x^{\alpha}}.$$
$$P_3 \le P(A_n A_{n-1} > x) = P(\max_{1\le i,j\le n} \gamma^{-i}\gamma^{-j}|\epsilon_i||\epsilon_j| > x)$$
$$\le \sum_{\substack{i=1\\ i\ne j}}^{n}\sum_{j=1}^{n} P(|\epsilon_i||\epsilon_j| > \gamma^{i+j}x)$$
$$\le \sum_{\substack{i=1\\ i\ne j}}^{n}\sum_{j=1}^{n} \frac{c(i+j)\log\gamma + c\log x}{\gamma^{(i+j)\alpha}x^{\alpha}} \le \frac{c_1\log x}{x^{\alpha}}$$

for all n.

Next we consider P_1. Define

$$W_i = \begin{cases} |\epsilon_i| & \text{for } |\epsilon_i| \le \gamma^i x^{\frac{1}{2}} \\ 0 & \text{otherwise.} \end{cases}$$

Then for large x and $0 < \alpha < 1$

$$\mathrm{E}W_i \sim c_1\gamma^{i(1-\alpha)}x^{\frac{1-\alpha}{2}} \tag{6.1}$$

and

$$\mathrm{E}W_i^2 \sim c_2\gamma^{i(2-\alpha)}x^{\frac{2-\alpha}{2}}. \tag{6.2}$$

Next we define $W_i = \mu_i + \sigma_i V_i$, where V_i, $i = 1,\dots,n$, are i.i.d. with $\mathrm{E}V_i = 0$ and $\sigma^2(V_i) = 1$. The asymptotics for μ_i and σ_i^2 are given in (6.1) and (6.2). We have

$$|Z_n| \le \sum_{\substack{i=1\\ i\ne j}}^{n}\sum_{j=1}^{n} \gamma^{-i}|\epsilon_i|\gamma^{-j}|\epsilon_j|$$
$$= \sum_{\substack{i=1\\ i\ne j}}^{n}\sum_{j=1}^{n} \gamma^{-i-j}\mu_i\mu_j + 2\sum_{\substack{i=1\\ i\ne j}}^{n}\sum_{j=1}^{n} \gamma^{-i-j}\mu_i\sigma_j V_j + \sum_{\substack{i=1\\ i\ne j}}^{n}\sum_{j=1}^{n} \gamma^{-i-j}\sigma_i\sigma_j V_i V_j.$$

Then

$$\sum_{\substack{i=1\\i\neq j}}^{n}\sum_{j=1}^{n}\gamma^{-i-j}\mu_i\mu_j \sim \sum_{\substack{i=1\\i\neq j}}^{n}\sum_{j=1}^{n}c_1^2\gamma^{-i\alpha}\gamma^{-j\alpha}x^{1-\alpha}$$

$$\sim cx^{1-\alpha} \qquad \text{for large } x \text{ and } n.$$

Also

$$\sum_{\substack{i=1\\i\neq j}}^{n}\sum_{j=1}^{n}\gamma^{-i-j}\sigma_i\sigma_j V_i V_j \sim \sum_{\substack{i=1\\i\neq j}}^{n}\sum_{j=1}^{n}c_2\gamma^{-\frac{i\alpha}{2}}\gamma^{-\frac{j\alpha}{2}}x^{\frac{2-\alpha}{2}}V_iV_j$$

$$\sim c_2x^{1-\frac{\alpha}{2}}\sum_{\substack{i=1\\i\neq j}}^{n}\sum_{j=1}^{n}\gamma^{-(i+j)\frac{\alpha}{2}}V_iV_j.$$

We noticed in the beginning of this section that $\sum_{\substack{i=1\\i\neq j}}^{n}\sum_{j=1}^{n}\gamma^{-(i+j)\frac{\alpha}{2}}V_iV_j$ converges a.s., for $n \to \infty$, to a random variable Z with $\mathrm{E}Z = 0$ and $\sigma^2(Z) < \infty$. Chebyshev's inequality gives us

$$P(cx^{1-\frac{\alpha}{2}}Z > x) \le c_3\frac{x^{2-\alpha}\mathrm{E}Z^2}{x^2} = c_4x^{-\alpha}.$$

Finally we have, for large n and x,

$$\sum_{\substack{i=1\\i\neq j}}^{n}\sum_{j=1}^{n}\gamma^{-i-j}\mu_i\sigma_jV_j \sim \Big(\sum_{i=1}^{n}\gamma^{-i}\mu_i\Big)\Big(\sum_{j=1}^{n}\gamma^{-j}\sigma_jV_j\Big),$$

where

$$\sum_{i=1}^{n}\mu_i\gamma^{-i} \sim \sum_{i=1}^{n}c_1\gamma^{-i\alpha}x^{\frac{1-\alpha}{2}} \sim cx^{\frac{1-\alpha}{2}}$$

and

$$\sum_{j=1}^{n}\sigma_jV_j\gamma^{-j} \sim \sum_{j=1}^{n}c_2^{\frac{1}{2}}\gamma^{-j\frac{\alpha}{2}}x^{\frac{2-\alpha}{4}}V_j \sim c_2^{\frac{1}{2}}x^{\frac{1}{2}-\frac{\alpha}{4}}\sum_{j=1}^{n}\gamma^{-j\frac{\alpha}{2}}V_j.$$

The r.v. $\sum_{j=1}^{n}\gamma^{-j\frac{\alpha}{2}}V_j$ converges a.s. for $n \to \infty$ to a r.v. with a normal distribution. Again we apply Chebyshev's inequality.

For $1 < \alpha < 2$ we define

$$W_i = \begin{cases}\epsilon_i & \text{for } |\epsilon_i| \le \gamma^i x^{\frac{1}{2}}\\ 0 & \text{otherwise}\end{cases}$$

and proceed in the same way as above.

For $\alpha = 1$ we have $\mu_i = 0$.

Finally we consider

$$P_2 = P(|Z_n| \geq x \wedge x^{\frac{1}{2}} \leq A_n \leq x(\log x)^{-\frac{1}{\alpha}} \wedge A_n A_{n-1} \leq x)$$
$$\leq \sum_{i=1}^{n} P(|Z_n| \geq x \wedge x^{\frac{1}{2}} \leq \gamma^{-i}|\epsilon_i| \leq x(\log x)^{-\frac{1}{\alpha}} \wedge A_n A_{n-1} \leq x).$$

We now write

$$Z_n = Z_n^{(1)} + Z_n^{(2)}$$

where the summands in $Z_n^{(2)}$ do not contain A_n and $Z_n^{(1)} = A_n D_n$. Here $D_n = \sum_{\substack{j=1\\ j\neq i}}^{n} \gamma^{-j}\epsilon_j$. Then

$$P(|Z_n^{(1)}| \geq x/2 \wedge x^{\frac{1}{2}} \leq \gamma^{-i}|\epsilon_i| \leq x(\log x)^{-\frac{1}{\alpha}} \wedge A_n A_{n-1} \leq x)$$
$$\leq P(|\epsilon_1\epsilon_2| \geq cx)$$

by using the stability property (1.3)

$$\leq c(\log x)x^{-\alpha}$$

from the tail expansion of $\epsilon_1\epsilon_2$ as given in proposition (2.1). Consider

$$P(|Z_n^{(2)}| \geq x/2 \wedge x^{\frac{1}{2}} \leq \gamma^{-i}|\epsilon_i| \leq x(\log x)^{-\frac{1}{\alpha}} \wedge A_n A_{n-1} \leq x). \tag{6.3}$$

Given $A_n = \gamma^{-i}|\epsilon_i| = y$ it follows that, for all $j \neq i$, $\gamma^{-j}|\epsilon_j| \leq xy^{-1}$. Then we define

$$W_j = \begin{cases} |\epsilon_j| & \text{for } |\epsilon_j| \leq \gamma^j xy^{-1} \\ 0 & \text{otherwise} \end{cases}$$

and we proceed as we have estimated P_1. Now we have for large n and x, $0 < \alpha < 1$,

$$\mathrm{E}W_j \sim c_1\gamma^{j(1-\alpha)}x^{1-\alpha}y^{-(1-\alpha)}$$

and

$$\mathrm{E}W_j^2 \sim c_2\gamma^{j(2-\alpha)}x^{2-\alpha}y^{-(2-\alpha)}.$$

Applying Chebyshev's inequality and integrating over y we obtain that the probability in (6.3) is $O(x^{-\alpha})$ for large x.

In the case $1 \leq \alpha < 2$ we proceed as mentioned before. ∎

References

Anderson, T.W. (1959). *On asymptotic distributions of estimates of parameters of stochastic difference equations*, Ann. Math. Statist. 30, 676–687.

Hu, Y. and W.A. Woyczynski (1995). *Limit behavior of quadratic forms of moving averages and statistical solutions of the Burgers' Equation*, J. Multiv. An. 52, 15–44.

Kwapień, S. and W. A. Woyczynski (1992). *Random Series and Stochastic Integrals, Single and Multiple*, Birkhäuser, Boston.

Mijnheer, J. (1991). *U-statistics and double stable integrals, Selected Proceedings of the Sheffield Symposium on Applied Probability*, eds. I.V. Basawa & R.L. Taylor. IMS Lecture Notes 18, 256–268.

Mijnheer, J. (1995). *On the influence of an extremal term in a (simple) U-statistic*, Theory of Probability and Mathematical Statistics **50**, 103–107.

Mijnheer, J. (1997). *Asymptotic inference for $AR(1)$ processes with (non-normal) stable errors III, Stable Ornstein-Uhlenbeck processes and integrals of stable $O-U$ processes*, Stochastic Models 13, 661–672.

Mijnheer, J. (1998a). *Asymptotic inference for $AR(1)$ processes with (non-normal) stable errors IV. A note on the case of a negative unit root*, J. Math. Sciences 92, 4035–4038.

Mijnheer, J. (1998b). *The asymptotic distribution of a sequential estimator for the parameter in an $AR(1)$ model with stable errors.* Proceedings of Conference „Asymptotic Methods in Probability and Statistics with Applications", St. Petersburg, August 1998. Publisher Birkhauser (in print).

Varberg, D.E. (1968). *Almost sure convergence of quadratic forms in independent random variables*, Ann. Math. Stat. 39, 1502–1506.

Williams, D. (1992). *Probability with Martingales*, Cambridge Math. Textbooks. Cambridge Univ. Press.

Zolotarev, V.M. (1986). *One-dimensional stable distributions*, vol. 65 of „Translations of Mathematical Monographs", Amer. Math. Soc. Translation from the original 1983 Russian edition.

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