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REMARKS ON THE SELFDECOMPOSABILITY AND NEW EXAMPLES

Dedicated to Professor Kazimierz Urbanik

Abstract. The analytic property of the *selfdecomposability* of characteristic functions is presented from stochastic processes point of view. This provides new examples or proofs, as well as a link between the stochastic analysis and the theory of characteristic functions. A new interpretation of the famous Lévy's stochastic area formula is given.

1. Introduction and notations

The class of *selfdecomposable* probability distributions, denoted as SD, (known also as the class L or Lévy class L distributions), appears in the theory of limiting distributions as laws of normalized partial sums of independent random variables but not necessarily *identically* distributed. However, the additional assumption of the *infinitesimality* of the summands guarantees their *infinite divisibility*; cf. Jurek & Mason (1993), Section 3.3.9.

All our random variables or stochastic processes are defined on a fixed probability space $(\Omega, \mathcal{F}, \mathcal{P})$. For a given random variable X (for short: rv) or its probability distribution $\mu = \mathcal{L}(X)$ or its probability density f, provided it exits (i.e.,$d\mu(x) = f(x)dx$), we define its *characteristic function* (in short: char.f.) $\phi_X(t) = \phi(t)$ as follows

$$\phi(t) = \phi_X(t) = \mathbb{E}[e^{itX}] = \int_\Omega e^{itX(\omega)}d\mathcal{P}(\omega) = \int_R e^{itx}d\mu(x), \qquad t \in R.$$

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We will say that a charactersitic function ϕ has *the selfdecomposability property* if

$$(1) \qquad \forall(0 < c < 1)\exists(\text{char.f. } \psi_c)\forall(t \in R) \ \ \phi(t) = \phi(ct)\psi_c(t).$$

In terms of a random variable X the above means that for any $0 < c < 1$ there exits a rv X_c such that

$$X \stackrel{\text{d}}{=} cX + X_c, \quad \text{with independent rv } X, X_c;$$

where $\stackrel{\text{d}}{=}$ means equality in distribution.

The class of all selfdecomposable char.f. (or probability distributions or rv.) we denote here by SD, although, it is often denoted by L and called *the Lévy class L*. It is known that all elements $\phi \in SD$ are *infinitely divisible*, i.e.,

$$\forall(n \geq 1)\exists(\text{char.f. } \phi_n)\forall(t \in R) \ \ \phi(t) = (\phi_n(t))^n.$$

The class of all infinitely divisible char.f. (or rv's or probability distributions) is denoted by ID. The classical Lévy-Khintchine Theorem says that

$$(2) \qquad \text{a function } \ \phi : R \to \mathbb{C} \ \text{ is an ID characteristic iff } \ \phi(t) = e^{\Phi(t)},$$

$$(3) \qquad \text{where } \ \Phi(t) = ita - \frac{1}{2}t^2\sigma^2 + \int_{R-\{0\}} [e^{itx} - 1 - \frac{itx}{1+x^2}]dM(x),$$

where $a \in R, \sigma^2 \geq 0$ and M is called Lévy spectral measure , i.e., M is finite measure ouside every neighbourhood of 0 and integrats x^2 in all neighbourhoods of 0. The triple $[a, \sigma^2, M]$ is uniquely determined by a char.f. ϕ from ID. Conversely, each triple gives an ID char.f. by (3); cf. Jurek & Mason (1993), Section 1.1.8. The function Φ is called *the Lévy exponent* of the infinitely divisible char.f. ϕ.

A stochastic processes $Y(t, \omega)$, $t \geq 0$, with stationary and idependent increments, starting from zero is called *a Lévy process*. Usually we may choose a version with *cadlag paths*. The law of $Y(.)$ is determined by the law of $Y(1)$ which is ID. Moreover, each infinitely divisible distribution μ can be inserted into a Lévy process Y such that $\mathcal{L}(Y(1)) = \mu$. The Lévy spectral measure $M(A)$, in (3), is the expected number of jumps of $Y(t)$, for $0 \leq t \leq 1$, whose sizes are in a set A.

We say that X has *the scaling* or *rescaling property* if for each $0 < c < 1$ there exixts a constant $h(c)$ such that

$$(4) \qquad X(ct) \stackrel{d}{=} h(c)X(t).$$

Some of self-similar processes have the scaling property. In general case one needs to add a deterministic function, depending on c, in (4).

For a Lévy process Y, it is easy to see that $Y(t+s) - Y(s)$, (s is fixed) $t \geq 0$, is another Lévy process with the same distribution, on the Skorochod space of *cadlag functions*, as the process $Y(t)$. Moreover, the second process is independent of σ field $\sigma(\{Y(u) : u \leq s\})$. More importantly, for any rv $T \geq 0$ we have

(5) $Y(t+T) - Y(T)$ and $Y(t), t \geq 0$ have the same probability distributions whenever $Y(.)$ and T are stochasticly independent.

This is so called *the strong Markov property* and it holds also for Markov stopping times τ with respect to the natural filtration associated with Y. Basic examples are the Brownian motion $B(t)$, and the stable process $\eta_p(t)$, where $0 < p \leq 2$ is the exponent of stability. The case $p = 2$ corresponds to Brownian motion.

1. Selfdecomposability and the strong Markov property

The following is a minor generalization of the observation in Bondesson (1992), p. 19. For future references we state it as follows:

PROPOSITION 1. *Let X be a process with independent increments, having the scaling and the strong Markow properties and let $T \geq 0$ be an independent of it selfdecomposable rv. If the scaling function is a homeomorphism of the unit interval, then for all $0 \leq c \leq 1$ we have*

(6) $X(T) \stackrel{d}{=} cX(T) + X_c(T)$ *with the two summands being independent, i.e., $X(T)$ is a selfdecomposable rv.*

Proof. Note that $X(T) = X(cT) + [X(T) - X(cT)] \stackrel{d}{=} h(c)X(T) + X_c(T)$, where $X_c(T) := [X(T) - X(cT)]$ is independent of $X(T)$; use conditioning on T. Putting for c values $h^{-1}(c)$ we get the selfdecomposability of $X(T)$.

Here are examples of SD rv which we obtained from Proposition 1 or via arguments as those in the proof of it.

EXAMPLE 1.

(a) For nonnegative $T \in SD$ that is independent of standard normal rv N and Brownian motion (B_t), we have that $N\sqrt{T} \stackrel{d}{=} B_T \in SD$.

(b) For a Brownian motion B, let T_a be the exit time from the interval $[-a, a]$, i.e., $T_a = inf\{t : |B(t)| = a\}$, and let g_{T_a} be its last zero before time T_a, i.e., $g_{T_a} = sup\{t < T_a : B(t) = 0\}$. Then for $a > 0$ we have that $g_{T_a} \in SD$. Furthermore, $N\sqrt{g_{T_a}} \stackrel{d}{=} B_{g_{T_a}}$ is in SD, and its characteristic function is $\tanh(at)/at, t \in R$.

(c) For Brownian motion $B(t)$ in R^d, $d \geq 3$ (the transience property holds) let $R(t) := ||B(t)||$ denotes the Bessel process (the distance from zero). Then

$$L_r := sup\{t : R(t) \leq r\}, \text{ and } \log L_r \text{ are both in } SD.$$

In fact, the law of L_r is equal to the law of $1/(2\gamma_{\frac{d-2}{2},r^2})$, where $\gamma_{\alpha,\lambda}$ is the gamma rv.

(d) For a normal rv Z and indepependent of it rv $\gamma_{\alpha,\lambda}$, the ratio $Z/\sqrt{\gamma_{\alpha,\lambda}} \stackrel{d}{=} B(1/\gamma_{\alpha,\lambda})$ is SD rv. In particular, any Student t-distribution is in SD.

(e) Let $\eta_p(t), t \geq 0$, be a symmetric stable process with the exponent $0 < p \leq 2$ and $\gamma_{\alpha,1}$ be independent of it rv. Then rv $\eta_p(\gamma_{\alpha,1})$ is in SD with the characteristic function $(1 + c_p|t|^p)^{-\alpha}$.

(f) For Brownian motion B_t on R, $b > 0$, $a \neq 0$, random variables

$$\int_0^\infty exp(aB(t) - bt)dt \text{ and } \log\Big(\int_0^\infty exp(aB(t) - bt)dt\Big) \text{ are both in } SD.$$

Proof. Notice that $N\sqrt{T} \stackrel{d}{=} B(T)$, which proves (a). For (b) first observe that $T_{ca} = inf\{t : |c^{-1}B(t)| = a\} \stackrel{d}{=} inf\{t : |B(t/c^2)| = a\} = c^2T_a$. For $0 < a < 1$, random variables $g_{T_a}, g_{T_1} - g_{T_a}$ are idependent and thus we have

$$g_{T_1} \stackrel{d}{=} g_{T_a} + g_{T_1} - g_{T_a} \stackrel{d}{=} a^2 g_{T_1} + [g_{T_1} - g_{T_a}]$$

which shows that g_{T_1} and thus g_{T_a} are in SD. Further, Proposition 1 gives that $B_{g_{T_a}} \in SD$ and use Yor (1997), Section 18.6, p.133.

(c) Note the scaling property $L_{ct} \stackrel{d}{=} c^2 L_t$ and increments independence of $L_t, t \geq 0$; cf. Getoor (1979). This and Proposition 1 shows that L_t is SD. Getoor (1979) also identified the law of L_t as the law of appropriate inverse of gamma rv. Furthermore, log-gamma is SD , cf. Jurek (1997), Example (c).

(d) From (c) we know that rv $1/\gamma_{\alpha,\lambda}$ is in SD. Taking independent of it BM (B_t) and stopping it at $1/\gamma_{\alpha,\lambda}$ we obtain SD distribution. Since t-distribution is defined as the ratio of a normal rv and square root of χ^2, which belongs to gamma family, we conclude the selfedecomposability of t-distributions. Comp. the original proof of Grosswald (1976).

(e) Symmetric stable Lévy process admits the scaling property (with $h(c) = c^{1/p}$) as well as the strong Markov property. Therefore the Proposition 1 gives the selfdecomposability. The remainder is a consequence of the equation

$$\eta_p(\gamma_{\alpha,1}) \stackrel{d}{=} \eta_p(1) \cdot \gamma_{\alpha,1}^{1/p},$$

where the two factors are independent. Note that the selfdecomposability of the characteristic functions in question, is also easy to obtain by checking the property (1) when $\alpha = 1$ (for all $p > 1$ Polya criterion implies that it is char.f.) and then using properties of the class SD.

(f) Dufresne (1990) (cf. also Yor (1992) and Urbanik (1992), Example 3.3, p.309) proved that the integral has probability distribution of an inverse of a gamma rv. Thus (c) gives that both rv are in SD.

3. Selfdecomposability and BDLPs

In this section we are focussing on the so called BDLPs or BDRVs. The following is *the random integral representation*

X has SD distribution iff there exists a unique, in distribution,

Lévy process Y such that

$$\mathbb{E}[\log(1+|Y(1)|)] < \infty \quad \text{and} \quad X \stackrel{d}{=} \int_0^\infty e^{-s} dY(s). \tag{7}$$

The process Y is refered to as the **b**ackground **d**riving **L**evy **p**rocess or, in short, BDLP for X. Similarly, $Y(1)$ is called the background driving random variable for X. Cf. Jurek and Mason (1993), Theorem 3.9.3. and the bibliografical comments there.

Here is a new method of finding the law of $Y(1)$ in (7).

PROPOSITION 2. *If $X_t := \int_0^t e^{-s} dY(s)$, for $t \geq 0$, then*

$$\mathcal{L}(X_t)^{*1/t} \Rightarrow \mathcal{L}(Y(1)), \quad as \quad t \to 0. \tag{8}$$

Proof. Note that Lemma 1.1 in Jurek (1985) gives

$$\begin{aligned} \mathcal{L}(X_t)^{*1/t} &\stackrel{d}{=} \int_0^t e^{-s} dY(s/t) \\ &= \int_0^1 e^{-tu} dY(u) \Rightarrow \mathcal{L}\Big(\int_0^1 dY(u)\Big) = \mathcal{L}(Y(1)), \end{aligned} \tag{9}$$

as $t \to 0$, which completes the proof.

REMARK 1. The above process X_t allows the identification of the law of $Y(1)$ (as $t \to 0$) as well it gives the random integral representation of SD rv (as $t \to \infty$); cf. Jurek and Mason (1993), Theorem 3.6.8 and 3.9.3.

For future references we need the following new description of the self-decomposability property.

PROPOSITION 3. *If ϕ is a class SD characteristic function then it is differentiable at $t \neq 0$, and*

$$(10) \qquad \psi(t) := exp[t\phi'(t)/\phi(t)] \text{ for } t \neq 0 \text{ and } \psi(0) := 1$$

is a characteristic function from the class ID_{log}.

Conversely, if ψ satisfies the above then ϕ is in the class SD.

[ψ or $Y(1)$ is referred to as the *background driving random variable* of SD char. f. ϕ; in short: BDRV.

In mathematical economy the expression $t\phi'(t)/\phi(t)$ is called *the elasticity of a function ϕ at a point t.* It represents the relative change in ϕ over relative change in argument t. Usually one is interested in the demand and supply functions].

Proof. In terms of charcteristic functions the random integral representation says that

$$\phi \in SD \quad \text{iff} \quad \log\phi(t) = \int_0^t \log\psi(u)\frac{du}{u},$$

where the characteristic function ψ corresponds to the distribution of $Y(1)$; cf. Jurek and Mason (1993), Remark 3.6.9(4), p.128. Hence we conclude the claim in Proposition 3.

COROLLARY 1. *A Lévy exponent Φ corresponds to a class SD characteristic function iff it is differentiable (in $R - \{0\}$), $lim_{t\to 0} t\Phi'(t) = 0$ and $t\Phi'(t)$ is a Lévy exponent of a characteristic function in ID_{log}.*

As we have seen the selfdecomposability is sometimes preserved by taking logarithm of a positive SD rv. Here we have a criterion for a such phenomena and at the same time we have a method of "producing" char. f. from a given SD char.f. .

COROLLARY 2. *Let $X > 0$ be an SD rv. Then $\log X$ is in SD iff the function*

$$t \to exp\{it\frac{\mathbb{E}[X^{it}logX]}{\mathbb{E}[X^{it}]}\} = exp[t\frac{d}{dt}(log\mathbb{E}[X^{it}])]$$

exists and is an infinitely divisible char.f. with a finite logarithmic moment.

Proof. Write the char. f. for $logX$ and use Proposition 3 for char. f. of X from the class SD.

REMARK 2. Using the property from Proposition 3 one can also get the criterion when SD rv X is such that $\exp(X)$ is again in SD. But as in the above Corollary 2 these are not easily applicable methods. On the other hand, if one knows that $X > 0$ and $\log(X)$ are in SD then the Corollary 2 "produces" and ID_{log} char. f.

EXAMPLE 2. Let T^{ν}, for a real $\nu > 0$, denotes the Student t-distribution with 2ν degress of freedom. It has the probability density function

$$f(x) = \frac{\Gamma(\nu+1/2)}{\sqrt{2\pi\nu}\Gamma(\nu)}(1+\frac{x^2}{2\nu})^{-\nu-1/2}, \text{ for } x \in R.$$

Hence its char. f. is equal to

$$\phi_{T^{\nu}}(t) = \frac{2^{1-\nu}}{\Gamma(\nu)}(\sqrt{2\nu}|t|)^{\nu}K_{\nu}(\sqrt{2\nu}|t|),$$

where K_{ν} is the Bessel function; cf. Grosswald(1976) or use Gradshteyn & Ryzhik (1994) formulae: 3.771(2) with 8.334(2).

From the Example (e) above we have that T^{ν} are selfdecomposable and therefore Proposition 3 and the property 8.486(12), in Gradshteyn & Ryzhik (1994), of Bessel funtions K_{ν} imply that

$$\psi_{T^{\nu}}(t) = \exp\left[\nu + \frac{|t|\sqrt{2\nu}K'_{\nu}(\sqrt{2\nu}|t|)}{K_{\nu}(\sqrt{2\nu}|t|)}\right] = \exp\left[-|t|\sqrt{2\nu}\frac{K_{\nu-1}(\sqrt{2\nu}|t|)}{K_{\nu}(\sqrt{2\nu}|t|)}\right], \quad t \neq 0, \tag{11}$$

is the BDRV for t-distribution. In particular, it is char. f. from ID_{log}. Because of properties of characteristic functions we have the following properties of Bessel functions at zero.

COROLLARY 3. *For Bessel functions K_{ν}, we have*

$$\text{(i)} \lim_{z\to 0}\frac{|z|K'_{\nu}(|z|)}{K_{\nu}(|z|}= -\nu. \qquad \text{(ii)} \lim_{z\to 0}\frac{|z|K_{\nu-1}(|z|)}{K_{\nu}(|z|)} = 0. \tag{12}$$

4. Two "curious" formulae

It is natural to define two "integral mappings": $\mathcal{I}$ from the class $ID_{\log}$ onto SD by

$$\mathcal{I}(\nu) := \mathcal{L}\Big(\int_0^{\infty} e^{-s}dY(s)\Big), \tag{13}$$

and similarly, $\mathcal{J}$ from ID onto $\mathcal{U}$ by

$$\mathcal{J}(\nu) := \mathcal{L}\Big(\int_0^{1} s dY(s)\Big), \tag{14}$$

where in both cases Y is a Lévy process such that $Y(1) = \nu$. More about the class $\mathcal{U}$ one can find in Jurek (1985). [Let us add here that $\mathcal{I}$ and $\mathcal{J}$ are isomorphisms between the corresponding topological convolution semigroups; Theorems 2.6 and 3.6 in Jurek (1985)]. Moreover, probability measures of the form

$$\mathcal{J}(\nu * \mathcal{I}(\nu)) \in SD, \text{ whenever } \nu \in ID_{log}. \tag{15}$$

Cf. Jurek (1985),Theorem 4.5. The argument of $\mathcal{J}$ above, $\nu * \mathcal{I}(\nu)$, which is *the convolution of SD distribution $\mathcal{I}(\nu)$ and its background driving distribution ν*, appears in some known formulae. Here are two occurences of such convolution products.

A. Let $B_t = (Z_t, \widetilde{Z}_t)$ be $\mathbf{R}^2$-Brownian motion and let

$$\mathcal{A}_u = \int_0^u Z_s d\widetilde{Z}_s - \widetilde{Z}_s dZ_s, u > 0,$$

be the Lévy's stochastic area integral. P. Lévy (1951) (see also Yor (1992a), p.19) has proved that

$$\mathbb{E}[e^{it\mathcal{A}_u} | B_u = a] = \frac{tu}{\sinh tu} exp[-\frac{|a|^2}{2u}(tu \coth tu - 1)], \quad t \in R, \tag{16}$$

where $a \in R^2$ and $u \geq 0$ are fixed. The family of characteristic functions $\frac{bt}{\sinh bt}$,($b \in R$ is a fixed parameter), is in SD and its BDRV/BDLP are of the form $exp[-2(bt \coth bt - 1)]$; cf. Jurek (1996), Corollary 3 and p. 182. Thus in (16) we have SD characteristic function and its BDRV/BDLP modulo a constant factor $2|a|^2/u$.

REMARK 3. From the formula (16) we infer that, conditionally, the stochastic area integral is infinitely divisible. In fact, the area integral $\mathcal{A}_u$ has char.f. $1/\cosh ut$, [cf. Lévy (1951), formula (1.3.5) or Yor (1992a), pp. 16-19 taking there in the formula (2.1): $\delta = 2, \alpha = 0$ and $x = 0$]. Thus the area integral itself has SD distribution and, in particular, it is infinitely divisible. (see the example **B** below for its BDLP/BDRV).

B. Let $B_t, 0 \leq t \leq 1$ be a Brownian motion and let N be an independent of it standard normal rv. From Wenocur (1986) (see also Yor (1992a), p.19) we infer that

$$\begin{aligned} \mathbb{E}[e^{-\frac{t^2}{2}\int_0^1 (B_s+sx)^2 ds} &= \mathbb{E}[e^{itN(\int_0^1 (B_s+sx)^2 ds)^{1/2}}] \\ &= \left(\frac{1}{\cosh t}\right)^{1/2} \exp\left[-\frac{x^2}{2t}\tanh t\right]. \end{aligned} \tag{17}$$

However, $1/\cosh t$ is SD characteristic function and its BDRV/BDLP is of the form $exp[-t\tanh t]$. Cf. Jurek (1996), Corollary 4 and an appropriate formula on p. 182. Thus again in (17) we see a product of SD distribution and its BDRV/BDLP.

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