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# FEW REMARKS ON INDIVIDUAL ERGODIC THEOREM AND SUMMABILITY METHODS

*Dedicated to Professor Kazimierz Urbanik on the occasion of his 70th birthday*

**1.** It is well-known that the asymptotic behaviour of the Cesàro means of a normal contraction operator $x$ in $L_2$ over a probability space depends on the local properties of the spectrum of $x$ near the value one.

Namely, Gaposhkin [2], [3] proved that if $E$ is the spectral measure of a normal contraction operator $x$ in $L_2(\Omega, \mathcal{F}, P)$ then, for $f \in L_2$, the ergodic averages

$$S_n(f) = \frac{1}{n} \sum_{k=0}^{n-1} x^k f \tag{1}$$

converge almost surely to $\tilde{f} \in L_2$ if and only if

$$E\{z : 0 < |1-z| < 2^{-n}\} f \to 0 \quad \text{a.s.}$$

The aim of this paper is to indicate some possibilities of extension of Gaposhkin's idea to the case of non-contractive normal operators by passing from the arithmetic means i.e. Cesàro averages $(C, 1)$ to other regular summability methods.

Take the Borel summability method, i.e. put

$$\lim_{n\to\infty} \xi_n = \xi(B) \quad \text{iff} \quad \lim_{t\to\infty} e^{-t} \sum_{k=0}^{\infty} \frac{t^k \xi_k}{k!} = \xi. \tag{2}$$

To unify the notation we shall write

$$x^k f \to E\{1\} f \ (C,1), \quad \text{a.s.}$$

to say that, for $S_n$ in (1), $S_n(f) \to E\{1\} f$ a.s.

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Similarly, we write

$$x^k f \to E\{1\}f\ (B), \quad \text{a.s.}$$

when (2) holds for $\xi_n = x^n f$ and $\xi = E\{1\}f$, almost everywhere.

One can show (it will also be clear from the forthcoming proof) that for the Borel method of summability the Gaposhkin's characterization can be formulated as follows. Let us put, for $\delta > 0$

$$D_\delta = \{z+1; z = re^{i\theta}, |\theta - \pi| < \frac{\pi}{2} - \delta, \ \ r \geq 0\}$$

($\delta$ may be taken as small as we want; it means that $D_\delta$ is very close to the whole half-plane $\{\text{Re } z < 1\}$.

Let $E$ be a spectral measure defined on a bounded Borel subset $\Delta$ of $D_\delta$. Then, for a normal operator $x = \int_\Delta zE(dz)$ and $f \in L_2(\Omega, \mathcal{F}, P)$, the following two conditions are equivalent

$$\text{(3)} \qquad x^k f \to E\{1\}f\ (B), \quad \text{a.s.}$$

$$\text{(4)} \qquad E(z \in \Delta; 0 < |1-z| < 2^{-n})f \to 0 \quad \text{a.s.}$$

A shortcoming of the above result is that the domain $D_\delta$ does not contain the whole disc ($|z| \leq 1$), so the above characterization does not embrace all contractive normal operators. We can overcome this disatvantage taking a modification of the classical Borel method. Namely, we put

$$\text{(5)} \qquad \xi_n \to \xi\ (B_{1/2}) \quad \text{iff} \quad \lim_{t\to\infty} \frac{1}{2} e^{-t} \sum \frac{t^{k/2}}{\Gamma(k/2+1)} \xi_k = \xi \quad \text{(cf. [5])}.$$

In the sequel we consider only "discrete" Borel methods, i.e. in (5) we take only $t = 1, 2, \ldots$ (instead of continuous parameter $t > 0$).

Let, for $0 < d < 1$, $D_d$ be the complement of the set

$$\left\{re^{i\theta} : r > \left(\frac{1-d}{\cos 2\theta}\right)^{1/2}, \ |\theta| < \pi/4\} \setminus \{|z| \leq 1\right\}.$$

In particular, $D_d$ contains the disc $\{|z| \leq 1\}$ and the whole half-plane $\{\text{Re } z < 0\}$.

We have the following theorem

**2. Theorem.** *Let*

$$x = \int_\Delta zE(dz)$$

*be the spectral representation of a normal operator, with spectral measure* $E$

*defined on a bounded set* $\Delta \subset D_d$. *Assume that*

$$\int_{|z|\le 1} \frac{||E(dz)f||^2}{|1-z|} < \infty. \tag{6}$$

*Then we have the following characterization*

$$x^k f \to \tilde{f}\ (B_{1/2}), \quad a.s. \tag{7}$$

*if and only if*

$$E(z \in \Delta; 0 < |1-z| < 2^{-n})f \to 0, \quad \text{a.s.} \tag{8}$$

## 3. Sketch of the proof

First we prove that, for every $f \in L_2$,

$$\frac{1}{2}e^{-2^n}\sum_{k=0}^{\infty}\frac{(2^n)^{k/2}}{\Gamma(k/2+1)}x^k f - E(z \in \Delta; 0 < |1-z| < 2^{-n})f \to 0 \quad \text{a.s.} \tag{9}$$

To this end let us put, for a sequence $\xi_n$ of numbers or vectors in $L_2$,

$$B_t(\xi_n) = \frac{1}{2}e^{-t}\sum_{k=0}^{\infty}\frac{t^{k/2}}{\Gamma(k/2+1)}\xi_k. \tag{10}$$

For the Mittag-Leffler's function $E_\alpha$, we have that, for $0 < \alpha < 1$,

$$E_\alpha(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+1)} \sim \alpha^{-1}\exp(z^{1/\alpha}), \tag{11}$$

when $z = re^{i\theta}$ tends to infinity in the angle $|\theta| \le \frac{\pi}{2}\alpha$ (cf [4], p. 198).

From (11) it follows, in particular, that

$$|B_t(z^n)| \le C\exp(-t(1 - \text{Re } z^2) \tag{12}$$

in the angle $|\theta| \le \frac{\pi}{4}$.

For the estimation of $|B_t(z^n)|$, $z \in \Delta$, we shall also need more concrete information.

Namely, we have the following very useful formula

$$B(t, z^n) = \frac{1}{2}e^{-t(1-z^2)}\left[1 + \frac{\epsilon(z)}{\Gamma(1/2)}\int_{[0,z^2t]} e^{-u}u^{-1/2}du\right], \tag{13}$$

where $\epsilon(z) = +1$ on the set $(z:\ \text{Re } z > 0) \cup (i\tau,\ \tau \ge 0)$, and $\epsilon(z) = -1$ elsewhere cf. [5], p. 160.

Moreover, for Re $w < 1$,

$$\left|\exp(t(w-1)\int_{[0,tw]} u^a e^{-u}du\right| \le |wt|^{a+1}\max(e^{-t}, e^{-t(1-\text{Re } w)}).$$

All this implies that, if $z \in \Delta$ (bounded set) and Re $z^2 < 1 - d$, $0 < d < 1$, then we have the estimation

$$(14) \qquad |B(t, z^n)| \le C \exp\left(-\frac{d}{2}t\right) \quad \text{(independent of } z\text{)}.$$

Here and always $C$ is a positive constant, in general different in different formulae.

Let us note that in the set $(|z| \le 1) \cap (\text{Re } z > 0)$, we have

$$1 - \text{Re } z^2 \ge 1 - \text{Re } z \ge \frac{1}{2}|1 - z|^2.$$

Consequently in this set

$$(15) \qquad |B(t, z^n)| \le Ce^{-t/2|1-z|^2}.$$

In the domain $D_0 = \{z = re^{i\theta}, \frac{3}{4}\pi < \theta < \frac{5}{4}\pi\}$, we have in (13), $\epsilon(z) = -1$.

Moreover, for $(\text{Re } z^2 \ge 1) \cap \Delta$,

$$(16) \qquad \left| \int_{[0,z^2t]} e^{-u}u^{-1/2}du - \int_{[0,|z^2|t]} e^{-u}u^{-1/2}du \right| \le Cte^{-t \text{ Re } z^2}.$$

These facts give the inequality

$$(17) \qquad (B(t, z^n)| \le Ce^{-t/2}, \quad \text{for} \quad (\text{Re } z^2 > 1) \cap \Delta.$$

Using (13), also for $z = 1$, we can show that

$$(18) \qquad |B(t, z^n - 1)| = |B(t, z^n) - B(t, 1)| \le Ct|1 - z|,$$

in the se t$(|z| \le 1) \cap (|1 - z|) \le 1/2)$. Finally, by (13) for $z = 1$, we have

$$B(t, 1) = 1 + \frac{1}{\Gamma(1/2)} \int_0^t e^{-u}u^{-1/2}du,$$

so we get

$$(19) \qquad 0 \le 1 - B(t, 1) \le Ce^{-t}, \quad \text{for } t \ge 1.$$

Thus, for any $\Delta \subset D_d$, we found rather strong estimations of $|B(t, z^n)|$ in the whole $\Delta$, and of $|B(t, z^n - 1)|$ in $\Delta$ near 1, good enough for our purpose.

Let us now put $\delta_n = B(2^n, x^k f) - E(z \in \Delta; 0 < |1 - z| < 2^{-n})$. Using (14), (15) and (17), (18) and (19) one can show that

$$\sum_n ||\delta_n||^2 < \infty. \tag{20}$$

Indeed, we have that

$$||\delta_n||^2 \leq C\Big[ \int_{(z:0<|1-z|\leq 2^{-n})\cap\Delta} |B(2^n, z^k - 1)|^2 (E(dz)f, f)$$

$$+ \int_{(z:|1-z|\geq 2^{-n})\cap\Delta} |B(2^n, z^k)|^2 (E(dz)f, f) + e^{-2^n}\Big].$$

Consequently, we can write

$$\sum_n ||\delta_n||^2 \leq C \int_\Delta g(z)(E(dz)f, f),$$

where

$$g(z) = |1-z|^2 \sum_{n:|1-z|\leq 2^{-n}} 2^{2n} + \sum_{n:|1-z|>2^{-n}} e^{-2^n(1-\mathrm{Re}\ z^2)} + C. \tag{21}$$

By the assumption (6), it is rather easy to check that the function $g$ in (21) is integrable with respect to $(E(dz)f, f)$ and (20) follows, so (9) is proved.

Putting

$$\beta_n = \max_{2^n\leq k<2^{n+1}} |B_k(x^\nu\xi) - B_{2^n}(x^\nu\xi)|$$

we show that

$$\sum_{n=1}^{\infty} ||\beta_n||^2 < \infty, \quad \text{so} \quad \beta_n \to 0, \quad \text{a.s.} \tag{22}$$

This can be done using a rather standard dyadic expansion method well-known in the theory of orthogonal series (cf. [1]) and using the estimations similar to (14), (15), (17), (18) and (19). We can omit rather long but standard elementary calculations. We can follow, to some extend, the way indicated by Gaposhkin [2].

From (9) and (22) Theorem follows. ∎

## 6. Final remarks

One can expect that taking the Borel summability methods $B_\alpha$ with $\alpha$ close to zero the domains $D_\alpha$ for which the Gaposhkin's characterization holds become close to the Mittag-Leffler's star for the function $z \to \frac{1}{1-z}$, i.e. to the domain $D = \mathbb{C} \setminus [1, \infty)$. In our opinion this situation deserves careful inspection. We hope that it will be done in the forthcoming papers.

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