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# 0-1 LAW FOR PRODUCT MEASURES IN PRODUCT OF GROUPS

**Abstract.** The 0-1 law for measurable subgroups and cosets in arbitrary product of Abelian groups with product measure is investigated.

## 1. Introduction

In this paper we study the 0–1 law for measurable cosets and subgroups in the product of groups with product measure. The 0–1 law in the product of linear spaces with product measure has been studied by a lot of authors (see e.g. [3–5], [7]). In [3] for example, the 0-1 law for linear subspaces in $R^\infty$ was investigated. In the present paper we shall consider more general situation, namely the case of an arbitrary product of Abelian groups with the product measure and the 0–1 law for measurable subgroups and cosets.

The main result of this paper (Theorem 1) generalizes the known 0-1 laws in product of linear spaces. It states namely, that the 0–1 law for measurable cosets holds true in the product $\prod_{t\in T} G_t$ of Abelian groups if and only if such law is valid in every group $G_t$.

As a corollary we obtain then the 0–1 law for measurable cosets on $R^T$ (Theorem 2), which extends the known similar facts for measurable linear subspaces (for example Th.3.1 in [3]).

## 2. Preliminaries

In this section we fix terminology and notation. Moreover we give some simple facts which we shall need to prove our main results.

By a group we will always understand in this paper an additively written Abelian group. Let $G$ be a group and $a$ an arbitrary element of $G$, and let $H$ be a subgroup of $G$. The set $a+H$, consisting of all elements $a+h$ with $h$ running over all elements of $H$, is said to be a *coset* of $G$ with respect to $H$.

LEMMA 1. *Let $G$ be a group and let $A$ be a subset of $G$. $A$ is a coset of $G$ (with respect to some subgroup) if and only if $x + y - z \in A$ for any $x, y, z \in A$.*

Proof. Necessity: Let $A$ be a coset of $G$ with respect to some subgroup $H$, that is $A = a + H$ for some $a \in G$. Assume that $x, y, z \in A$, i.e. $x = x_1 + a$, $y = y_1 + a$, $z = z_1 + a$, where $x_1, y_1, z_1 \in H$. Then $x+y-z = x_1+y_1-z_1+a$. Since $H$ is a subgroup, $x_1 + y_1 - z_1 \in H$. Therefore $x + y - z \in A$.

Sufficiency: Let $A$ be a subset of $G$ and let $a \in A$. Put $H = A - a$. Then $A = H + a$. To prove that $A$ is a coset of $G$ it is sufficient to show that $H$ is a subgroup of $G$, i.e. that $x - y \in H$ if $x, y \in H$. Let therefore $x, y \in H$. Then $x = x_1 - a, y = y_1 - a$ where $x_1, y_1 \in A$. Hence $x - y = x_1 - a - (y_1 - a) = x_1 + a - y_1 - a$. Since $x_1, y_1, a \in A$ then from the assumption we have that $x_1 + a - y_1 \in A$ and consequently $x - y \in H$, what completes the proof of Lemma 1.

Let $G_t$ be a group for each $t$ of some set $T$. The product space $G = \prod_{t \in T} G_t$ itself is a group with addition defined in the canonical way by $(x_t)_{t \in T} + (y_t)_{t \in T} = (x_t + y_t)_{t \in T}$.

LEMMA 2. *Let $G_1$ and $G_2$ be groups. If $H \subset G_1 \times G_2$ is a coset of the group $G_1 \times G_2$, then for every $y \in G_2$ the $y$-section of $H$, i.e. the set $H_y = \{x \in G_1 : (x, y) \in H\}$ is either an empty set or a coset of the group $G_1$.*

Proof. Suppose that for $y \in G_2$ $H_y \neq \emptyset$ and let $x_1, x_2, x_3 \in H_y$. Then $(x_1, y) \in H, (x_2, y) \in H, (x_3, y) \in H$. Since $H$ is a coset of $G_1 \times G_2$, then by virtue of Lemma 1 we obtain that $(x_1, y) + (x_2, y) - (x_3, y) \in H$, i.e. $(x_1 + x_2 - x_3, y) \in H$. Hence $x_1 + x_2 - x_3 \in H_y$, what means that $H_y$ is a coset of $G_1$.

At the end of this section we give some well known facts about the 0-1 law in the product of arbitrary probability spaces.

Let for any $n = 1, 2, \ldots$ $(G_n, \mathcal{B}_n, \mu_n)$ be an arbitrary probability space and let $(G, \mathcal{B}, \mu) = (\prod_{n=1}^{\infty} G_n, \prod_{n=1}^{\infty} \mathcal{B}_n, \prod_{n=1}^{\infty} \mu_n)$. A set $A \subset G$ is called a *tail event* if $A$ satisfies the condition:

(1) If $x = (x_n) \in G$ and $x_i = y_i$ for all $i \geq n_0$, for some $y = (y_n) \in A$ and some $n_0 \geq 1$, then $x \in A$.

It is well known that $\mu(A)$ is either 0 or 1 if $A$ is a measurable tail event. This follows immediately from the Kolmogorov zero–one law (see [6], p.241).

Let $(G, \mathcal{B}, \mu)$ be a probability space and let $G$ be a group. We will say that *the 0-1 law for measurable cosets (subgroups)* holds in the space $(G, \mathcal{B}, \mu)$ if for any measurable coset (subgroup) $A$ either $\mu(A) = 0$ or $\mu(A) = 1$.

## 3. Main results

Let for each $t$ of some set $T$, $G_t$ be a group, $\mathcal{B}_t$ a $\sigma$–algebra on $G_t$ and $\mu_t$ a probability measure on $\mathcal{B}_t$. We shall here not assume the connection between the structures of a group and a space with measure. Let $G = \prod_{t\in T} G_t$ and denote by $\mathcal{B}$ the product $\sigma$–algebra $\prod_{t\in T} \mathcal{B}_t$, and by $\mu$ the product measure $\prod_{t\in T} \mu_t$ on $\mathcal{B}$.

LEMMA 3. *Suppose that for any countable set $\{t_1, t_2, \ldots\} \subset T$ the 0–1 law holds in the space $(\prod_{n=1}^{\infty} G_{t_n}, \prod_{n=1}^{\infty} \mathcal{B}_{t_n}, \prod_{n=1}^{\infty} \mu_{t_n})$ for measurable cosets. Then the 0–1 law for measurable cosets holds also in the space $(G, \mathcal{B}, \mu)$.*

Proof. Let $A \in \mathcal{B}$ be a coset of $G$. It follows from the definitions of the product $\sigma$–algebra and the product measure, that there exists a countable subset $\{t_1, t_2, \ldots\}$ of elements of $T$ and a set $\bar{A} \in \prod_{t\in T} \mathcal{B}_{t_n}$ such that $A$ is a cylinder in $G$ with basis $\bar{A}$, i.e. $A = \{(x_t)_{t\in T} \in G : (x_{t_n}) \in \bar{A}\}$ and

$$\mu(A) = \prod_{n=1}^{\infty} \mu_{t_n}(\bar{A}). \tag{2}$$

Since $A$ is a coset of $G$ then obviously $\bar{A}$ is a coset of $\prod_{n=1}^{\infty} G_{t_n}$. Therefore by virtue of the assumption we obtain that either $\prod_{n=1}^{\infty} \mu_{t_n}(\bar{A}) = 0$ or $\prod_{n=1}^{\infty} \mu_{t_n}(\bar{A}) = 1$. Thus from (2) it follows that also either $\mu(A) = 0$ or $\mu(A) = 1$, which completes the proof of Lemma 3.

Now we are ready to present the main result of this paper.

THEOREM 1. *Let for each $t \in T$ $G_t$ be an Abelian group endowed with the $\sigma$–algebra $\mathcal{B}_t$ and let $\mu_t$ be a probability measure on $\mathcal{B}_t$. Then the 0-1 law for measurable cosets holds in the space $(G, \mathcal{B}, \mu) = (\prod_{t\in T} G_t, \prod_{t\in T} \mathcal{B}_t, \prod_{t\in T} \mu_t)$ if and only if the 0-1 law for measurable cosets holds in each space $(G_t, \mathcal{B}_t, \mu_t)$.*

Proof. The necessity of the condition in this theorem is evident. Indeed, if $t_0$ is fixed and $A$ is a measurable coset of $G_{t_0}$, then $\bar{A} = A \times \prod_{t\neq t_0} G_t$ is a measurable coset of $G$ and $\mu(\bar{A}) = \mu_{t_0}(A)$. Since by the assumption $\mu(\bar{A}) = 0$ or $\mu(\bar{A}) = 1$ then also $\mu_{t_0}(A) = 0$ or $\mu_{t_0}(A) = 1$.

To prove the sufficiency we may assume, by virtue of Lemma 3, that the set $T$ is countable. Moreover, without loss of generality we may suppose for simplicity that $T = \{1, 2, \ldots\}$.

Let therefore $A$ be a measurable coset in the space $(G, \mathcal{B}, \mu) = (\prod_{n=1}^{\infty} G_n, \prod_{n=1}^{\infty} \mathcal{B}_n, \prod_{n=1}^{\infty} \mu_n)$ which has positive measure. We must prove that $\mu(A) = 1$.

First of all we show that for each finite set $\{n_1, \ldots, n_k\}$ of indexes there exists a measurable coset $A_{n_1,\ldots,n_k}$ in $(G, \mathcal{B}, \mu)$ such that the following two conditions are fulfilled:

(3) if $x = (x_n) \in A$, $x \in A_{n_1,\ldots,n_k}$, and for some $y = (y_n) \in A_{n_1,\ldots,n_k}$ $x_n = y_n$ for any $n \neq n_1, \ldots, n_k$, then $y \in A$.

(4) $A_{n_1,\ldots,n_k} = \prod_{n=1}^{\infty} A_n^{(n_1,\ldots,n_k)}$, where for any $n$ $A_n^{(n_1,\ldots,n_k)}$ is a measurable coset in $(G_n, \mathcal{B}_n, \mu_n)$ such that $\mu_n(A_n^{(n_1,\ldots,n_k)}) = 1$.

We show the above fact in two steps. In the first place we prove that for any $n_0 = 1, 2, \ldots$ there exists a measurable coset $A_{n_0}$ of $G$ which satisfies conditions (3) and (4). Of course, we may identify the space $(G, \mathcal{B}, \mu)$ with the product space $(G_{n_0} \times G_0, \mathcal{B}_{n_0} \times \mathcal{B}_0, \mu_{n_0} \times \mu_0)$, where, $G_0 = \prod_{n \neq n_0} G_n$, $\mathcal{B}_0 = \prod_{n \neq n_0} \mathcal{B}_n$, $\mu_0 = \prod_{n \neq n_0} \mu_n$. Furthermore, we may treat the set $A$ as a measurable coset of the group $G_{n_0} \times G_0$.

Since $\mu_{n_0} \times \mu_0(A) > 0$ then from the definition of the product measure it follows that there is an element $y_0 \in G_0$ such that $A_{y_0} \in \mathcal{B}_{n_0}$ and $\mu_{n_0}(A_{y_0}) > 0$, where $A_{y_0}$ is a $y_0$–section of $A$, i.e. $A_{y_0} = \{x \in G_{n_0} : (x, y_0) \in A\}$. But from Lemma 2 we have that $A_{y_0}$ is a coset of $G_{n_0}$. Hence, in view of the assumption of this theorem, we obtain that $\mu_{n_0}(A_{y_0}) = 1$. Therefore if we put $A_{n_0} = A_{y_0} \times G_0$, then $A_{n_0}$ is a coset of $G$, and the condition (4) is fulfilled.

We prove that condition (3) is also satisfied. Let thus $(x_1, y_1) \in A$, $(x_1, y_1) \in A_{n_0}$ and $(x_2, y_1) \in A_{n_0}$, where $x_1, x_2 \in G_{n_0}$ and $y_1 \in G_0$. We must show that also $(x_2, y_1) \in A$. From the definition of the set $A_{n_0}$ we have that $x_1 \in A_{y_0}$ and $x_2 \in A_{y_0}$, whence it follows that $(x_1, y_0) \in A$ and $(x_2, y_0) \in A$. Since $A$ is a coset of $G_{n_0} \times G_0$ we obtain by virtue of Lemma 1 that $(x_2, y_0) + (x_1, y_1) - (x_1, y_0) \in A$, i.e. $(x_2 + x_1 - x_1, y_0 + y_1 - y_0) \in A$, and consequently $(x_2, y_1) \in A$. Thus the set $A_{n_0}$ satisfies in fact the condition (3).

In the second step of our proof we show, for every finite set $\{n_1, \ldots, n_k\}$ of indexes, the existence of the measurable coset $A_{n_1,\ldots,n_k}$ of $G$, which satisfies conditions (3) and (4). Let for any $n = 1, 2, \ldots$ $A_n$ be a coset of $G$ constructed in the first step of the proof and put $A_{n_1,\ldots,n_k} = \bigcap_{i=1}^{k} A_{n_i}$. Each set $A_{n_i}$ is of the form $A_{y_0^{(i)}} \times \prod_{n \neq n_i} G_n$, where $A_{y_0^{(i)}}$ is a coset of $G_{n_i}$, such that $\mu_{n_i}(A_{y_0^{(i)}}) = 1$. Hence $A_{n_1,\ldots,n_k} = \prod_{i=1}^{k} A_{y_0^{(i)}} \times \prod_{n \neq n_i} G_n$, and consequently the set $A_{n_1,\ldots,n_k}$ satisfies the condition (4). We prove that the condition (3) is also fulfilled. Without loss of generality we restrict to the case $k = 2$; the transition to the general case can be made by induction over k. Let therefore $A_{n_1,n_2} = A_{y_0^{(1)}} \times A_{y_0^{(2)}} \times \prod_{n \neq n_1,n_2} G_n$ and assume that $(x_1, y_1, z_1) \in A, (x_1, y_1, z_1) \in A_{n_1,n_2}$, and $(x_2, y_2, z_1) \in A_{n_1,n_2}$, where $x_1, x_2 \in A_{y_0^{(1)}}$ $y_1, y_2 \in A_{y_0^{(2)}}$ and $z_1 \in \prod_{n \neq n_1,n_2} G_n$. We have to show that also $(x_2, y_2, z_1) \in A$. Since $x_2 \in A_{y_0^{(1)}}$ and $y_1 \in A_{y_0^{(2)}}$ then $(x_2, y_1, z_1) \in$

$A_{n_1,n_2}$, and because $A_{n_1,n_2} \subset A_{n_1}$ then we have that $(x_1, y_1, z_1) \in A_{n_1}$ and $(x_2, y_1, z_1) \in A_{n_1}$ Hence, taking into account that $(x_1, y_1, z_1) \in A$ we obtain that also $(x_2, y_1, z_1) \in A$. On the other hand from the inclusion $A_{n_1,n_2} \subset A_{n_2}$ it follows that $(x_2, y_2, z_1) \in A_{n_2}$ and $(x_2, y_1, z_1) \in A_{n_2}$. Therefore we receive that also $(x_2, y_2, z_1) \in A$, what completes the proof of the existence of the sets $A_{n_1,\dots,n_k}$ with properties (3) and (4).

Let us remark that from the condition (4) it follows that for each set $A_{n_1,\dots,n_k}$ $\mu(A_{n_1,\dots,n_k}) = 1$.

Now we put

$$A_0 = \bigcap_{k=1}^{\infty} \bigcap_{n_1,\dots,n_k} A_{n_1,\dots,n_k}.$$

Then $A_0$ is a measurable coset of $G$. From (3) we obtain the following property of $A_0$ :

(5) if $x = (x_n) \in A$, $x \in A_0$ and $\{n_1, \dots, n_k\}$ is an arbitrary finite subset of indexes, and for some $y = (y_n) \in A_0$ $x_n = y_n$ for any $n \neq n_1, \dots, n_k$, then $y \in A$.

Moreover from the definition of the set $A_0$ and from (4) we have that

$$A_0 = \bigcap_{k=1}^{\infty} \bigcap_{n_1,\dots,n_k} \Big( \prod_{n=1}^{\infty} A_n^{(n_1,\dots,n_k)} \Big) = \prod_{n=1}^{\infty} \Big( \bigcap_{k=1}^{\infty} \bigcap_{n_1,\dots,n_k} A_n^{(n_1,\dots,n_k)} \Big).$$

If we now put

$$A_n^0 = \bigcap_{k=1}^{\infty} \bigcap_{n_1,\dots,n_k} A_n^{(n_1,\dots,n_k)},$$

then from (4) it follows that:

(6) $A_0 = \prod_{n=1}^{\infty} A_n^0$, where for any $n$ $A_n^0$ is a measurable coset in $(G_n, \mathcal{B}_n, \mu_n)$ such that $\mu_n(A_n^0) = 1$.

From (6) we infer also that $\mu(A_0) = 1$. Denote by $\mu_0$ a restriction of the measure $\mu$ to $A_0$. Then it is easy to see that $\mu = \prod_{n=1}^{\infty} \mu_n^0$, where for any $n$ $\mu_n^0$ is a restriction of $\mu_n$ to $A_n^0$. Observe now that the condition (5) means that the set $A \cap A_0$ is a tail event in the product space $A_0$ with the measure $\mu_0$. Therefore $\mu_0(A \cap A_0) = 1$ or $\mu_0(A \cap A_0) = 0$. Hence $\mu(A) = 1$ or $\mu(A) = 0$. But, by the assumption, $\mu(A) > 0$. Thus $\mu(A) = 1$, and this is what we had to show. The theorem is thus proved.

REMARK 1. The above theorem is not true in the case of subgroups. More precisely, if in the space $(G, \mathcal{B}, \mu)$ the 0-1 law holds for measurable subgroups then, of course, the same law holds also in each space $(G_t, \mathcal{B}_t, \mu_t)$. But, as opposed to the case of cosets, from the validity of the 0-1 law for subgroups in each space $(G_t, \mathcal{B}_t, \mu_t)$ does not follow that such law holds also in $(G, \mathcal{B}, \mu)$. This is shown in the following simple example.

EXAMPLE 1. Let $T = \{1, 2\}$ and $G_1 = G_2 = R$ be real lines. Denote by $\delta_x$ the probability measure on $R$ concentrated at the point $x$ $(x \in R)$ and put $\mu_1 = \delta_1$, $\mu_2 = 1/2(\delta_{-1} + \delta_1)$. Then for measures $\mu_1$ and $\mu_2$ the 0-1 law for measurable subgroups holds true. For $\mu_1$ it is evident, and we show this fact for $\mu_2$. Let therefore $H$ be an arbitrary subgroup of $R$. If $1 \in H$ then also $-1 \in H$, and consequently $\mu_2(H) = 1$. Similarly, if $-1 \in H$ then $1 \in H$ and also we have that $\mu_2(H) = 1$. If, on the other hand, $-1 \notin H$ and $1 \notin H$ then $\mu_2(H) = 0$. But in the product $G_1 \times G_2 = R^2$ with the product measure $\mu_1 \times \mu_2$ the 0-1 law for measurable subgroups does not hold. Indeed, if for example $H = \{(x, y) : x = y\}$, then $H$ is a subgroup of $R^2$, but $\mu_1 \times \mu_2(H) = 1/2$.

At the end of this paper we consider one important case of the product of groups, namely the situation when for each $t \in T$ $G_t$ is a real line $R$ with the Borel $\sigma$–algebra $\mathcal{B}(R)$ and $\mu_t$ is a probability measure on $R$. It is easy to see that if $\nu$ is a Borel probability measure on $R$, which is absolutely continuous with respect to the Lebesgue measure, then for $\nu$ the 0-1 law for Borel cosets of $R$ holds true. This follows from the following well known statement (see e.g. [1], Remark 4.2).

LEMMA 4. *Let $\nu$ be a Borel measure on a real line $R$, absolutely continuous with respect to the Lebesgue measure $m(\nu \ll m)$. If $H$ is a Borel coset of $R$ then either $\nu(H) = 0$ or $H = R$.*

Proof. Suppose in the first place that $\nu$ itself is the Lebesgue measure (i.e. that $\nu = m$), and that $H$ is a subgroup of $R$. Assume that $m(H) > 0$. We must show that $H = R$. It is well known that if $m(H) > 0$ then the difference set $H - H$ contains an open interval containing the origin (see [2, p. 68, Th. B]). Therefore there exists $\delta > 0$ such that $(-\delta, \delta) \subset H - H$. But $H$ is a subgroup, whence $H - H \subset H$. Thus $(-\delta, \delta) \subset H$. Let $t$ be an arbitrary real number. We may choose a sufficiently large positive integer $n$, so that $t/n \in (-\delta, \delta)$, whence $t/n \in H$. Since $H$ is a group, this implies that $t = n \cdot t/n = t/n + \ldots + t/n \in H$. Thus $H = R$.

Let now $H$ be an arbitrary Borel coset of $R$ such that $m(H) > 0$. Then $H = G + a$, where $G$ is a subgroup of $R$ and $a \in R$. Since $m(H) = m(G)$, then $m(G) > 0$ and from the first part of this proof it follows that $G = R$. Thus also $H = R$, what completes the proof in the case when $\nu = m$.

If $\nu$ is an arbitrary Borel measure on $R$ such that $\nu \ll m$, and $H$ is a Borel coset of $R$ with $\nu(H) > 0$, then also $m(H) > 0$, and from the first part of the proof we receive that $H = R$. The lemma is thus proved.

Of course, the 0-1 law for Borel cosets of $R$ is valid if the measure $\nu$ on $R$ is concentrated at one point. Therefore from Theorem 1 we receive the following theorem.

THEOREM 2. *Let for each $t$ of some set $T$ $\mu_t$ be a probability measure defined on the Borel $\sigma$-algebra $\mathcal{B}(R)$ of a real line $R$, and let $(R^T, \mathcal{B}^T, \mu) = (\prod_{t\in T} R_t, \prod_{t\in T} \mathcal{B}_t, \prod_{t\in T} \mu_t)$, where $R_t = R$ and $\mathcal{B}_t = \mathcal{B}(R)$ for any $t \in T$. If for each $t \in T$ the measure $\mu_t$ is absolutely continuous with respect to the Lebesgue measure on $R$, or $\mu_t$ is concentrated at one point, then for every measurable coset $A$ of $R^T$ either $\mu(A) = 0$ or $\mu(A) = 1$.*

The above theorem extends some known 0-1 laws for product measures in $R^T$, especially in the case of a countable set $T$. In this case, i.e. if $T = \{1, 2, \ldots\}$ we denote $R^T$ by $R^\infty$. See for example [3], where the 0-1 law for linear subspaces in $R^\infty$ was received.

Let us remark at the end of this paper that in the case of the space $R^\infty$ the product $\sigma$-algebra $\mathcal{B}^\infty$ is equal to the Borel $\sigma$-algebra of $R^\infty$. Thus from Theorem 2 we have the following assertion which extends the 0-1 law for measurable linear subspaces in $R^\infty$ (see [3], Th. 3.1).

COROLLARY. *Let for any $n = 1, 2, \ldots$ $\mu_n$ be a Borel probability measure on $R$ absolutely continuous with respect to the Lebesgue measure, or concentrated at one point. If $\mu = \prod_{n=1}^{\infty} \mu_n$, then for every Borel coset (in particular subgroup or linear subspace) $A$ of $R^\infty$ either $\mu(A) = 0$ or $\mu(A) = 1$.*

REMARK 2. The author does not know, if the assumptions in the Theorem 2 are also necessary for the validity of the 0-1 law for measurable cosets in $R^T$ (even in the case of the measure $\mu$ on $R$).

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