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Bohdan Mincer

THE EXISTENCE OF ONE-PARAMETER SEMIGROUPS AND CHARACTERIZATIONS OF OPERATOR-LIMIT DISTRIBUTIONS

Dedicated to Professor Kazimierz Urbanik
in honour of fiftieth anniversary
of his work at Wrocław University

1. Throughout this paper we shall work with a Banach space X with the norm $\|\cdot\|$. We write $B(X)$ for the algebra of continuous linear operators on X with the norm topology. By a *semigroup* we mean a subsemigroup of $B(X)$ under the composition operation.

In the theory of operator-limit distributions some compact semigroups associated with probability measures play a very essential role. Furthermore, it is a great importance whether there exist one-parameter semigroups in semigroups in question (cf., e.g.,[2]). The existence of one parameter semigroups was intensively investigated in the theory of compact semigroups (cf., e.g., [1] Chapter B, Section 3). The results proved there are concentrated on purely topological semigroups problems. Moreover, it seems rather complicated to apply these results in the probability on Banach spaces setting. In the paper we give a construction of one-parameter semigroups in some compact semigroups of $B(X)$. The result could be applied directly to a problem occuring in the theory of operator-limit distributions.

2. Given a subset $\mathcal{A}$ of $B(X)$, by $Sem(\mathcal{A})$ we shall denote the closed subsemigroup of $B(X)$ spanned by $\mathcal{A}$. By a *projector* we mean an idempotent from $B(X)$. As usual, I and 0 will denote the unit and the zero operators, respectively. Further, let $U(X)$ denote the group of all invertible operators

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from X onto X. The following consequence of the Numakura theorem will be useful in the sequel (cf. [2], Corollary 1.1.3).

PROPOSITION 2.1. *Let $A \in B(X)$. If the monothetic semigroup $Sem(\{A\})$ is compact, then the set of limit points of the sequence $\{A^n\}$ form a group G_A. Moreover, G_A is the minimal ideal of $Sem(\{A\})$ and the unit of G_A is the only idempotent in $Sem(\{A\})$.*

It is clear that G_A is a commutative group.

REMARK 2.1. Suppose S is a compact subsemigroup of $B(X)$ and let $A \in S$. Then for every projector P from $B(X)$ the following are equivalent:

(i) $P \in G_A$;

(ii) $AP = PA$, the restriction $A|ImP$ belongs to $U(ImP)$, $(A|ImP)^{-1}P \in S$ (and, consequently, $(A|ImP)^{-1}P \in G_A$) and $A^n(I - P) \to 0$.

Indeed, the implication $(i) \Rightarrow (ii)$ is an easy consequence of Proposition 2.1. To prove the converse we may assume without loss of generality that $P = I$. Thus we have an operator A from $U(X)$ such that the semigroup $Sem(\{A, A^{-1}\})$ is compact. By using Proposition 2.1 one easily obtain that we then have $A^n \to I$.

Let S be a compact subsemigroup of $B(X)$ and let P be a projector from S. We use $\mathcal{G}_P$ to denote the largest group with the identity P such that it is contained in S. Notice that $\mathcal{G}_P$ is a compact group.

From Remark 2.1 we get

REMARK 2.2. $\mathcal{G}_P = \bigcup G_A$, where the summation runs over all groups $G_A \subset S$ such that $P \in G_A$. P.

REMARK 2.3. Suppose $0, I \in S$. Then $\mathcal{G}_0 = \{0\}$ and $\mathcal{G}_I = \{A \in S : A \in U(X)$ and $A^{-1} \in S\}$. In particular, $\mathcal{G}_I$ is a compact subgroup of $U(X)$.

Suppose the unit operator is the only idempotent in S. Then S is a compact subgroup of $U(X)$.

3. Denote by $\mathcal{S}$ the set of all compact semigroups $S \subset B(X)$ containing the operators 0 and I and satisfying the condition: the component of the identity (with respect to S) contains 0.

Let S be a compact subsemigroup of $B(X)$ with $0, I \in S$. Observe that $S \in \mathcal{S}$ if and only if for any real-valued continuous function f on S such that $f(0) = 0$ and $f(I) = 1$ we have $[0,1] \subset f(S)$.

Let $\{a_n\}$ be a sequence of real numbers from the open interval $(0,1)$ such that $a_n \to 1$. Then obviously, $Sem(\{a_n\}) = [0,1]$. The following example gives a generalization of this statement.

EXAMPLE 3.1. Suppose $S \subset B(X)$ is a compact semigroup such that there is a sequence $\{C_n\} \subset S$ satisfying the conditions $C_n \to I$ and for each n $\lim_{k\to\infty} C_n^k = 0$. Then $S \in \mathcal{S}$.

Indeed, given a real-valued continuous function f on S with $f(0) = 0$, $f(I) = 1$ and $0 < a < 1$ we choose a sequence $\{k_n\} \subset \mathbb{N}$ such that $f(C_n^{k_n}) \geq a$ and $f(C_n^{k_n+1}) \leq a$ for sufficiently large n. Let C_a be any limit point of the sequence $\{C_n^{k_n}\}$. Then we have $f(C_a) = a$.

Notice that there are semigroups from the class $\mathcal{S}$ without the above property. Consider, e.g., the semigroup

$$S = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\} \cup \left\{\begin{pmatrix} \exp(-t) & 0 \\ 0 & 0 \end{pmatrix} : t \geq 0\right\} \cup \left\{\begin{pmatrix} 1 & 0 \\ 0 & \exp(-t) \end{pmatrix} : t \geq 0\right\} \subset B(\mathbb{R}^2).$$

EXAMPLE 3.2. Suppose $\{D_n\}$ is a sequence of $B(X)$ such that:

(i) the semigroup $Sem(\{D_n\})$ is compact;
(ii) $D_n \to I$;
(iii) $D_n D_{n-1} \ldots D_1 \to 0$.

Denote be S the closed semigroup spanned by the limit points of sequences of the form $\{D_n D_{n-1} \ldots D_{k_n}\}$, where $\{k_n\} \subset \mathbb{N}$ and $k_n \leq n$. Then $S \in \mathcal{S}$.

Indeed, let f and a be as in Example 3.1. We choose a sequence $\{k_n\} \subset \mathbb{N}$, $k_n \leq n$ such that $f(D_n D_{n-1} \ldots D_{k_n}) \geq a$ and $f(D_n D_{n-1} \ldots D_{k_n} D_{k_{n-1}}) \leq a$ a for sufficiently large n. By (iii) we have $k_n \to \infty$. Let C_a be any limit point of the sequence $\{D_n D_{n-1} \ldots D_{k_n}\}$. Since $D_{k_{n-1}} \to I$, we have $f(C_a) = a$.

REMARK 3.1. Instead of (ii) in Example 3.1 one can assume a weaker condition: the limit points of the sequence $\{D_n\}$ belong to a compact group $G \subset U(X)$.

Indeed, we define by induction a sequence $\{U_n\} \subset G$ such that $U_n D_n U_{n-1}^{-1} \to I$. We put $U_0 = I$. Assume that the operator U_{n-1} is defined. We choose as U_n an operator from G for which

$$\|U_n D_n U_{n-1}^{-1} - I\| = \inf\{\|U D_n U_{n-1}^{-1} - I\| : U \in G\}.$$

Now, consider the sequence $\{U_n D_n U_{n-1}^{-1}\}$ instead of $\{D_n\}$. Then the assumptions (i)–(iii) are satisfied. Denote by S' the new semigroup S. Since $G \subset S$, we have $S' \subset S$.

REMARK 3.2. Instead of (iii) in Example 3.1 suppose that $D_1 D_2 \ldots D_n \to 0$. Then we get that the closed semigroup spanned by the limit points of sequences of the form $\{D_{k_n} D_{k_n+1} \ldots D_n\}$, where $\{k_n\} \subset \mathbb{N}$ and $k_n \leq n$

belongs to the class $\mathcal{S}$. However, in the sequel we shall deal with the situation of Example 3.1.

4. Throughout this section X will stand for a real separable Banach space. By a *measure* we shall understand a probability measure defined on the class of Borel subsets of X. We write $P(X)$ for the space of all measures with the weak topology. A measure from $P(X)$ is said to be *full* if its support is not contained in any proper hyperplane of X.

Let $\mu \in P(X)$ and $A \in B(X)$. We say that the measure μ is *A-decomposable* if there exists a measure $\mu_A \in P(X)$ for which the equality $\mu = A\mu * \mu_A$ holds. The asterics denotes here the convolution of measures and $A\mu(E) = \mu(A^{-1}(E))$ for all Borel subsets E of X.

In the study of limit distributions [3] K. Urbanik introduced the concept of *decomposability semigroup* $D(\mu)$ of linear operators associated with the probability measure μ. Namely, $D(\mu)$ consists of all operators A from $B(X)$ for which μ is A-decomposable. It is clear that $D(\mu)$ is a closed semigroup containing the zero and the unit operators.

It has been shown that some purely probabilistic properties of measures are equivalent to some algebraic and topological properties of their decomposability semigroups (cf., e.g., [2]). In particular, a measure μ on $\mathbb{R}^n$ is full if and only if $D(\mu)$ is compact ([2], Theorem 2.3.1). Further by $A(\mu)$ we denote the subset of $D(\mu)$ consisting of those operators A for which $\mu = A\mu*\delta_x$ for some $x \in X$. Here δ_x denotes the probability measure concentrated at the point x. Notice that $A(\mu)$ is a semigroup and $A(\mu) \cap U(X)$ is the largest group in $D(\mu) \cap U(X)$ (cf. [2], Proposition 2.3.4). If μ is a full measure on $\mathbb{R}^n$, then $A(\mu)$ is a compact subgroup of $U(\mathbb{R}^n)$ ([2], Corollary 2.3.2). Furthermore, if $\mu \in P(X)$ is a full measure, $A \in A(\mu)$ and the semigroup $Sem(\{A\})$ is compact, then $A \in U(X)$, $A^{-1} \in A(\mu)$ and for any compact semigroup $S \subset B(X)$ $S \cap A(\mu)$ is a compact subgroup of $U(X)$. This follows from Remark 2.3 (if μ is full, then the identity is the only idempotent in $A(\mu)$).

Let μ be a measure on X. μ is called *operator-selfdecomposable* if there are sequences $\{A_n\} \subset U(X)$, $\{\mu_n\} \subset P(X)$, $\{x_n\} \subset X$ with

(i) the semigroup $S = Sem(\{A_m A_n^{-1} : n = 1, 2, \ldots m;\ m = 1, 2, \ldots\})$ is compact,

(ii) the triangular collection of measures $A_n\mu_j$ ($j = 1, 2, \ldots, n$; $n = 1, 2, \ldots$) is uniformly infinitesimal, that is $A_n\mu_{j_n} \to \delta_0$ for each choice of j_n, $j_n \leq n$ such that

$$A_n(\mu_1 * \mu_2 * \ldots * \mu_n) * \delta_{x_n} \Rightarrow \mu.$$

K. Urbanik in [4] gave a complete characterization of full operator-selfdecomposable measures on X. Namely, he has proved that a full measure μ on X is operator-selfdecomposable if and only if its decomposability semigroup contains a one-parameter semigroup $\exp(tB)$, where $B \in B(X)$ and $\exp(tB) \to 0$ when $t \to \infty$ ([4], Theorem 4.1).

We note that for full measures on $\mathbb{R}^n$ the compactness condition (i) can be omitted ([2], Corollary 3.2.4). The same is true if all A_n are multiples of the unit operator and μ is not concentrated at a single point.

Our aim is to give a tool which will be useful in finding one-parameter semigroups in the decomposability semigroups of operator-decomposable measures.

Suppose μ is a full operator-selfdecomposable measure on X and let

$$\mu = \lim_n A_n(\mu_1 * \mu_2 \ldots \mu_n) * \delta_{x_n},$$

where the sequence $\{A_n\} \subset U(X)$ have properties (i) and (ii), $\{\mu_n\} \subset P(X)$, $\{x_n\} \subset X$. Let S be defined as in condition (i). Then we have

PROPOSITION 4.1. *The semigroup* $S \cap D(\mu) \in \mathcal{S}$.

Proof. See [4], Proposition 3.1, Lemma 3.1 and Lemma 3.2. It is shown that:

1) $A_n \to 0$;

2) if $m_n \geq k_n$ and $k_n \to \infty$, then all limit points of the sequence $\{A_{m_n} A_{k_n}^{-1}\}$ belong to $D(\mu)$ (and thus belong to $S \cap D(\mu)$);

3) all limit points of the sequence $\{A_{n+1} A_n^{-1}\}$ belong to $A(\mu)$ (and thus belong to $S \cap A(\mu)$).

Now we put $G = S \cap A(\mu)$. Since S is a compact semigroup, G is a group. Put now $D_n = A_{n+1} A_n^{-1}$ $(n = 1, 2, \ldots)$ and apply Example 3.2 and Remark 3.1. The proposition is thus proved.

It is shown in [4] (Proposition 3.2) that we can choose sequence $\{A_n\}$ with the property $A_{n+1} A_n^{-1} \to I$. It follows from Remark 3.1 that we do not need this statement for our purposes.

5. Suppose P, Q are projectors on X. We write $P \leq Q$ if $PQ = QP = P$. We use the notation $P < Q$ if $P \leq Q$ and $P \neq Q$. Further, suppose S is a subset of $B(X)$ and $P < Q$. We say that *there are no idempotents from* S *between* P *and* Q if there are no projectors $R \in S$ such that $P < R < Q$.

REMARK 5.1. Assume S is a compact semigroup and let $P < Q$ be projectors from S. Then there exists a finite collection of projectors $P_1 < P_2 < \ldots < P_N$ from S such that $P_1 = P$, $P_N = Q$ and for every $i = 1, 2, \ldots, N-1$ there are no idempotents from S between P_i and P_{i+1}.

It follows easily from the fact that every family of commuting projectors belonging to a compact semigroup is finite.

The essential step in proving our main result is the following

LEMMA 5.1. *Suppose S is a compact semigroup from $\mathcal{S}$ and let $P < Q$ be projectors from S such that there are no idempotents from S between P and Q. Then there exists an operator $B \in B(X)$ such that $PB = BP$, $QB = BQ$, the semigroup*

$$\{P + \exp(tB)(Q - P) : t \geq 0\} \subset S$$

and

$$\lim_{t \to \infty} \exp(tB)(Q - P) = 0.$$

Proof. *Step I.* We shall prove that there exists a sequence of operators $\{C_n\}$ such that $C_n Q = QC_n$ and $C_n P = P$ $(n = 1, 2, \ldots)$, $C_n \to Q$ and $P \in G_{C_n}$, where the group G_{C_n} is defined in Proposition 2.1. To do this consider the function

$$f(A) = d((Q - P)A(Q - P) + P, \mathcal{G}_Q) \quad (A \in S),$$

where $d(\cdot, \mathcal{G}_Q)$ is the distance function from the set $\mathcal{G}_Q$ and the group $\mathcal{G}_Q$ is defined in section 2. We have $f(0) = d(P, \mathcal{G}_Q) > 0$ and $f(I) = d(Q, \mathcal{G}_Q) = 0$. Since $S \in \mathcal{S}$, there exists a sequence $\{D_n\} \subset S$ such that $f(D_n) = f(0)/n$, $n = 1, 2, \ldots$ $(g(A) = (f(0) - f(A))/f(0)$ is a continuous function on S and $g(0) = 0$, $g(I) = 1)$. Passing, if necessary, to a subsequence we may assume without loss of generality that there exists an operator U from $\mathcal{G}_Q$ such that $(Q - P)D_n(Q - P) + P \to U$.

Since $\mathcal{G}_Q$ is a group with the identity Q, there exists $V \in \mathcal{G}_Q$ such that $VU = UV = Q$. We put $C_n = V(Q - P)D_n(Q - P) + VP$ $(n = 1, 2, \ldots)$. Obviously $C_n \to Q$. Moreover we have $C_n Q = QC_n = C_n$ for each n. Hence, in particular, $Q \notin G_{C_n}$ (G_{C_n} is an ideal of $Sem(\{C_n\})$ and we have $d(UC_n, \mathcal{G}_Q) > 0$, where $U \in \mathcal{G}_Q$). Further, since $PU = UP = P$, we have $PV = VP = P$. This implies that $PC_n = P$ and $C_n P = P$ for each n. Finally, since there are no idempotents from S between P and Q, we conclude that $P \in G_{C_n}$.

Step II. Now we give a construction of the semigroup $\{P + \exp(tB)(Q - P) : t \geq 0\}$.

Suppose $\{C_n\}$ is the sequence constructed in Step I. We may assume without loss of generality that $Q = I$ (consider the sequence $\{C_n | ImQ\}$). Thus we have $C_n \to I$, $C_n = P + C_n(I - P)$ and $P \in G_{C_n}$ for each n. At the same time there are no idempotents from S between P and I.

Since $C_n \to I$, for sufficiently large n we have $\|C_n - I\| < 1$ and, consequently, $C_n = \exp(\widetilde{B}_n)$ with $\widetilde{B}_n \in B(X)$. Obviously, $\widetilde{B}_n \to 0$ and $\widetilde{B}_n P = P\widetilde{B}_n = 0$.

We choose a subsequence $\{j_n\} \subset \mathbb{N}$ such that $\|C_{j_n} - I\| < 1$ for each n and $n!\widetilde{B}_{j_n} \to 0$. We put $B_n = n!\widetilde{B}_{j_n}$ $(n = 1, 2, \ldots)$. Then we have

$$\exp(B_n),\ \exp\left(\frac{1}{n!}B_n\right) \in S \text{ for every } n, \tag{1}$$

$\exp(B_n) \to I$ and for each n $B_nP = PB_n = 0$ and

$$P \in G_{\exp(B_n)} \quad \text{for all } n. \tag{2}$$

It follows from (2) that for all n

$$\liminf_{j\to\infty} \|\exp(jB_n) - I\| \geq d(I, \mathcal{G}_P) > 0. \tag{3}$$

Let now $0 < a < \min\{d(I, \mathcal{G}_P), 1\}$. Passing, if necessary, to a subsequence we may assume without loss of generality that $\|\exp(B_n) - I\| \leq a$ for all n. At the same time, by (2), $\liminf_{j\to\infty} \|\exp(jB_n) - I\| > a$. Denote by k_n the least such j that $\|\exp((j+1)B_n) - I\| > a$. Suppose D is a limit point of the sequence $\{\exp(k_nB_n)\}$.

We have $\|D - I\| = a < 1$. Hence $D = \exp(B)$, for some $B \in B(X)$. Moreover, $DP = PD = P$ and, consequently, $BP = PB = O$. At the same time, $I \notin G_{\exp(B)}$. Indeed, for $j > 1$ we have $\|\exp(jk_nB_n) - I\| > a > 0$. Since there are no idempotents between P and I, we thus obtain that $P \in G_{\exp(B)}$. In particular we have $\exp(nB)(I - P) \to 0$ (see Remark 2.1(ii)).

Let now r be a positive rational number. Since $rn! \in \mathbb{N}$ for sufficiently large n, it follows by (3) that $\exp(rB) \in S$. Hence we get $\{\exp(tB) : t \geq 0\} \subset S$.

At the same time we have $\exp([t]B)(I - P) \to 0$, where $[t]$ denotes the greatest integer less than or equal to t. Hence we conclude that $\exp(tB)(I - P) \to 0$. The lemma is thus proved.

Our main result is the following

THEOREM 5.2. *Suppose S is a compact subsemigroup of $B(X)$ with $0, I \in S$. Then the following are equivalent:*

(i) *$S \in \mathcal{S}$;*

(ii) *there exist a finite family of projectors $Q_1, Q_2, \ldots, Q_N$ from $B(X)$ with $Q_iQ_j = Q_jQ_i = 0$ for $i \neq j$, $Q_1 + Q_2 + \ldots + Q_N = I$ and operators $B_1, B_2, \ldots, B_N$ from $B(X)$ satisfying the conditions $B_jQ_j = Q_jB_j$ and $\lim_{t\to\infty} \exp(tB_j)Q_j = 0$ $(j = 1, 2, \ldots, N)$ such that the semigroups $\{\exp(tB_1)Q_1 : t \geq 0\}$ and $\{Q_1 + \ldots + Q_{j-1} + \exp(tB_j)Q_j : t \geq 0\}$ for $j > 1$ are contained in S.*

Proof. The implication $(i) \Rightarrow (ii)$ follows easily from Remark 5.1 and Lemma 5.1. Indeed, let $0 = P_0 < P_1 < \ldots < P_N = I$ be projectors from S such that for $i = 0, 1, \ldots, N-1$ there are no idempotents from S between P_i and P_{i-1}. Apply Lemma 5.1 for each pair $P_i < P_{i-1}$ and put $Q_1 = P_1$, $Q_2 = P_2 - P_1, \ldots, Q_N = P_N - P_{N-1}$.

Since the implication $(ii) \Rightarrow (i)$ is obvious, the theorem is thus proved.

REMARK 5.2. Suppose X is a real separable Banach space and let $\mu \in P(X)$.

Let $Q_1, Q_2, \ldots, Q_N$ be commuting projectors from $D(\mu)$ with the property $Q_j Q_i = 0$ for all $i \neq j$. Then for every collection $A_1, A_2, \ldots, A_N$ from $D(\mu)$ such that $A_j Q_j = Q_j A_j$ $(j = 1, 2, \ldots, N)$ we have $A_1 Q_1 + A_2 Q_2 + \ldots + A_N Q_N \in D(\mu)$ (cf. [2], Theorem 2.3.6(c)).

Assume now a measure μ is full and operator-selfdecomposable. By Proposition 4.1 a subsemigroup of $D(\mu)$ belongs to the class $\mathcal{S}$. An as application of Theorem 5.2 and Remark 5.2 we obtain that $D(\mu)$ contains a one-parameter semigroup $\{\exp(tB) : t \geq 0\}$ with $\exp(tB) \to 0$ as $t \to \infty$ (put $B = B_1 Q_1 + B_2 Q_2 + \ldots + B_N Q_N$, where $B_1, B_2, \ldots, B_N$ and $Q_1, Q_2, \ldots, Q_N$ are as in Theorem 5.2).

References

[1] K. H. Hofmann and P. S. Mostert, *Elements of Compact Semigroups*, Ch. E. Merill Books, Columbus, Ohio, 1966.

[2] Z. J. Jurek and J. D. Mason, *Operator-limit Distributions in Probability Theory*, J. Wiley, New York, 1993.

[3] K. Urbanik, *Lévy's probability measures on Euclidean spaces*, Studia Math. 44 (1972), 119–148.

[4] K. Urbanik, *Lévy's probability measures on Banach spaces*, Studia Math. 63 (1978), 283–308.

INSTITUTE OF MATHEMATICS
WROCŁAW UNIVERSITY
Pl. Grunwaldzki 2/4
50-384 WROCŁAW, POLAND
E-mail: mincer@math.uni.wroc.pl

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