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GENERALIZED TRANSLATION OPERATORS AND MARKOV PROCESSES

Dedicated to Professor K. Urbanik on his Seventies

Abstract. We study the relationship between generalized translation operators and stochastic convolutions on locally compact spaces. We prove that stochastic convolution semigroups can generate Lévy type processes which are strong Markov Feller processes and, as an example, we study the Bingham convolution and its dual on integers.

1. Notations and preliminaries

Let E denote a locally compact separable topological space and let $\mathcal{P}$ and Q denote the class of p.m.'s and sub-p.m.'s on E, respectively. The one-point compactification of E is denoted by $\overline{E}$. Let ∞ denote the isolated point of $\overline{E}$. The convergence of p.m.'s resp. sub-p.m.'s will be understood in the weak and vague sense, respectively.

Let C_b denote the Banach space all bounded continuous real valued functions on E and $C_0 := \{f \in C_b : f(\infty) = 0\}$.

Let τ^x, $x \in E$, denote the class of bounded linear operators on C_b such that the following conditions are satisfied :

(i) There exists an element $e \in E$ such that $\tau^e = I$, the identity operator.

For any $x, y \in E$, $f \in C_0$,

(ii) $\tau^x f(y) = \tau^y f(x)$.

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The function $F(x,y) := \tau^x f(y)$ is continuous in the product space and vanishes whenever x or y tends to the isolated point ∞.

(iii) $\tau^x \tau^y = \tau^y \tau^x$.

(iv) Positivity : If $f \geq 0$ then $\tau^x f \geq 0$.

(v) Sub-Markovian property: If $0 \leq f \leq 1$ then $0 \leq \tau^x f \leq 1$.

Operators $\{\tau^x]$ with properties (i)–(v) stand for a subclass of generalized translation operators (g.t.o.'s) (cf. Levitan [10]) which appear in the theory of DE's and PDE's...)

By virtue of (ii), (iv) and (v) it follows that the functional $\tau^x f(y)$ (x, y) are given) is positive which implies that there exists a unique sub- p.m., say $\delta_x \circ \delta_y$, s.t. for every $f \in C_b$.

$$\tau^x f(y) = \int f(u)\delta_x \circ \delta_y(du) \tag{1.1}$$

where the symbol $\int$ denotes the integration over E.

Put, for $\mu, \nu \in Q$

$$\mu \circ \nu(.) = \int \delta_x \circ \delta_y(.)\mu(dx)\nu(dy). \tag{1.2}$$

Then we get a binary operation $\circ$ on Q with the following properties:

(a) $(Q, \circ)$ is a commutative topological semigroup with δ_e as the unit element.The topology in Q is understood in the vague sense.

(b) For any μ, ν, γ and $\alpha, \beta \geq 0$, $\alpha + \beta \leq 1$ we get $\gamma \circ (\alpha\mu + \beta\nu) = \alpha\gamma \circ \mu + \beta\gamma \circ \nu$.

(c) If E is non-compact then δ_∞ is the only one cluster point of the set $\{\delta_x \circ \delta_y : x, y \in E\}$.

Conversely, if $\circ$ is a binary operation defined on Q such that the conditions (a), (b) and (c) are satisfied then the formula (1.1) defines a system of g.t.o.'s with the properties (i)–(v).

Thus we have proved the following

1.1. PROPOSITION. *Generalized translation ooperators satisfy conditions* (i)–(v) *if and only if there exists a binary operation* $\circ$ *on the set* Q *of sub-p.m.'s on* E *such that the conditions* (a), (b) *and* (c) *are satisfied.*

The equation (1.1) *determines a one to one correspondence between g.t.o.'s and operation* $\circ$.

In the sequel we shall confine ourselves to the case when the relation (1.1) defines a p.m. $\delta_x \circ \delta_y$, $x, y \in E$. It is easy to prove the following :

1.2. PROPOSITION. *Suppose that g.t.o.'s* $\{\tau^x\}$ *satisfy conditions* (i)–(v). *Then for any* $\mu, \nu \in \mathcal{P}$ *the measure* $\mu \circ \nu$ *in* (1.2) *is a p.m. if and only if for every* $x \in E$.

(vi) $\tau^x \mathbb{I} = \mathbb{I}$,
where $\mathbb{I}$ *denotes the function identically equal to 1.*

In what follows we shall consider g.t.o.'s $\{\tau^x\}$ such that the conditions (i)–(vi) are satisfied which implies that the set $\mathcal{P}$ is closed under the operation $\circ$.

Following Vol'kovich [16], we will call the operation $\circ$ on $\mathcal{P}$ a **stochastic convolution**. It is evident that the stochastic convolution $\circ$ is continuous in the weak topology and hence $(\mathcal{P}, \circ)$ becomes a topological semigroup. From now on we will call $(\mathcal{P}, \circ)$ a **stochastic semigroup** on E.

1.3. EXAMPLES. Let us consider the case $E = R_+ = [0, \infty)$. Then, $\overline{E} = \overline{R}_+ = [0, \infty]$. Let $\circ$ denote a regular Urbanik convolution operation defined on p.m'.s and let $\Omega(t)$, $t \in R_+$, denote the kernel for its characteristic function (cf. [14]). Since, for any $t, x, y \in R_+$, $\int \Omega(tu)\delta_x \circ \delta_y(du) = \Omega(tx)\Omega(ty)$ it follows that if $\Omega \in C_0$ then the Urbanik convolution $\circ$ satisfies our conditions (a), (b) and (c). As examples of such convolutions one can take α-convolutions and Kingman convolutions.

The symmetric convolution $*_{1,1}$ is defined by

$$\delta_x *_{1,1} \delta_y = \tfrac{1}{2}(\delta_{x+y} + \delta_{|x-y|}).$$

It is a Urbanik convolution with the kernel $\Omega(t) = \cos t$ and does not satisfy (c).

3. Lévy type processes

Suppose that $(\mathcal{P}, \circ)$ is a stochastic semigroup on E. The operation $\circ$ can be extended to the set $\overline{\mathcal{P}}$ of all p.m.'s on $\overline{E}$. Namely, every p.m. on $\overline{E}$ is of the form

$$\alpha\mu + (1-\alpha)\delta_\infty \tag{2.1}$$

where $0 \le \alpha \le 1$ and $\mu \in P$. Let $\beta\nu + (1-\beta)\delta_\infty$ $(0 \le \beta \le 1,\ \nu \in \mathcal{P})$ be another p.m. on $\overline{E}$. Define

$$[\alpha\mu + (1-\alpha)\delta_\infty] \circ [\beta + (1-\beta)\delta_\infty] = \alpha\beta\mu \circ \nu + (1-\alpha\beta)\delta_\infty. \tag{2.2}$$

Then, by virtue of (ii), (2.1) and (2.2), the pair $(\overline{\mathcal{P}}, \circ)$ becomes a new stochastic semigroup with the unit element δ_e.

Let $\{\mu_t\} \subset \overline{\mathcal{P}}$ be a continuous semigroup w.r.t. the operation $\circ$ (shortly, $\circ$ - semigroup) i.e. for any $t, s \ge 0$,

$$\mu_t \circ \mu_s = \mu_{t+s}, \tag{2.3}$$

$$\lim_{t \to 0} \mu_t = \delta_e. \tag{2.4}$$

It is evident that every p.m. μ_t is i.d. in the following sense

$$\mu_t = \mu_{t/n} \circ \ldots \circ \mu_{t/n} \ \ (n\text{-times}) = \mu_{t/n}^{\circ n}.$$

Let $\overline{\mathcal{B}}$ denote the Borel σ-algebra of subsets of $\overline{E}$, respectively. Given $x \in \overline{E},\ t \geq 0,\ B \in \overline{\mathcal{B}}$ we put

$$P(t,x,B) = (\mu_t \circ \delta_x)(B). \tag{2.5}$$

It is easy to prove the following

2.1. PROPOSITION. *The function $P(.,.,.)$ defined by* (2.5) *satisfies the Chapman–Kolmogorov equation i.e.*

$$\bar{\int} P(t,x,dy)P(s,y,B) = P(t+s,x,B), \tag{2.6}$$

where $t,s \geq 0,\ x \in \overline{E}$ and $B \in \overline{\mathcal{B}}$ and $\bar{\int}$ denotes the integration over $\overline{E}$.

Consequently, there exists an $\overline{E}$-valued Markov process $\{X_t\}$ with the transition probability $P(t,x,B)$ given by (2.5) and

$$P(t,x,B) = P(X_t \in B \mid X_0 = x). \tag{2.7}$$

It should be noted that if $\circ = *$, the ordinary convolution on the real line, then the process $\{X_t\}$ in Proposition 2.1. stands for a classical Lévy process i.e. a stochastic process with stationary independent increments (see Sato [11]). Hence and in the sequel, the time homogenuous Markov process induced by an $\circ$-semigroup $\{\mu_t\}$ will be called a **Lévy type process** or shortly, $\circ$-**Lévy process**. The Lévy type processes (or generalized Lévy processes) related to Urbanik convolution $\circ$ [15] were first introduced and studied by Thu [13].

2.2. LEMMA. *Under conditions* (i)–(vi) *g.t.o.'s $\{\tau^x\},\ x \in \overline{E}$, transform C_0 into C_0.*

Proof. Given $f \in C_0$ and $x \in \overline{E}$ we consider the family of p.m.'s $\{\delta_x \circ \delta_y\}$ in the compact space $\overline{E}$. Naturally, it is relatively compact and if y tends to ∞ we get $\delta_x \circ \delta_y$ weakly converges to δ_∞. Consequently, regarding f as a function on $\overline{E}$ with $f(\infty) = 0$, we have $\lim_{y\to\infty} \bar{\int} f(u)\delta_x \circ \delta_y(du) = f(\infty) = 0$ which implies that $\tau^x f$ belongs to C_0.

Lemma 2.2 allows us to extend g.t.o's to the space $\overline{E}$. Namely, we put $\tau^x f(y) = f(\infty)$ where $x,y \in \overline{E}$ with $x = \infty$ or $y = \infty$ and $f \in C(\overline{E})$. Then, $\{\tau^x\},\ x \in \overline{E}$, becomes a new system of g.t.o's satisfying conditions (i)-(vi). Moreover, formula (1.1) holds also on the space $\overline{E}$. Namely, for any $x,y \in \overline{E}$ and $f \in C(\overline{E})$,

$$\tau^x f(y) = \bar{\int} f(u)\delta_x \circ \delta_y. \tag{1.1$'$}$$

Let μ be a p.m. on $\overline{E}$. We put

$$\tau^\mu f(x) = \bar{\int} f(u)\mu \circ \delta_x(du) = \bar{\int} \tau^x f(u)\mu(du) \tag{2.8}$$

for $x \in \overline{E}$ and $f \in C(\overline{E})$.

2.3. LEMMA. *For every $\mu, \nu \in \overline{\mathcal{P}}$ τ^μ transforms C_0 into C_0 (here C_0 is regarded as a subspace of $C(\overline{E})$). Moreover, we get*

$$\tau^\mu \tau^\nu = \tau^{\mu \circ \nu}. \tag{2.9}$$

Proof. The fact that τ^μ transforms C_0 into C_0 follows immediately from Lemma 2.2. Equation (2.9) is an easy consequence of (1.1') and (2.8).

From Lemma 2.3 and (2.8) we have the following:

2.4. LEMMA. *Let $\{\mu_t\} \subset \overline{\mathcal{P}}$ be a continuous $\circ$-semigroup. The formula*

$$S_t := \tau^{\mu_t} \quad (t \geq 0) \tag{2.10}$$

defines a strongly continuous contraction semigroup on C_0.

Let $\{X_t\}$ be an $\overline{E}$-valued Markov process with the transition probability $P(.,.,.)$ defined by (2.5). Then we have

$$S_t f(x) = E^x f(X_t). \tag{2.11}$$

Since, by Lemma 2.3 and Lemma 2.4, $\{S_t\}$ is a strongly continuous semigroup we infer that $\{X_t\}$ is a Feller process. Moreover, since the function $(t, x, f) \to S_t f(x)$ is continuous, it follows that the process is a strong Markov process (cf. Blumenthal and Getoor [2], p. 41). Hence and by Proposition 2, p.50 and Theorem 6 p.54 in Chung [5] the following theorem holds:

2.5. THEOREM. *Every $\circ$-Lévy process on E is a strong Markov Feller process. Consequently, it is stochastically continuous and has a version with right continuous paths having left limits.*

2.6. REMARK. Theorem 2.6. in Thu [7] is a special case of Theorem 2.5.

3. Bingham convolution

This section is concerned with the compact space case $E = [-1, 1]$ and g.t.o'.s τ^x, $x \in E$ defined for $f \in C(E)$ by

$$\tau^x f(y) = \int_{-1}^{1} f(u)\delta_x \circ \delta_y(du) = \int_{-1}^{1} G_{\nu-1/2}(d\lambda) f(xy + \lambda\sqrt{(1-x^2)(1-y^2)}) \tag{3.1}$$

where $0 \leq v \leq \infty$,

$$G_\nu(d\lambda) = \frac{\Gamma(\alpha+1)}{\pi^{1/2}\Gamma(\alpha+1/2)}(1-\lambda^2)^{\alpha-1}d\lambda, \quad (\nu \in (-1/2, \infty)),$$

$$G_{-1/2}(d\lambda) = \frac{1}{2}(\delta_1 + \delta_{-1}) \qquad (\nu = -1/2),$$

$$G_\infty = \delta_0 \qquad (\nu = \infty).$$

To study the convolution (3.1) we consider classical ultraspherical or Gegenbauer polynomials $W_m^v(x),\ x \in E,\ n = 0,1,2,\dots v \in (0,\infty)$

$$\int_{-1}^{1} W_m^\nu(x) W_n^\nu(x) G_\nu(dx) = \delta_{mn/\omega_n^v}$$

where

$$\omega_n^v = \frac{n+1}{v} . \frac{\Gamma(n+2v)}{n!\Gamma(2v)},$$

$$W_n^v(x) = \int_{-1}^{1} [x + i\lambda(1-x^2)^{1/2}]^n G_{\nu-1/2}(d\lambda).$$

By the classical multiplication theorem of Gegenbauer ([17], p.369) we have

$$W_n^\nu(x) W_n^\nu(y) = \int_{-1}^{1} W_n^\nu(xy + \lambda(1-x^2)^{1/2}(1-y^2)^{1/2} G_{\nu-\frac{1}{2}}(d\lambda) \tag{3.2}$$

$(x,y \in [-1,1],\ \nu \in [0,\infty])$. Consequently,

$$\tau^x W_n^\nu(y) = W_n^\nu(x) W_n^\nu(y). \tag{3.3}$$

By a similar way as in Thu ([13], formula 3.1) one can introduce the following generalized differential operator

$$D^\circ f(x) = \lim_{y \to 1-} \frac{\tau^y f(x) - f(x)}{1-y} \tag{3.4}$$

where $\circ$ and τ^x are defined by (3.1) and the convergence is taken in $C(E)$-norm.

3.1. Lemma. *D° is densely defined in $C(E)$ and its domain $\mathcal{D}(D^\circ)$ contains all Gegenbauer polynomials $W_n^\nu(x)$, $n = 0,1,2,\dots$ Moreover, we get the formula*

$$D^\circ W_n^\nu(x) = \frac{n(n+2\nu)}{1+2\nu} W_n^\nu(x). \tag{3.5}$$

Proof. By a result of Bingham ([1], formula (8)) it follows that

$$\lim_{y \to 1} \frac{1 - W_n^\nu(y)}{1-y} = \frac{n(n+2\nu)}{(1+2\nu)} \tag{3.6}$$

which together with (3.3) and (3.4) implies (3.5).

By virtue of (3.2) and (3.3) it follows that the map π_ν, defined by

$$\pi_\nu(P) = \left\{ \int_{-1}^{1} W_n^\nu(x) P(dx) \right\}_{n=0}^{\infty} \tag{3.7}$$

($P \in \mathcal{P}$) provides a homomorphism of the semigroup $(\mathcal{P}, \circ)$ to a certain class $\mathcal{P}_\nu$ of sequences under termwise multiplication.

If $\mathcal{P}_\nu$ is equipped with term-wise convergence then, under π_ν, $(\mathcal{P}, \circ)$ is isomorphic to $\mathcal{P}_\nu$ (cf. Bingham [1], Proposition 1. a).

3.2. LEMMA. *Let $\{\mu_t\}$ be an $\circ$-semigroup, where $\circ$ is given by (3.1). Then there exists a p.m. $H \in \mathcal{P}$ such that*

$$(3.8) \qquad (1-y)t^{-1}\mu_t(dy) \longrightarrow H \quad \text{weakly as} \quad t \to 0.$$

Proof. By a result of Bochner [3] the i.d. elements $\{c_n\}$ of $\mathcal{P}_\nu$ are of the form

$$(3.9) \qquad c_n = \exp\Big[\int_{-1}^{1} \frac{1 - W_n^\nu(x)}{1-x} H(dx)\Big]$$

with $H \in \mathcal{P}$. In particular, $H = \delta_x$, $x \in [-1, 1)$ correspond to Poisson measures and $H = \delta_1$ corresponds to the Gaussian measure. Consequently, since μ_1 is i.d. in $(\mathcal{P}, \circ)$, $\pi_\nu(\mu_1)$ is of the form (3.9) and thus

$$(3.10) \qquad \pi_\nu(\mu_t) = \Big\{\exp\Big(-t\int_{-1}^{1} \frac{1 - W_n^\nu(x)}{1-x} H(dx)\Big)\Big\}.$$

Let $m_t = t^{-1}(1-y)\mu_t(dy)$ for $t > 0$. Then

$$\Big\{\int_{-1}^{1} \frac{W_n^\nu(x) - 1}{1-x} m_t(dx)\Big\} = t^{-1}\big(\pi_\nu(\mu_t) - 1\big)$$

$$\longrightarrow \Big\{\int_{-1}^{1} \frac{W_n^\nu(x) - 1}{1-x} H(dx)\Big\} \qquad (t \to 0)$$

which implies the weak convergence

$$\lim_{t\to 0} m_t = H.$$

3.3. THEOREM. *Let A be the infinitesimal generator of an $\circ$-Lévy process $\{\xi_t\}$ corresponding to $\{\mu_t\}$. Then, the following inclusion holds*:

$$(3.11) \qquad \mathcal{D}(D^\circ) \subset \mathcal{D}(A).$$

Moreover, for every $f \in \mathcal{D}(D^\circ)$, we have

$$(3.12) \qquad Af(x) = \int_{-1}^{1} \frac{\tau^y f(x) - f(x)}{1-y} H(dy)$$

where H is a p.m. in $\mathcal{P}$ and the integrand assumes the value $D^\circ f(x)$ at $y = 1$. The measure H is uniquely determined.

Proof. Suppose that $f \in \mathcal{D}(D^\circ)$. Then, by (3.4), $\frac{\tau^y f(x) - f(x)}{1-y}$ assumes the value $D^\circ f(x)$ at $y = 1$ and is a continuous function on the compact product space $E \times E$. Moreover, by (3.8), we have

$$Af(x) = \lim_{t\to 0} t^{-1}(\tau^\mu t f(x) - f(x)) = \lim_{t\to 0} \int_{-1}^{1} (\tau^y f(x) - f(x)) t^{-1} \mu_t(dy)$$

$$= \lim_{t\to 0} \int_{-1}^{1} \frac{\tau^y f(x) - f(x)}{1-y} . (1-y) t^{-1} \mu_t(dy) = \int_{-1}^{1} \frac{\tau^y f(x) - f(x)}{1-y} H(dy)$$

where $H \in \mathcal{P}$.

Note that the last expression is a continuous function in E and, since the convergence is boundedly pointwise, the limit can be taken in $C(E)$-norm by the use of a general theory of Dynkin (cf. [6], Lemma 2.11). This shows that the inclusion (3.11) is true and hence (3.12) is proved.

By Lemma 3.1 all Gegenbauer polynomials $W_n^\nu(x)$, $n = 0, 1, 2, \ldots$, belong to $\mathcal{D}(D^\circ)$ and consequently they belong to $\mathcal{D}(A)$. Finally, the uniqueness of representation (3.12) follows from the uniqueness of representation (3.9) (cf. Bochner [3]).

3.4. THEOREM. *Let $A, \{\mu_t\}$ and $\{\xi_t\}$ be the same as in Theorem 3.3.*

If $\{\mu_t\}$ is "Gaussian" (w.r.t. Bingham convolution) then there exists a constant $a > 0$ such that

$$A = aD^\circ. \tag{3.13}$$

Consequently, for every $n = 0, 1, 2, \ldots$ and $\nu \in [0, \infty]$

$$AW_n^\nu = \frac{an(n+2\nu)}{1+2\nu)} W_n^\nu. \tag{3.14}$$

Proof. If $\{\mu_t\}$ is "Gaussian", then so is the corresponding sequence $\pi_\nu(\mu_t)$ and therefore measure H in (3.9) becomes δ_1 which together with (3.5) implies (3.13) and (3.14).

4. Ultraspherical generating functions

Suppose μ is a p.m. on Z_+. Write

$$\mu = \sum_{n=0}^{\infty} \alpha_n \delta_n \tag{4.1}$$

where $\alpha_n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = 1$.

The **ultraspherical generating function**, say $\hat{\mu}(x)$, of μ is defined on $[-1, 1]$ by

$$\hat{\mu}(x) = \sum_{n=0}^{\infty} \alpha_n W_n^{\nu}(x) \tag{4.2}$$

with W_n^{ν}, $n = 0, 1, 2 \ldots,$ being Gegenbaure polynomials. It is evident that $\hat{\mu}$ uniquely dermines μ.

Let $\mathcal{G}_{\nu}$ denote the set of all ultraspherical generating functions. The particular case $v = \infty$ $\mathcal{G}_{\nu}$ becomes the set of the classical generating functions.

4.1. THEOREM. *For every $v \in [0, \infty]$ $\mathcal{G}_{\nu}$ is closed under uniform convergence and convex combinations and pointwise multiplications.*

Proof. The fact that $\mathcal{G}_{\nu}$ is closed under convex combinations and uniform convergence is clear.

Suppose that $n, m = 0, 1, 2, \ldots$ are given. Then the product $\Pi_{n,m}(x)$ of two Gegenbauer polynomials $W_n^{\nu}(x)$ and $W_m^{\nu}(x)$ stands for an element of $L^2([-1, 1], G_{\nu}(dx))$ and admits a linearization (cf. Lasser [8], p. 299)

$$\Pi_{n,m}(x) = \sum_{k=0}^{2\min(n,m)} \Pi(n, m, k) W_{n+m-k}^{\nu}(x) \tag{4.3}$$

which together with the linearization formula in (Bressound [4], Th.1) implies that the quantity

$$\Pi(n, m, k) = \int_{-1}^{1} W_n^{\nu}(x) W_m^{\nu}(x) W_k^{\nu}(x) G_{\nu}(dx) \tag{4.4}$$

is nonnegative and moreover, for any n, m.

$$\sum_{k=0}^{2\min(n,m)} \Pi(n, m, k) = 1. \tag{4.5}$$

Consequently, $\Pi_{n,m}(x)$ belongs to $\mathcal{G}_{\nu}$. Thus, by an easy reasoning, we infer that the set $\mathcal{G}_{\nu}$ is closed under pointwise multiplications. The proof is complete.

By Theorem 4.1 it follows that for any $n, m = 0, 1, 2, \ldots$ there exists a p.m. say $\delta_n \Box \delta_m$ on Z_+ with the ultraspherical generating function (4.3).

Hence we get a binary operation $\Box$ on point measures δ_n which can be easily extended to discrete p.m.'s on Z_+. This convolution on Z_+ can be considered as a natural dual convolution of the Bingham convolution on $[-1, 1]$ defined by (3.1). For an alternative study of the convolution $\Box$ on Z_+ the reader is refered to Vol'kovich [16].

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References

[1] N. H. Bingham, *Random walks on spheres*, Z. Wahrscheinlichkeitstheorie Verw. Geb., **22** (1972), 169–192.

[2] P. M. Blumenthal and R. K. Getoor, *Markov Process and Potential Theory*, Academic Press, New York, (1968).

[3] S. Bochner, *Sturm-Liouville and heat equations whose eigenfunctions are ultraspherical polynomials and associated Bessel functions*, Proc. Conference on Differential Equations Maryland, 1955.

[4] D. M. Bressound, *Linearization and related formulas for q-ultraspherical polynomials*, SIAM J. Math. Anal. 12 (1981), 161–168.

[5] K. L. Chung, *Lectures from Markov Process to Brownian Motion*, Springer, New York, 1982.

[6] E. B. Dynkin, *Markov Processes, I*, Springer 1965.

[7] H. Heyer, *Probability theory on hypergroups: A servey*, In H. Heyer (Ed.) Probability measures on groups VII, Lecture Notes in Mathematics, Vol. 1064, Springer, Berlin, pp. 481–550, 1984.

[8] R. Lasser, *On the Lévy-Hinčin formula for a commutative hypergroups*, in: H. Heyer (Ed.), Probability measures on groups VII. Lecture Notes in Mathematics, Vol. 1064, Springer, Berlin, pp. 298–308, 1984.

[9] R. Lasser, *Bochner Theorems for hypergroups and their applications to orthogonal polynomial expansions*, J. Approx. Theor. 37 (1983), 311–325.

[10] B. M. Levitan, *Generalized translation operators and some of their applications*, Israel Program for Scientific Translations, Jerusalem 1962.

[11] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, 1999.

[12] T. Shiga and S. Wantanabe, *Bessel diffusions as a one-parameter family of diffusion processes*, Z. Wahrscheinlichkeitstheorie Verw. Geb. 27 (1973), 34–46.

[13] N. V. Thu, *Generalized independent increments processes*, Nagoya Math. J. 133 (1994), 155–175.

[14] K. Urbanik, *Generalized convolutions*, Studia Math. 23 (1964), 217–245.

[15] V. E. Vol'kovich, *On an analytical description of Urbanik algebras*, Izv. Akad. Nauk. UzSSR Ser. Fiz. Math. Nauk, 5 (1979), 12–17.

[16] V. E. Vol'kovich, *On symmetric stochastic convolutions*, J. Theor. Prob. 5, No 3 (1992), 117–430.

[17] G. N. Watson, *A Treatise on Bessel Functions*, Sec. Ed. Cambridge University Press, 1966.

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