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ON A STATIONARY PROCESS INDUCED BY AN ALMOST PERIODICALLY CORRELATED PROCESS

Dedicated to Professor Kazimierz Urbanik

Abstract. Each periodically correlated (PC) processes induces a certain infinite dimensional stationary process which reflects the properties of an underlying PC process. The induced process technique is an efficient tool in analysis of PC processes and sequences. In this note we will define a stationary process induced by a uniformly continuous almost periodically correlated process.

1. Introduction

Throughout the paper $\mathcal{R}$ and $\mathcal{Z}$ denote the sets of real numbers and integers, respectively, and H and K stand for complex Hilbert spaces. A complex function f on $\mathcal{R}$ is said to be *almost periodic* (AP) if f is a uniform limit of trigonometric polynomials. Each AP function is bounded, continuous, and admits the *mean*

$$\mathcal{M}(f) = \lim_{T\to\infty} (1/T) \int_0^T f(t)dt.$$

Every AP function f is characterized by its means $\hat{f}(\lambda) = \mathcal{M}(f(\cdot)e^{-i\lambda\cdot})$, $\lambda \in \mathcal{R}$, and only countably many of them are nonzero. The set $\Lambda(f)$ of all $\lambda \in \mathcal{R}$ for which $\hat{f}(\lambda) \neq 0$ is referred to as the set of *nonzero frequencies* of f. The set of all almost periodic functions whose nonzero frequencies are contained in a fixed set $\Lambda \subseteq \mathcal{R}$ will be denoted $AP(\Lambda)$.

Any measurable function $x : \mathcal{R} \to H$ is called *a process.* The function $R_x(s,t) = (x(s), x(t))$, $s,t \in \mathcal{R}$, where $(\cdot,\cdot)$ denotes the inner product in H,

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is referred to as *the correlation function* of the process x. We also denote

$$B_x(t,u) = R_x(t+u,u), \quad t,u \in \mathcal{R}.$$

Any two processes with the same correlation function will be identified. A process x is called *uniformly almost periodically correlated* (UAPC), if R_x is uniformly continuous and for every $t \in \mathcal{R}$ the function $u \to B_x(t,u)$ is almost periodic. A process x is called *periodically correlated* (PC)with period $T > 0$, if x is locally square integrable and for every $t \in \mathcal{R}$, $B_x(t,u) = B_x(t, T+u)$ du-a.e. A process x is called *continuous periodically correlated* (CPC) with period T if x is continuous and for every $t \in \mathcal{R}$ the function $B_x(t,u)$ is periodic in u with the same period $T > 0$. For an UAPC process x we denote

$$a_\lambda(t) = \mathcal{M}(B_x(t,\cdot)e^{-i\lambda\cdot}), \quad \lambda, t \in \mathcal{R}, \tag{1}$$

The coefficients $a_\lambda(t)$ uniquely determine the correlation function of an UAPC process, and hence they determine the process. If x is UAPC and Λ_t is the set of nonzero frequencies of $B_x(t,\cdot)$, then the smallest discrete subgroup Λ_x of $\mathcal{R}$ that contains $\bigcup_t \Lambda_t$ will be called the *frequency group* of x. Gladyshev [1] proved that if x is UAPC, then Λ_x is countable. If Λ is a fixed discrete subgroup of $\mathcal{R}$, then $UAPC(\Lambda)$ will stand for the set of all UAPC processes such that $\Lambda_x \subseteq \Lambda$. Note, that if x is CPC with period T, then x is UAPC, $\Lambda_x \subseteq 2\pi\mathcal{Z}/T = \{2\pi j/T : j \in \mathcal{Z}\}$, and

$$a_{2\pi j/T}(t) = \frac{1}{T}\int_0^T B_x(t,u)e^{-2\pi i j u/T}du, \quad t \in \mathcal{R}, j \in \mathcal{Z}.$$

A process x is called stationary, if it is continuous and $B_x(t,u)$ does not depend on u. Each stationary process is CPC with an arbitrary period $T > 0$. A family (y^λ), $\lambda \in \Lambda$, of processes in H is said to be an *infinite dimensional stationary (IDS)* process in H indexed by Λ, if all $y^\lambda(\cdot)$, $\lambda \in \Lambda$, are continuous, uniformly bounded, and

$$(y^\lambda(s), y^\mu(t)) = (y^\lambda(s-t), y^\mu(0)) \quad \text{for every } \lambda, \mu \in \Lambda \text{ and } s,t \in \mathcal{R}.$$

If (y^λ) is IDS, then the matrix function $[R_y^{\lambda,\mu}(s)]_{\lambda,\mu\in\Lambda}$ with entries $R_y^{\lambda,\mu}(t) = (y^\lambda(t), y^\mu(0))$ is called the *correlation function* of (y^λ). If (y^λ) is IDS, then there is a unique matrix measure $[\Gamma_x^{\lambda,\mu}]_{\lambda,\mu\in\Lambda}$ on $\mathcal{R}$ such that

$$R_y^{\lambda,\mu}(t) = \int_{\mathcal{R}} e^{ixt}\Gamma_y^{\lambda,\mu}(dx), \quad \lambda,\mu \in \Lambda, t \in \mathcal{R}.$$

Each PC process x with period T and values in H induces an IDS process (z^k), $k \in \mathcal{Z}$, which takes values in $K = L^2([0,T), dt, H)$ and is defined by the formula

$$z^k(t)(u) = x(t+u)e^{-2\pi ik(t+u)/T}, \quad k \in \mathcal{Z}, t \in \mathcal{R}, u \in [0,T).$$

The induced process has proven to be very useful in analysis of periodically correlated processes and sequences (see [4]–[7]). The purpose of this note is to define an IDS process induced by an UAPC process.

Let x be an UAPC process in H and let Λ be a discrete subgroup of $\mathcal{R}$ that contains the frequency group Λ_x of x. Let K_0 be the linear space spanned by the functions $e^{i\lambda\cdot}x(t+\cdot)$, $\lambda \in \Lambda$, $t \in \mathcal{R}$, and for any two functions $y, w \in K_0$ let

$$(y,w)_0 = \mathcal{M}\left((y(\cdot), w(\cdot))\right).$$

Note that if $w \in K_0$ and $||w||_0^2 = (w,w)_0 = 0$, then $\mathcal{M}\left(||w(\cdot)||^2\right) = 0$, and hence $||w(t)||^2 \equiv 0$ because $||w(\cdot)||^2$ is almost periodic. Therefore $(\cdot,\cdot)_0$ is an inner product in K_0. Let K denotes the completion of K_0 with respect to the norm $||\cdot||_0$.

For every $\lambda \in \Lambda$ and $t \in \mathcal{R}$ let $z^\lambda(t)$ be an element of K defined as

$$z^\lambda(t) = e^{-i\lambda(t+\cdot)}x(t+\cdot). \tag{2}$$

THEOREM 1. *Let x be an UAPC proces in H, Λ be a discrete countable subgroup of $\mathcal{R}$ which contains the frequency group Λ_x of x, and let K and $z^\lambda(t)$, $\lambda \in \Lambda$, $t \in \mathcal{R}$, be defined as above. Then*

1. *The family (z^λ), $\lambda \in \Lambda$, is an IDS process in K indexed by Λ.*
2. *The correlation function of (z^λ) is given by*

$$R_z^{\lambda,\mu}(t) = e^{-i\lambda t}a_{\lambda-\mu}(t), \tag{3}$$

where a_λ's are defined in (1).

Proof. Straightforward computation shows that

$$\begin{aligned}(z^\lambda(t), z^\mu(s))_0 &= \mathcal{M}(e^{-i\lambda(t+\cdot)}e^{i\mu(s+\cdot)}(x(t+\cdot), x(s+\cdot)))\\ &= e^{-i\lambda(t-s)}\mathcal{M}(e^{-i(\lambda-\mu)\cdot}(x(t-s+\cdot), x(\cdot)))\\ &= e^{-i\lambda(t-s)}a_{\lambda-\mu}(t-s)\end{aligned}$$

depends only on $t-s$. Moreover $||z^\lambda(t)||_0^2 = a_0(0)$ for all $\lambda \in \Lambda$ and $t \in \mathcal{R}$, and hence $z^\lambda(t)$, $\lambda \in \Lambda$, $t \in \mathcal{R}$, are uniformly bounded. Finally each z^λ is continuous because

$$||z^\lambda(t) - z^\lambda(s)||^2 = 2a_0(0) - 2\Re(e^{-i\lambda(t-s)}a_0(t-s))$$

and a_0 is continuous. ∎

As consequences of the construction we obtain easy proofs of the following two known facts.

COROLLARY 1 (cf. Hurd [2]). *If x is UAPC and Λ is as in Theorem 1, then for every $\lambda \in \Lambda$ there is a complex measure γ_λ of $\mathcal{R}$ such that*

$$a_\lambda(t) = \int_{\mathcal{R}} e^{itx} \gamma_\lambda dx, \ \ t \in \mathcal{R},$$

where a_λ's are defined in (1).

COROLLARY 2 (cf. Gladyshev [1]). *If x is UAPC and Λ is as in Theorem 1, then the matrix function with entries*

$$R^{\lambda,\mu}(t) = e^{-i\lambda t} a_{\lambda-\mu}(t), \quad \lambda, \mu \in \Lambda,$$

is positive definite, that is for any choice of $\lambda_1, \dots, \lambda_n \in \Lambda$, $t_1, \dots, t_n \in \mathcal{R}$ and complex numbers $c_1, \dots, c_n$

$$\sum_j \sum_k c_j \overline{c_k} R^{\lambda_j, \lambda_k}(t_j - t_k) \geq 0.$$

Corollary 1 follows immediately from the formula (3) which says that

$$a_\lambda(t) = R_z^{0,-\lambda}(t) = \int_{\mathcal{R}} e^{ixt} \Gamma_z^{0,-\lambda}(dx).$$

Corollary 2 follows from the fact that the correlation function of any IDS process is positive definite. Corollary 2 is the *only if* part of Theorem 2 from [1] that was proved in the case when Λ is the set of all rational multiples of nonzero frequencies of x.

We will finish this note with two comments which underline differences between periodically and almost periodically correlated processes and point out to some open questions.

Comments. Note that from (3) it follows that the correlation function $R^{\lambda,\mu}(t)$ of an induced process has the property that for every $t \in \mathcal{R}$

(4) $$e^{i\lambda t} R^{\lambda,\mu}(t) \quad \text{depends only on} \quad \lambda - \mu.$$

Since the coefficients $a_\lambda(t)$ uniquely determine an UAPC process, the correspondence

$$\mathcal{I} : \ x \longrightarrow (z^\lambda)$$

defined in (2) is an injection from the set $UAPC(\Lambda)$ to the set $IS(\Lambda)$ of all IDS processes indexed by Λ and having the property (4). In [5] it was shown that if $\Lambda = 2\pi\mathcal{Z}/T$, then every process in $IS(2\pi\mathcal{Z}/T)$ is induced by some PC process, however the latter does not have to be continuous. This shows that the mapping $\mathcal{I} : UAPC(\Lambda) \to IS(\Lambda)$ is not *onto*, and making it a surjection (if possible) would require a proper extension of the notion of an almost periodically correlated process. In this context the *if* part of mentioned Theorem 2 from [1] states that:

If $(w^\lambda) \in IS(\Lambda)$, Λ *is closed under multiplication by rationals, and if there is a uniformly continuous function* $R(s,t)$ *with* $R(s+\cdot, t+\cdot) \in AP(\Lambda)$, $s,t \in \mathcal{R}$, *such that*

$$\mathcal{M}\left(R(t+\cdot,\cdot)e^{i\lambda\cdot}\right) = R_w^{0,\lambda}(t),\ \lambda \in \Lambda,\ t \in \mathcal{R},$$

then (w^λ) *is induced by some* $x \in UAPC(\Lambda)$ *and moreover* $R = R_x$.

The property (4) of an induced process indicates also that its structure involves interplay of two unitary groups. Indeed, if $(w^\lambda) \in IS(\Lambda)$, where Λ is a discrete subgroup of $\mathcal{R}$, then in the space M_w spanned by $w^\lambda(t)$, $\lambda \in \Lambda$, $t \in \mathcal{R}$, one can define two unitary groups:

- a unitary representation U^s of $\mathcal{R}$ defined by $U^s w^\lambda(t) = w^\lambda(t+s)$, $\lambda \in \Lambda$, $s,t \in \mathcal{R}$, and
- a unitary representation S^μ of Λ defined by $S^\mu w^\lambda(t) = e^{i\mu t} w^{\lambda+\mu}(t)$, $\mu, \lambda \in \Lambda$, $t \in \mathcal{R}$.

The pair (U^s, S^μ) satisfies the commutation relation

$$S^\mu U^s = e^{i\mu s} U^s S^\mu, \quad \mu \in \Lambda, s \in \mathcal{R}. \tag{5}$$

This commuting property was used in [5] to reconstruct a PC process that induces (w^λ) and, as a byproduct, to obtain a description of the structure of (not necessary continuous) PC processes. Pairs of representations with the property (5) appear in Quantum Theory (see for example [8]), and in the case when Λ is the dual of a quotient group $\mathcal{R}/K$, where K is a closed subgroup of $\mathcal{R}$, they are characterized by so called Mackey's Imprimitivity Theorem. If x is PC, then $\Lambda_x = 2\pi\mathcal{Z}/T$ and hence $\hat{\Lambda}_x = \mathcal{R}/K$ with $K = \{nT : n \in \mathcal{Z}\}$. In the UAPC case, however, the group Λ_x is the dual of the Bohr compactification of $\mathcal{R}$ with respect to Λ_x and the latter is not a quotient of $\mathcal{R}$, unless process is periodically correlated. The open questions about the structure of almost periodically correlated processes (see [3]) seem therefore to be related to the questions regarding the structure of pairs of representations commuting in the sense (5), or equivalently, the structure of the corresponding to U^s unitary cocycles.

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