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LIMITING DISTRIBUTIONS OF DIFFERENCES BETWEEN SOME GENERALIZED ORDER STATISTICS

Dedicated to Professor Kazimierz Urbanik

Abstract. Let $n \in \mathbb{N}$, $\widetilde{m} = (m_1, \ldots, m_{n-1})$, $k \geq 1$. Define for all $i \in \{1, \ldots, n-1\}$ $\gamma_i = k + n - i + M_i$, where $M_i = \sum_{j=i}^{n-1} m_j$. Let $X(1, n, \widetilde{m}, k), \ldots, X(n, n, \widetilde{m}, k)$ denote the generalized order statistics in the sense of Kamps [3]. We study the limit distributions of the random variables $W(i, n, \widetilde{m}, k) = \gamma_{i+1}[X(i+1, n, \widetilde{m}, k) - X(i, n, \widetilde{m}, k)]$, when the generalized order statistics are Pfeifer's records, k_n-records and sequential order statistics, and summarize previously known results of that type for order statistics and k-th record values.

1. Introduction

Let F be an absolutely continuous distribution function (df) with density f and let $n \in \mathbb{N}$, $\widetilde{m} = (m_1, \ldots, m_{n-1})$, $k \geq 1$, be parameters such that for all $i \in \{1, \ldots, n-1\}$,

$$\gamma_i = k + n - i + M_i \geq 1,$$

where $M_i = \sum_{j=i}^{n-1} m_j$. The random variables

$$X(1, n, \widetilde{m}, k), \ldots, X(n, n, \widetilde{m}, k)$$

are generalized order statistics from an absolutely continuous common distribution function (cdf) F, if their joint density function is of the form

$$f^{X(1,n,\widetilde{m},k),\ldots,X(n,n,\widetilde{m},k)}(x_1, \ldots, x_n)$$
$$= k\Big(\prod_{j=1}^{n-1} \gamma_j\Big)\Big(\prod_{i=1}^{n-1}[1 - F(x_i)]^{m_i} f(x_i)\Big)[1 - F(x_n)]^{k-1} f(x_n),$$

for $F^{-1}(0) \leq x_1 \leq \ldots \leq x_n \leq F^{-1}(1)$.

The notion of generalized order statistics introduced by Kamps provides a unified approach to some distributional and moment properties of ordered random variables. The model of generalized order statistics contains many models of ordered random variables as special cases, e.g.

1. order statistics $X_{1:n}, \ldots, X_{n:n}$ of a sample $(X_1, \ldots, X_n)$ of size n from cdf F are generalized order statistics with parameters $m_1 = \ldots = m_{n-1} = 0$ and $k = 1$,

2. k-th record values $Y_1^{(k)}, \ldots, Y_n^{(k)}$ of a sequence $\{X_n, n \geq 1\}$ of independent identically distributed random variables are generalized order statistics with parameters $m_1 = \ldots = m_{n-1} = -1$ and $k \in \mathbb{N}$.

Other models will be discussed in detail in Section 3.

First note that if $X(i, n, \tilde{m}, k)$, $1 \leq i \leq n$, are generalized order statistics based on the distribution function $F(x) = 1 - e^{-x}$, $x \geq 0$, then for $i = 1, 2, \ldots, n-1$, the random variables

$$W(i, n, \tilde{m}, k) = \gamma_{i+1}\{X(i+1, n, \tilde{m}, k) - X(i, n, \tilde{m}, k)\}$$

are independent and identically distributed according to F (cf. [3], Th. 3.3.5). The aim of this paper is to examine the asymptotic distributions of $W(i, n, \tilde{m}, k)$ as $n \to \infty$ or $k \to \infty$, depending on the model considered. In Section 2 we give formulae for distributions of the differences $X(i+1, n, \tilde{m}, k) - X(i, n, \tilde{m}, k)$, and in the following sections we study behaviour of $W(i, n, \tilde{m}, k)$ in some special cases.

2. Distributions of differences of successive generalized order statistics

Let F be a df. Let $n \in \mathbb{N}$, $\tilde{m} = (m_1, \ldots, m_{n-1})$, $M_i = \sum_{j=i}^{n-1} m_j$, $k \geq 1$, be parameters such that $\gamma_i = k + n - i + M_i \geq 1$, for $i \in \{1, \ldots, n-1\}$ and let $X(i, n, \tilde{m}, k)$, $1 \leq i \leq n$, be the associated generalized order statistics defined in [3]. For $1 \leq i < n$, we denote by

$$Z(i, n, \tilde{m}, k) = X(i+1, n, \tilde{m}, k) - X(i, n, \tilde{m}, k)$$

the difference of successive generalized order statistics.

Lemma 2.1. *If F is concentrated on some interval $S \subset \mathbb{R}$, then the distribution function of $Z(i, n, \tilde{m}, k)$, $1 \leq i < n$, has the form*

$$F^{Z(i,n,\tilde{m},k)}(x) = 1 - \int_S \left(\frac{1 - F(x+u)}{1 - F(u)}\right)^{\gamma_{i+1}} dF^{X(i,n,\tilde{m},k)}(u), \quad x \geq 0,$$

where $F^{X(i,n,\tilde{m},k)}(u)$ is the distribution function of $X(i, n, \tilde{m}, k)$.

Proof. We use the fact that generalized order statistics form a Markov chain with transition probabilities

$$P[X(i+1,n,\tilde{m},k) > t \mid X(i,n,\tilde{m},k) = s] = \left(\frac{1-F(t)}{1-F(s)}\right)^{\gamma_{i+1}},$$

$t \geq s$, $1 \leq i < n$. Thus, for $x \geq 0$,

$$\begin{aligned}
F^{Z(i,n,\tilde{m},k)}(x) &= 1 - P\{Z(i,n,\tilde{m},k) > x\} \\
&= 1 - \int_S P\{Z(i,n,\tilde{m},k) > x \mid X(i,n,\tilde{m},k) = u\} dF^{X(i,n,\tilde{m},k)}(u) \\
&= 1 - \int_S P\{X(i+1,n,\tilde{m},k) > x+u \mid X(i,n,\tilde{m},k) = u\} dF^{X(i,n,\tilde{m},k)}(u) \\
&= 1 - \int_S \left(\frac{1-F(x+u)}{1-F(u)}\right)^{\gamma_{i+1}} dF^{X(i,n,\tilde{m},k)}(u),
\end{aligned}$$

which ends the proof.

Now we prove a lemma, which gives an estimate for the df $F^{W(i,n,\tilde{m},k)}$ of the random variable

$$W(i,n,\tilde{m},k) = \gamma_{i+1} Z(i,n,\tilde{m},k).$$

Let f be the density of F and let the hazard rate be

$$r(x) = \frac{f(x)}{1-F(x)}.$$

LEMMA 2.2. *If $r(x)$ is a differentiable function with bounded first derivative*

$$|r'(x)| \leq M, \quad x \in S, \tag{1}$$

then

$$H(x)\exp\left(-\frac{Mx^2}{\gamma_{i+1}}\right) \leq 1 - F^{W(i,n,\tilde{m},k)}(x) \leq H(x)\exp\left(\frac{Mx^2}{\gamma_{i+1}}\right), \quad x \geq 0,$$

where

$$H(x) = H^{(i,n,\tilde{m},k)}(x) = \int_S \exp(-xr(u)) dF^{X(i,n,\tilde{m},k)}(u), \quad x \geq 0.$$

Proof. By Lemma 2.1

$$1 - F^{W(i,n,\tilde{m},k)}(x) = \int_S \left(\frac{1-F(u+\frac{x}{\gamma_{i+1}})}{1-F(u)}\right)^{\gamma_{i+1}} dF^{X(i,n,\tilde{m},k)}(u).$$

Applying Taylor's formula to the function

$$g(u) = \log(1-F(u))$$

and the differentiability of r we obtain for $x \geq 0$

$$\log \frac{1-F(u+\frac{x}{\gamma_{i+1}})}{1-F(u)} = g\left(u+\frac{x}{\gamma_{i+1}}\right) - g(u) =$$

$$= -r(u)\frac{x}{\gamma_{i+1}} - r'\left(u+\frac{\theta x}{\gamma_{i+1}}\right)\frac{x^2}{\gamma_{i+1}^2},$$

where $0 < \theta < 1$. Therefore

$$1 - F^{W(i,n,\widetilde{m},k)}(x)$$

$$= \int_S \exp(-xr(u)) \exp\left(-r'\left(u+\frac{\theta x}{\gamma_{i+1}}\right)\frac{x^2}{\gamma_{i+1}}\right) dF^{X(i,n,\widetilde{m},k)}(u).$$

Then Lemma 2.2 follows from (1). ∎

3. Limiting distributions of $Z(i,n,\widetilde{m},k)$

Now we give limiting distributions of $Z(i,n,\widetilde{m},k)$ in some special cases of generalized order statistics. Throughout this section we assume that F fulfills the assumptions of Lemma 2.2, i.e. F is concentrated on some interval $S \subset \mathbb{R}$, and such that the hazard rate $r(x)$ is differentiable with bounded first derivative.

3.1. Order statistics

Let $m_i = 0$, $1 \leq i < n$, $k = 1$. Then the generalized order statistics are order statistics $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ of a sample of size n from F. Moreover, $\gamma_i = n - i$. By Lemma 2.1, the distribution function of

$$W_{i,n} = (n-i)(X_{i+1:n} - X_{i:n})$$

is

$$F_{W_{i,n}}(x) = 1 - \int_S \left(\frac{1-F(u+\frac{x}{n-i})}{1-F(u)}\right)^{n-i} dF_{X_{i:n}}(u).$$

By Lemma 2.2 the following inequality holds:

$$H_{i,n}(x)\exp\left(-\frac{Mx^2}{n-i}\right) \leq 1 - F_{W_{i,n}}(x) \leq H_{i,n}(x)\exp\left(\frac{Mx^2}{n-i}\right),$$

where

$$H_{i,n}(x) = \int_S \exp(-xr(u)) dF_{X_{i:n}}(u).$$

Note that if $x_0 = \inf S$, then

$$F_{X_{i:n}}(x) \to \begin{cases} 0, & \text{for } x < x_0, \\ 1, & \text{for } x > x_0, \end{cases} \qquad n \to \infty.$$

Moreover, if $\{i_n, n \geq 1\}$ is any sequence of positive integers such that

$$\frac{i_n}{n} \to p \in (0,1), \ n \to \infty,$$

and x_p is a quantile of order p of F, then

$$F_{X_{i_n:n}}(x) \to \begin{cases} 0, & \text{for } x < x_p, \\ 1, & \text{for } x > x_p, \end{cases} \quad n \to \infty.$$

Therefore $H_{i,n}(x) \to \exp(-\lambda x)$, as $n \to \infty$, $x \geq 0$. This proves the following theorem (cf. Theorem 5.1 of [5]).

THEOREM 3.1. *For any $i \in \mathbb{N}$ the sequence of random variables*

$$n(X_{i+1:n} - X_{i:n}), \ n > i,$$

converges weakly to an exponential distribution with parameter

$$\lambda = \lim_{x \to x_0^+} \frac{f(x)}{1 - F(x)}. \tag{2}$$

Moreover, if $\{i_n, n \geq 1\}$ is any sequence of positive integers such that $i_n \leq n$, $n \geq 1$, and

$$\frac{i_n}{n} \to p \in (0,1), \ n \to \infty,$$

then the sequence of random variables

$$(n - i_n)(X_{i_n+1:n} - X_{i_n:n}), \ n \geq 1,$$

converges weakly to an exponential distribution with parameter

$$\mu = \lim_{x \to x_p} \frac{f(x)}{1 - F(x)}.$$

3.2. Sequential order statistics

Suppose that we are given some triangular array of independent random variables

$$\{Y_j^{(i)}; \ 1 \leq i \leq n, \ 1 \leq j \leq n - i + 1\},$$

such that for any $i = 1, 2, \ldots, n$, random variables $Y_1^{(i)}, Y_2^{(i)}, \ldots, Y_{n-i+1}^{(i)}$ have a common distribution function F_i. We assume that $F_1, \ldots, F_n$ are strictly increasing and continuous and $F_1^{-1}(1) \leq \ldots \leq F_n^{-1}(1)$. Let

$$X_j^{(1)} = Y_j^{(1)}, \quad 1 \leq j \leq n,$$
$$X_{\square,n}^{(1)} = \min(X_1^{(1)}, \ldots, X_n^{(1)}),$$

and for $i = 2, 3, \ldots, n$,

$$X_j^{(i)} = F_i^{-1}(F_i(Y_j^{(i)})(1 - F_i(X_{\square,n}^{(i-1)})) + F_i(X_{\square,n}^{(i-1)})),$$
$$X_{\square,n}^{(i)} = \min(X_1^{(i)}, \ldots, X_{n-i+1}^{(i)}).$$

The random variables $X^{(1)}_{\square,n}, \dots, X^{(n)}_{\square,n}$ are called *sequential order statistics* (cf. [3]). They form a Markov chain with transition probabilities

$$P[X^{(i)}_{\square,n} > t \mid X^{(i-1)}_{\square,n} = s] = \left(\frac{1 - F_i(t)}{1 - F_i(s)}\right)^{n-i+1}, \quad t \geq s.$$

If $F_i(x) = 1 - (1 - F(x))^{\alpha_i}$ where $\alpha_i > 0$ for $i \in \mathbb{N}$, then the sequential order statistics are generalized order statistics with parameters

$$m_i = (n - i + 1)\alpha_i - (n - i)\alpha_{i+1} - 1, \quad 1 \leq i < n,$$

and $k = \alpha_n$. Then $\gamma_i = (n - i + 1)\alpha_i$. By Lemma 2.1

$$W_n = (n - i)\alpha_{i+1}(X^{(i+1)}_{\square,n} - X^{(i)}_{\square,n})$$

has distribution function

$$F_{W_n}(x) = 1 - \int_S \left(\frac{1 - F(u + \frac{x}{(n-i)\alpha_{i+1}})}{1 - F(u)}\right)^{(n-i)\alpha_{i+1}} dF_{X^{(i)}_{\square,n}}(u), \quad x \geq 0,$$

where $F_{X^{(i)}_{\square,n}}$ is the df of $X^{(i)}_{\square,n}$. By Lemma 2.2 we obtain the following inequalities

$$H_n(x)\exp\left(-\frac{Mx^2}{(n-i)\alpha_{i+1}}\right) \leq 1 - F_{W_n}(x) \leq H_n(x)\exp\left(\frac{Mx^2}{(n-i)\alpha_{i+1}}\right),$$

where

$$H_n(x) = \int_S \exp(-xr(u))dF_{X^{(i)}_{\square,n}}(u), \; x \geq 0.$$

Thus we have proved the following theorem.

THEOREM 3.2. *For any $i \in \mathbb{N}$ the sequence $\{W_n, \; n > i\}$ of random variables*

$$W_n = (n - i)\alpha_{i+1}(X^{(i+1)}_{\square,n} - X^{(i)}_{\square,n}),$$

converges weakly to an exponential distribution with parameter

$$\lambda = \lim_{x \to x_0^+} \frac{f(x)}{1 - F(x)},$$

where $x_0 = \inf S$.

3.3. k-th record values

Let $\{X_i, i \geq 1\}$ be a sequence of i.i.d. random variables with cdf F. For any fixed $k \geq 1$ we define the *k-th (upper) record times* $U_k(n)$, $n \geq 1$, of the sequence $\{X_i, i \geq 1\}$, inductively by

$$U_k(1) = 1,$$
$$U_k(n+1) = \min\{j > U_k(n) : X_{j:j+k-1} > X_{U_k(n):U_k(n)+k-1}\}, \; n \geq 1,$$

and the *k-th (upper) record values* by $Y_n^{(k)} = X_{U_k(n):U_k(n)+k-1}$, $n \geq 1$ [1].

We see that k-th record values are generalized order statistics with parameters $m_i = -1$, $1 \le i < n$, $k \in \mathbb{N}$, while $\gamma_i = k$, for $i = 1, 2, \ldots, n-1$. Using the above arguments we obtain Theorem 2.2 of Gajek [2].

THEOREM 3.3. *The sequence of random variables*

$$k(Y_{n+1}^{(k)} - Y_n^{(k)}), \quad k \ge 1,$$

converges weakly, as $k \to \infty$, to an exponential distribution with parameter λ given by (2).

3.4. Pfeifer's record values

Consider a double sequence $\{X_i^{(j)}, i \ge 1, j \ge 1\}$ of independent random variables such that $X_i^{(j)}$, $i \ge 1$, are identically distributed with distribution function F_j, $j \ge 1$. Define *Pfeifer's inter-record times* as

$$\begin{cases} \Delta_1 = 1, \\ \Delta_{n+1} = \min\{j \in \mathbb{N} : X_j^{(n+1)} > X_{\Delta_n}^{(n)}\}, \ n \ge 1, \end{cases}$$

and *Pfeifer's record values* as $X_{\Delta_n}^{(n)}$ (cf. [4]).

If $F_j(x) = 1 - (1 - F(x))^{\beta_j}$ where F is absolutely continuous and $\beta_j > 0$ for $j \ge 1$, then Pfeifer's record values are generalized order statistics with parameters $m_i = \beta_i - \beta_{i+1} - 1$, $k = \beta_n$. Moreover $\gamma_{i+1} = \beta_{i+1}$. Hence, using Lemma 2.1, the distribution function of

$$W_n = \beta_{n+1}(X_{\Delta_{n+1}}^{(n+1)} - X_{\Delta_n}^{(n)})$$

is of the form

$$F_{W_n}(x) = 1 - \int_S \left(\frac{1 - F(u + \frac{x}{\beta_{n+1}})}{1 - F(u)}\right)^{\beta_{n+1}} dF_{X_{\Delta_n}}^{(n)}(u), \ x \ge 0,$$

and by Lemma 2.2,

$$H_n(x) \exp\Big(-\frac{Mx^2}{\beta_{n+1}}\Big) \le 1 - F_{W_n}(x) \le H_n(x) \exp\Big(\frac{Mx^2}{\beta_{n+1}}\Big),$$

where

$$H_n(x) = \int_S \exp(-xr(u)) dF_{X_{\Delta_n}}^{(n)}(u), \ x \ge 0.$$

Note that if $x_0 = \sup S$, $\beta_n \to \infty$ as $n \to \infty$, and

$$\lambda = \lim_{x \to x_0^-} r(x), \tag{3}$$

then $H_n(x) \to \exp(-\lambda x)$, $n \to \infty$. Therefore we get the following theorem.

THEOREM 3.4. *If F_j, $j \geq 1$, fulfill the above assumptions, and if $\beta_n \to \infty$, $n \to \infty$, then the sequence $\{W_n, n \geq 1\}$ of random variables*

$$W_n = \beta_{n+1}(X^{(n+1)}_{\Delta_{n+1}} - X^{(n)}_{\Delta_n})$$

converges weakly to an exponential distribution with parameter λ given by (3).

3.5. k_n-records

Similarly to the Pfeifer's records case, consider a double sequence $\{X_i^{(j)}, i \geq 1, j \geq 1\}$ of independent random variables such that $X_i^{(j)}$, $i \geq 1$, are identically distributed with distribution function F_j, $j \geq 1$. Let $\{k_n, n \geq 1\}$ be a sequence of positive integers.

k_n-inter-record times are defined inductively by

$$\Delta_1 = 1,$$
$$\Delta_{n+1} = \min\{j \in \mathbb{N} : X^{(n+1)}_{j:j+k_{n+1}-1} > X^{(n)}_{\Delta_n:\Delta_n+k_n-1}\},\ n \geq 1,$$

and *k_n-record values* by

$$X^{(n)}_{\Delta_n,k_n} = X^{(n)}_{\Delta_n:\Delta_n+k_{n+1}-1},\ n \geq 1 \quad \text{(cf. [3])}.$$

Note that if $k_n = 1$ for $n \geq 1$, then k_n-records become Pfeifer's records, and if $k_n = k$, $F_n = F$ for $n \geq 1$, then k_n-records are k-th record values.

It is known that k_n-records $X^{(1)}_{\Delta_1,k_1}, X^{(2)}_{\Delta_2,k_2}, \ldots$ form a Markov chain with the transition probabilities

$$(4) \quad P[X^{(n)}_{\Delta_n,k_n} > t \mid X^{(n-1)}_{\Delta_{n-1},k_{n-1}} = s] = \left(\frac{1-F_n(t)}{1-F_n(s)}\right)^{k_n},\ t \geq s,\ n \geq 2.$$

If

$$(5) \qquad F_j(x) = 1 - (1-F(x))^{\beta_j},\ j \geq 1,$$

where $\{\beta_j, j \geq 1\}$ is a sequence of positive numbers, then k_n-records are generalized order statistics with parameters $m_i = \beta_i k_i - \beta_{i+1}k_{i+1} - 1$, $1 \leq i < n$, and $k = \beta_n k_n$. Moreover $\gamma_i = \beta_i k_i$ for $1 \leq i < n$. By Lemma 2.1 for $n \geq 1$

$$W_n = \beta_{n+1}k_{n+1}(X^{(n+1)}_{\Delta_{n+1},k_{n+1}} - X^{(n)}_{\Delta_n,k_n})$$

has df

$$F_{W_n}(x) = 1 - \int_S \left(\frac{1-F(u+\frac{x}{k_{n+1}\beta_{n+1}})}{1-F(u)}\right)^{k_{n+1}\beta_{n+1}} dF_{X^{(n)}_{\Delta_n,k_n}}(u),\ x \geq 0,$$

and by Lemma 2.2

$$H_n(x)\exp\left(-\frac{Mx^2}{k_{n+1}\beta_{n+1}}\right) \leq 1 - F_{W_n}(x) \leq H_n(x)\exp\left(\frac{Mx^2}{k_{n+1}\beta_{n+1}}\right),$$

where

$$H_n(x) = \int_S \exp(-xr(u))dF_{X^{(n)}_{\Delta_n,k_n}}(u), \ x \geq 0.$$

Note that if $x_0 = \sup S$, $k_n\beta_n \to \infty$ as $n \to \infty$, and λ is as in (3), then $H_n(x) \to \exp(-\lambda x)$, $n \to \infty$. Therefore we get the following generalization of Theorem 3.4.

THEOREM 3.5. *With F_j as above, if $\{\beta_n, n \geq 1\}$ and $\{k_n, n \geq 1\}$ are sequences of numbers such that $\beta_n k_n \to \infty$, $n \to \infty$, then the sequence $\{W_n, n \geq 1\}$ of random variables*

$$W_n = \beta_{n+1}k_{n+1}(X^{(n+1)}_{\Delta_{n+1},k_{n+1}} - X^{(n)}_{\Delta_n,k_n}), \quad n \geq 1,$$

converges weakly to an exponential distribution with parameter λ given by (3).

Note that if in the general case the F_j are not of the form (5) then the k_n-records are not generalized order statistics. However, if we assume that $k_n = k$ does not depend on n, then we can prove the following result.

Let F_n, $n \geq 1$, be absolutely continuous distribution functions such that F_n is concentrated on the interval S_n with density f_n and the hazard function r_n. We consider k_n-records, as defined above, but with $k_n = k$, $n \geq 1$. Using the fact that k_n-records form a Markov chain with transition probabilities given by (4), we can deduce that the distribution function of

$$W^{(n)}_k = k(X^{(n+1)}_{\Delta_{n+1},k} - X^{(n)}_{\Delta_n,k})$$

is of the form

$$F_{W^{(n)}_k}(x) = 1 - \int_{S_n} \left(\frac{1 - F_{n+1}(u + \frac{x}{k})}{1 - F_{n+1}(u)}\right)^k dF_{X^{(n)}_{\Delta_n,k}}(u).$$

Fix $n \geq 1$ such that r_n is a differentiable function with bounded first derivative

$$|r'_n(x)| \leq M_n, \quad x \in S_n.$$

Analogously to the proof of Lemma 2.2 we obtain the estimate

$$H_k(x)\exp\left(-\frac{M_n x^2}{k}\right) \leq 1 - F_{W^{(n)}_k}(x) \leq H_k(x)\exp\left(\frac{M_n x^2}{k}\right),$$

where

$$H_k(x) = \int_{S_n} \exp(-xr_{n+1}(u))dF_{X^{(n)}_{\Delta_n,k}}(u).$$

Moreover, we see that $X^{(n)}_{\Delta_n,k} \xrightarrow{D} x_{0,n}$ as $k \to \infty$, where $x_{0,n} = \inf S_n$. This establishes

THEOREM 3.6. *For any $n \geq 1$ such that $r_n(x)$ is a differentiable function with bounded first derivative, the sequence $\{W_k^{(n)},\ k \geq 1\}$ of random variables*

$$W_k^{(n)} = k(X_{\Delta_{n+1},k}^{(n+1)} - X_{\Delta_n,k}^{(n)}),\ k \geq 1,$$

converges weakly as $k \to \infty$, to an exponential distribution with parameter

$$\lambda_n = \lim_{x \to x_{0,n}^+} r_n(x).$$

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