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ON CONVERGENCE OF BOX DIMENSIONS OF FRACTAL INTERPOLATION STOCHASTIC PROCESSES

Abstract. We introduce fractal interpolation processes determined by n-dimensional random vectors. We examine convergence of their box dimensions and trajectories. We prove, in particular, that box dimensions and trajectories of fractal interpolations of α-fractional Brownian motion converge to those of the interpolated process.

1. Introduction

During last years there has been great interest in modelling real phenomena using self-similar processes (for instance see [15] for applications to economy and analysis of stock market behaviour). Estimation of fractal properties and simulation of trajectories from a given set of sample points has attracted considerable attention. The literature with short reviews of methods of estimating fractal dimension include [7],[13]. For simulation of trajectories see [25], [6]. We focus on simulation of self-similar processes based on fractal interpolation. Fractal interpolations were introduced by Barnsley ([1]) and investigated in by others (see [3], [4], [5], [9],[10], [11],[18],[19], [20]). Fractal interpolation is a continuous function which interpolates a data. The graph of fractal interpolation is the attractor of a finite family of affine transformations in R^2 each of which has a free parameter which controls vertical scaling. Strahle ([23]) proposed a method which generates a unique fractal interpolation for a given set of data, by seting free parameters to ensure that fractal interpolation has the correct values at the midpoints. This method was extended by Chao and Leu in [8]. We apply fractal interpolations to stochastic processes. In our method, like in Strahle one, vertical scaling factors are uniquely defined by n-tuples of equally spaced sample points and determine a unique fractal interpolation.

Key words and phrases: fractal interpolation, interpolation dimension, box dimension, stochastic process, fractional Brownian motion, stationary increments, self-similar.

We prove that for some class of α - self similar processes which includes α-fractional Brownian motions (α-fBm's) trajectories of such fractal interpolations converge to a trajectory of the interpolated process. Moreover we show that the respective sequence of box dimensions of graphs of fractal interpolations converge to 2-α i.e. the box dimension of a typical trajectory of α - self similar process (for the relationship between the index α of self-similarity and the dimension of trajectory for α - self similar processes see [24] and [27]). Since the formula for box dimension is very simple in case of fractal interpolation, we get in this way a very simple estimator of index α for α-fBm or more general for some α - self similar processes.

A *set of data* is a set of points $\{(x_i, y_i) \in R^2 : i = 0, 1, \dots, n\}$, where $x_0 < x_1 \dots < x_n$. Let $w_i : R^2 \longrightarrow R^2$, $i = 1, 2, \dots, n$, be affine transformations defined as follows. For every (x, y) in R^2

$$w_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_i & 0 \\ c_i & d_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_i \\ f_i \end{pmatrix},$$

where

$$a_i = \frac{x_i - x_{i-1}}{x_n - x_0}, \qquad c_i = \frac{y_i - y_{i-1}}{x_n - x_0} - d_i \frac{y_n - y_0}{x_n - x_0},$$

$$e_i = \frac{x_n x_{i-1} - x_0 x_i}{x_n - x_0}, \qquad f_i = \frac{x_n y_{i-1} - x_0 y_i}{x_n - x_0} - d_i \frac{x_n y_0 - x_0 y_n}{x_n - x_0}.$$

According to [2, section 6.2, Theorem 2] we get

1.1. *If the vertical scaling factors d_i obey $0 \leq d_i < 1$, then the attractor of a family $\{w_i : i = 1, 2, \dots, n\}$ exists and is the graph of a continuous real function defined on $[x_0, x_n]$.*

The function defined by 1.1 interpolates the set of data $\{(x_i, y_i) \in R^2 : i = 0, 1, \dots, n\}$. We call it a *fractal interpolation with scaling factors* $(d_i)_{i=1}^n$ and denote by $F(\{(x_i, y_i) : i = 0, 1, \dots, n\}; (d_i)_{i=1}^n)$.

The graph of a fractal interpolation has box dimension $\dim_B$ determined by the vertical scaling factors d_i (comp. [3] and [10, Example 11.4]). For equally spaced interpolated points the formula for the box dimension of fractal interpolation is of the following form.

1.2. *If $a_i = \frac{1}{n}$ for $1 \leq i \leq n$, $0 \leq d_i < 1$ and $\sum_{i=1}^n d_i \geq 1$, then the box dimension D of the graph of fractal interpolation satisfies*

$$D = 1 + \frac{\ln \sum_{i=1}^n d_i}{\ln n}.$$

We shall generalize the notion of fractal interpolation to stochastic processes. Let $(\Omega, \mathcal{F}, P)$ be some probability space and let T be a subset of R. We call X a *stochastic process* from T to R if $X(t)$ is a random variable for each $t \in T$. The value $X(t)(\omega)$ will be denoted by $X(t,\omega)$. Let $X_{t_i} : \Omega \longrightarrow R$ be random variables for $t_0 < t_2 < \ldots < t_n$ and let $d_i : \Omega \longrightarrow R$ be measurable functions for $i = 1, 2, \ldots, n$. Let $\Omega' \subset \Omega$ be defined as follows

$$\omega \in \Omega' :\Leftrightarrow 0 \leq d_i(\omega) < 1 \text{ for } i = 1, 2, \ldots, n.$$

By 1.1, if $\omega \in \Omega'$, then $F\left(\{(t_i, X_{t_i}(\omega)) : i = 0, 1, \ldots, n\}; (d_i(\omega))_{i=1}^n\right)$ exists.

A *fractal interpolation of* $(X_{t_0}, \ldots, X_{t_n})$ with *scaling factors* $(d_1, \ldots, d_n)$ is a 'partial' process $F_{(X_{t_i})(d_i)} : [t_0, t_n] \longrightarrow R$, depending on sequences (X_{t_i}) and (d_i), such that

$$F_{(X_{t_i})(d_i)}(t,\omega) = F\left(\{(t_i, X_{t_i}(\omega)) : i = 0, 1, \ldots, n\}; (d_i(\omega))_{i=1}^n\right)(t)$$

for every $\omega \in \Omega'$ and $t \in [t_0, t_n]$.

1.3 Lemma. *$F_{(X_{t_i})(d_i)}(t, \cdot)$ is a measurable function for every t.*

Proof. For every $\omega \in \Omega'$ let the graph of $F_{(X_{t_i})(d_i)}(t,\omega)$ be the attractor of a family $\{w_1(\omega), \ldots, w_n(\omega)\}$ of affine transformations. Set $l_i(t,\omega) := a_i(\omega)t + e_i(\omega)$ and $f_i(t,x)(\omega) := c_i(\omega)t + d_i(\omega)x + f_i(\omega)$. Let $T_k := \{l_{i_1} \circ l_{i_2} \circ \cdots \circ l_{i_k}(t_0)\colon i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, n\}\}$. Denote $l_{i_1} \circ l_{i_2} \circ \cdots \circ l_{i_k}(t_0)$ by $t_{i_1 \ldots i_k}$. Take $t = t_{i_1 \ldots i_k} \in T_k$. Then $F_{(X_{t_i})(d_i)}(t,\omega) = f_{i_1} \circ w_{i_2} \circ \ldots \circ w_{i_k}(t_0, X(t_0,\omega))$ so $F_{(X_{t_i})(d_i)}(t, \cdot)$ is measurable. If t is an arbitrary point in $[t_0, t_n]$, then there is a sequence (t_k), such that $t = \lim_{k\to\infty} t_k$ and $t_k \in T_k$. In that case $F_{(X_{t_i})(d_i)}(t, \cdot)$ is a pointwise limit of measurable functions $F_{(X_{t_i})(d_i)}(t_k, \cdot)$ so also is measurable. ∎

The function $D_{(d_i)} : \Omega \longrightarrow [0, 2]$ given by

$$D_{(d_i)}(\omega) = \begin{cases} 1 + \dfrac{\ln \sum_{i=1}^n d_i(\omega)}{\ln n} & \text{if } \sum\limits_{i=1}^n d_i(\omega) > 0 \\ 1 & \text{the otherwise.} \end{cases}$$

is the *interpolation dimension with scaling factors* (d_i). Let us note that $D_{(d_i)}$ is defined independently of the existence of fractal interpolation. For equally spaced random variables (X_{t_i}) and $\omega \in \Omega'$ we have simply

$$D_{(d_i)}(\omega) = \dim_B \operatorname{graph} F_{(X_{t_i})(d_i)}(\omega).$$

We shall define now a fractal interpolation fully determined by a set of data. For a sequence $(X_{t_0}, X_{t_1}, \ldots, X_{t_n})$ of random variables we define scaling factors $d_1, d_2 \ldots, d_n$ by

$$d_i = \frac{|X_{t_i} - X_{t_{i-1}}|}{\max\{X_{t_k} : 0 \leq k \leq n\} - \min\{X_{t_k} : 0 \leq k \leq n\}} \quad \text{for} \quad i = 1, \ldots, n.$$

It is obvious that $0 \leq d_i \leq 1$ and $\sum_{i=1}^n d_i \geq 1$.

If there exists a fractal interpolation that corresponds to scaling factors $d_1, d_2 \ldots, d_n$ defined above, it depends only on the sequence (X_{t_i}) and is called *fractal interpolation of* (X_{t_i}). The corresponding interpolation dimension will be referred to as the *interpolation dimension of* $(X_{t_0}, X_{t_1}, \ldots, X_{t_n})$ and will be denoted by $D_{(X_{t_i})}$.

We shall consider sequences of fractal interpolation processes in two cases.

CASE 1 (time series). For a given stochastic process $\{X_t\}_{t\in[0,\infty]}$ and $\Omega_n \subset \Omega$ we take fractal interpolations F_n of $(X_i : i = 0, \ldots, n)$, $n \in N$, with some scaling factors $(d_i^{(n)})$.

CASE 2. For a given stochastic process $\{X_t\}_{t\in[0,1]}$ and $\Omega_n \subset \Omega$ we take fractal interpolations F_n of $(X_{i/n} : i = 0, \ldots, n)$, $n \in N$, with some scaling factors $(d_i^{(n)})$.

In both cases interpolated random variables are equally spaced, thus the box dimension of F_n depends only on the sequence $(d_i^{(n)})$ of scaling factors $(1 \le i \le n)$. We shall denote it by $D_n(d_i^{(n)})$. For the interpolation dimension with scaling factors determined by a process $\{X_t\}$, more strictly, by sequences $(X_i)_{0\le i\le n}$ in case 1, and $(X_{i/n})_{0\le i\le n}$ in case 2, we use the notation $D_n(X_i)$ or $D_n(X_{\frac{i}{n}})$, respectively. For a fixed process we write it simply D_n when no confusion can arise. In the paper we shall only deal with the case $\lim_{n\to\infty} P(\Omega_n) = 1$.

It is clear that if we interpolate points of a process with known box dimension α then we can easily find scaling factors to obtain fractal interpolations with box dimensions convergent to α.

1.4. *Let $1 \le \alpha < 2$. If an interpolation F_n is given by equally spaced random variables and constant (with probability 1) identical scaling factors $d_i^{(n)} = 1/n^{2-\alpha}$, for $i = 1, 2 \ldots, n$ and every $n \in N$, then $D_n(d_i^{(n)})$ converges to α with probability 1.*

Proof. Notice that factors $d_i^{(n)}$ satisfy assumptions of 1.1 an 1.2. The convergence of $D_n(d_i^{(n)})$ to α is obvious. ∎

Let us remind shortly (see [22]) that a process $(X(t))$is *self-similar with index* α (α-ss) if for any $a > 0$, the finite dimensional distributions of $(X(at))$ are the same as those of $a^\alpha X(t)$. The process is $\alpha - sssi$ if it is self-similar with index α and has stationary increments.

A Gaussian α-sssi process, $0 < \alpha < 1$, is called *α-fractional Brownian motion* (α-fBm). Fractional Brownian motion is symmetric and has continuous trajectories with probability 1. The increments of fractional Brownian motion make a stationary and ergodic process. The (1/2)-fBm process is called *Brownian motion.*

We say that a sequence (X_n) of random variables satisfies *the law of large numbers* if with probability 1

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n} = E(X_1).$$

In particular, if (X_n) is stationary and ergodic then (X_n) and $(|X_n|)$ satisfy the law of large numbers ([21]).

In section 2 we examine convergence of interpolation dimensions and in section 3 we prove the convergence of trajectories of fractal interpolations of processes with continuous trajectories.

2. Convergence of interpolation dimensions

Let $\{X_t\}_{t\in[0,\infty)}$ be a stochastic process. We assume that

$$P\Big\{\omega : \sum_{i=1}^{n} |X_i(\omega) - X_{i-1}(\omega)| > 0\Big\} = 1.$$

We shall examine convergence of $D_n(X_i)$ for processes with stationary increments. For self-similar processes it will give also convergence of $D_n(X_{i/n})$.

Let $X^{(n)}_{\max} = \max\{X_i : 0 \le i \le n\}$ and $X^{(n)}_{\min} = \min\{X_i : 0 \le i \le n\}$.

2.1 Theorem. *If the sequence* $(|X_i - X_{i-1}|)$ *satisfies the law of large numbers and* $\lim_{n\to\infty} \frac{\ln(X^{(n)}_{max} - X^{(n)}_{min})}{\ln n} = l$ *with probability 1 (in probability), then* $\lim_{n\to\infty} D_n(X_i) = 2 - l$ *with probability 1 (in probability).*

Proof. By the assumptions,

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} |X_i - X_{i-1}|}{n}$$

converges to a positive constant with probability 1. Thus, with probability 1 (in probability),

$$\lim_{n\to\infty} D_n = 1 + \lim_{n\to\infty} \frac{\ln \dfrac{\sum_{i=1}^{n} |X_i - X_{i-1}|}{X^n_{max} - X^n_{min}}}{\ln n} =$$

$$2 + \lim_{n\to\infty} \frac{\ln \frac{\sum_{i=1}^{n} |X_i - X_{i-1}|}{n}}{\ln n} - \lim_{n\to\infty} \frac{\ln(X^n_{max} - X^n_{min})}{\ln n} = 2 - l. \qquad \blacksquare$$

For processes which satisfy some kind of iterated logarithmic law we have the following result.

2.2 Collorary. *Let* $\{X_n\}$ *be a symmetric process for which the sequence* $(|X_i - X_{i-1}|)$ *satisfies the law of large numbers. If there exist functions* f_i *for* $i = 1, 2$ *such that* $\lim_{n\to\infty} \ln f_i(n)/\ln n = 0$, $\limsup_{n\to\infty} X^{(n)}_{\max}/(n^\alpha f_1(n)) = c_1$, *and* $\liminf_{n\to\infty} X^{(n)}_{\max}/(n^\alpha f_2(n)) = c_2$, *with probability 1, for some positive constants* c_1, c_2 *and* $0 < \alpha < 1$, *then* $\lim_{n\to\infty} D_n = 2 - \alpha$ *with probability* 1.

Proof. By Theorem 2.1 it is enough to prove that $\lim_{n\to\infty} \frac{\ln(X^n_{max} - X^n_{min})}{\ln n} = \alpha$ with probability 1. By the assumptions

$$\limsup_{n\to\infty} \frac{\ln(X^{(n)}_{max} - X^{(n)}_{min})}{\ln n} = \limsup_{n\to\infty} \frac{\ln \frac{2X^{(n)}_{max}}{n^\alpha f_1(n)} + \ln n^\alpha f_1(n)}{\ln n} = \alpha$$

with probability 1. Similarly $\liminf_{n\to\infty} \frac{\ln(X^{(n)}_{max} - X^{(n)}_{min})}{\ln n} = \alpha$ with probability 1. ∎

Since fractional Brownian motions satisfy the assumptions of Collorary 2.2 we have the following result.

2.3 Collorary. *The n-th interpolation dimension* D_n *of an* $\alpha - fBm$ *process tends with probability 1 to* $2 - \alpha$ *(i.e. to the box dimension of a typical trajectory).*

Proof. Let $\{X_t\}_{t\in[0,\infty)}$ be a fractional Brownian motion with index α. The sequence $|X_i - X_{i-1}|$ of increments is stationary and ergodic thus satisfies the law of large numbers . Moreover (X_n) is symmetric and satisfies the iterated logarithmic law so also $(X^{(n)}_{max})$ does and we take $f_1(n) = \sqrt{\ln\ln n}$ (compare [12],[14]) . Finally, by [26], there exists a positive constant c such that

$$\liminf_{n\to\infty} \frac{X^{(n)}_{\max}}{n^\alpha(\ln\ln(n))^{-\alpha}} = c$$

with probability 1. Thus we can define $f_2(n) = (\ln\ln(n))^{-\alpha}$. ∎

All the above results were obtained for stochastic time series. As could be expected, for self-similar processes the convergence of $D_n(X_i)$ is equivalent to the convergence of $D_n(X_{\frac{i}{n}})$. The next theorem clarifies this dependence.

2.4 Theorem. *Let a process* $\{X_t\}_{t\in[0,\infty)}$ *be self-similar and let* $\alpha \in R$. *Then*

$$D_n(X_i) \longrightarrow \alpha \text{ in probability 1} \iff D_n(X_{i/n}) \longrightarrow \alpha \text{ in probability 1.}$$

Proof. Let α be the index of self-similarity. Since for all $a > 0$, the finite dimensional distributions of $\{X(at)\}$ are identical to the finite dimensional distributions of $\{a^\alpha X(t)\}$ we have

$$(X_1/n^\alpha, X_2/n^\alpha, \ldots, X_n/n^\alpha) \overset{d}{=} (X_{1/n}, X_{2/n}, \ldots, X_{n/n}).$$

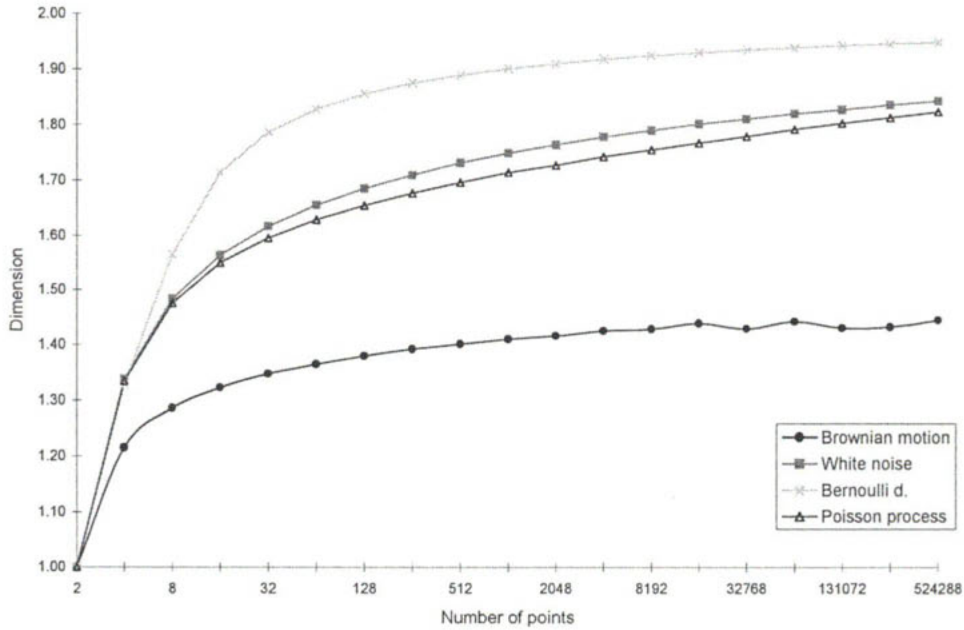

Fig. 1. Mean interpolation dimensions

This implies

$$D_n(X_i) \stackrel{d}{=} D_n(X_{i/n}).$$

The convergence in distribution to a contant α implies the convergence in probability. ■

We are mainly interested in interpolating self-similar processes with continuous trajectories. Nevertheless, we can apply the notion of interpolation dimension to other processes and even to sequences of independent random variables. As an example we calculate interpolation dimension for Gaussian white noise and for the Poisson process.

2.5 Collorary. *Let $X_1, X_2, \dots$ be a sequence of independent Gaussian random variables with expectation 0 and variance 1. Then $\lim_{n\to\infty} D_n(X_i) = 2$ in probability.*

Proof. Since $X_1, X_2, \dots$ are symmetric, by Theorem 2.1, it is enough to prove that

$$\lim_{n\to\infty} \frac{\ln(X_{max}^{(n)})}{\ln n} = 0 \text{ in probability.}$$

Take an $\varepsilon > 0$.

$$P\left(\frac{\ln(X_{max}^{(n)})}{\ln n} > \varepsilon\right) \le P(X_1 > n^\varepsilon) + P(X_2 > n^\varepsilon) + \cdots + P(X_n > n^\varepsilon) =$$

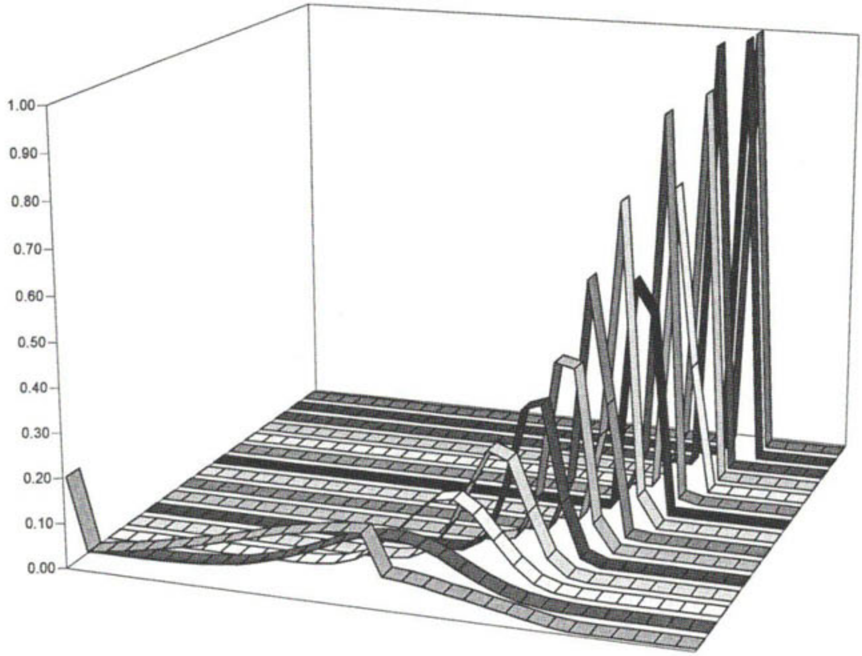

Fig. 2. Density of interpolation dimension of the Brownian motion

$$nP(X_1 > n^{\varepsilon}) = n \int_{n^{\varepsilon}}^{\infty} e^{\frac{-x^2}{2}}\, dx = n \int_{n^{\varepsilon}}^{\infty} e^{\frac{-x^2}{4}} \cdot e^{\frac{-x^2}{4}}\, dx \le ne^{-\frac{n^{2\varepsilon}}{4}} c$$

for some constant c. Thus

$$\lim_{n\to\infty} P\left(\frac{\ln(X_{max}^{(n)})}{\ln n} > \varepsilon\right) = 0. \qquad \blacksquare$$

2.6 Collorary. *If* $\{X_t\}$ *is the Poisson process then* $\lim_{n\to\infty} D_n(X_i) = 1$ *in probability.*

Proof. Since (X_n) is an increasing sequence and $X_0 = 0$ with probability 1, by Theorem 2.1 and Chebyshev's inequality, we get

$$\lim_{n\to\infty} P\left(\left|\frac{\ln(X_{max}^{(n)} - X_{min}^{(n)})}{\ln n} - 1\right| > \varepsilon\right) =$$

$$\lim_{n\to\infty} P(X_n > n^{\varepsilon+1}) \le \lim_{n\to\infty} \frac{E(X_n)}{n^{\varepsilon+1}} = 0.$$

We shall compare the above theoretical results with the results of the experiment consisting in measuring mean interpolation dimensions:

We generate sample points $X_0, X_1, \ldots, X_n$ of a process, for every fixed n being a power of 2 between 2^2 and 2^{15}. We repeat the experiment $(2^{20}/n)$

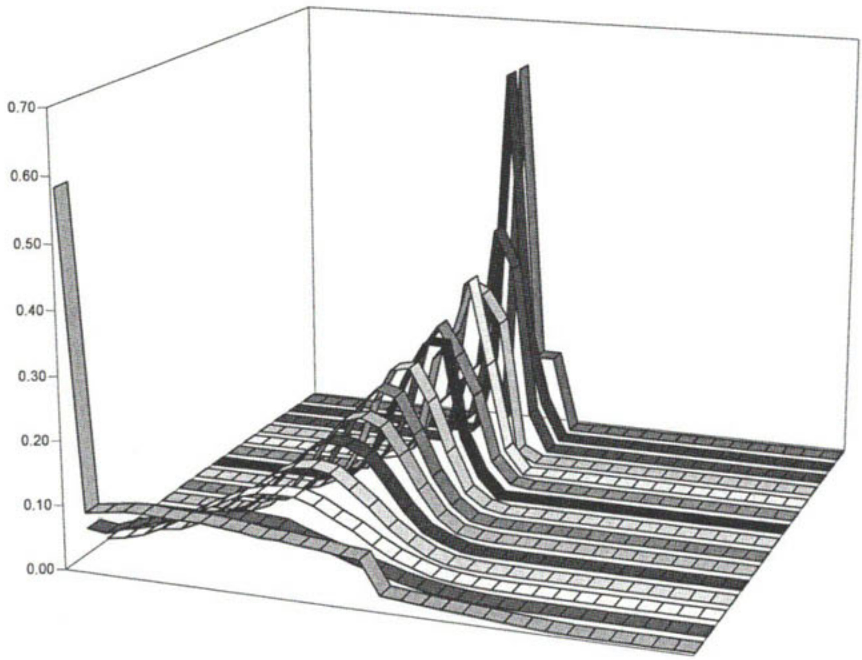

Fig. 3. Density of interpolation dimension of the white noise

times and calculate the arithmetic mean of interpolation dimensions obtained in the series.

The results for Brownian motion and the Gauss distribution (white noise), are shown in Figure 1.

To illustrate better the convergence of interpolation dimensions, we have also found experimentally the density functions of interpolation dimensions: (Figures 2–3).

We divide the interval $I = [1, 2]$ into 32 subintervals $I_1, \dots, I_{32}$ of equal length. For every fixed number n being a power of 2 between 2^2 and 2^{15}, we generate points $X_0, X_1, \dots, X_n$ of a process and calculate the interpolation dimension D_n. We repeat the experiment $(2^{20}/n)$ times for every n and measure the number k_i of occurrences of D_n in the interval I_i for $1 \leq i \leq 32$. In three dimensional coordinate system we mark on the x-axis numbers s between 2 and 15, where $n = 2^s$ is the number of generated sample points. On the y-axis there are midpoints of intervals I_i, and on the z-axis we mark numbers k_i/n for $i = 1, 2 \dots, 32$ and $n = 4, 8, 16, \dots, 2^{15}$. The density lines are linear interpolations of all points graphed in each of the the s-series, $2 \leq s \leq 15$.

3. Convergence of trajectories

Fractal interpolations of processes with continuous trajectories have additional great advantage. Under some, not very restrictive assumptions, their trajectories converge uniformly to trajectories of the initial process. Thus fractal interpolations can be used to approximate processes.

In order to prove the convergence of trajectories of fractal interpolation processes we first show a technical theorem about fractal interpolations.

3.1 Theorem. *Let* $E_0 = \{(i/n, X_{i/n}) : 0 \leq i \leq n\}$ *be a set of data. Let* $\max_i\{X_{i/n}\} - \min_i\{X_{i/n}\} \neq 0$. *If* F *is a fractal interpolation of* E_0 *with scaling factors* $d_i = \frac{|X_{i/n} - X_{(i-1)/n}|}{\max_i\{X_{i/n}\} - \min_i\{X_{i/n}\}}$ *and* $\max_i\{d_i\} < 1/2$, *then for every* $i \in \{1, \dots, n\}$

$$\sup_{(i-1)/n \leq t \leq i/n} F(t) - \inf_{(i-1)/n \leq t \leq i/n} F(t) \leq \frac{3 \max_i\{|X_{i/n} - X_{(i-1)/n}|\}}{1 - 2\max_i\{d_i\}}.$$

Proof. Let $A := \max_i\{|X_{i/n} - X_{(i-1)/n}|\}$, $M_0 := \max_i\{X_{i/n}\}$ and $m_0 = \min_i\{X_{i/n}\}$. The interpolation F is determined by a family of affine transformations, say $w_i : R^2 \longrightarrow R^2$, $1 \leq i \leq n$. For $(t, x) \in R^2$,

$$w_i \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 1/n & 0 \\ c_i & d_i \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} + \begin{pmatrix} (i-1)/n \\ f_i \end{pmatrix},$$

where

$$c_i = X_{i/n} - X_{(i-1)/n} - d_i(X_1 - X_0),$$
$$f_i = X_{(i-1)/n} - d_i X_0$$

and

$$d_i = \frac{|X_{i/n} - X_{(i-1)/n}|}{M_0 - m_0}.$$

Let W be the Hutchinson operator defined on the set of subsets of R^2 by

$$W(E) = \bigcup_{i=1}^{n} w_i(E) \text{ for an arbitrary } E \subset R^2.$$

Denote the composition $\underbrace{W \circ W \cdots \circ W}_{k-times}$ by W^k. Since the graph of F is the attractor for $\{w_i\}$, it follows that $\mathrm{graph}F = \lim_H W^k(E_0)$, where $\lim_H$ is the Hausdorff limit. Let $E_k = W^k(E_0)$. Write

$M_k = \max\{x : (t, x) \in E_k\}$, $m_k = \min\{x : (t, x) \in E_k\}$,

$M_k^{(i)} = \max\{x : (t, x) \in E_k, (i-1)/n \leq t \leq i/n\}$

and
$m_k^{(i)} = \min\{x : (t,x) \in E_k, (i-1)/n \le t \le i/n\}$.

Notice at first that for every $i \in \{0, \dots, n\}$ and $k \in N$

$$M_k^{(i)} - m_k^{(i)} \le 2|X_{i/n} - X_{(i-1)/n}| + d_i(M_{k-1} - m_{k-1}). \tag{1}$$

To show (1) we shall consider four cases.

CASE 1. $X_{i/n} - X_{(i-1)/n} \ge 0$ and $X_1 - X_0 \ge 0$.

Then $c_i \ge 0$ and, since

$$M_k^{(i)} \le c_i \cdot 1 + d_i \cdot M_{k-1} + f_i$$

and

$$m_k^{(i)} \ge c_i \cdot 0 + d_i \cdot m_{k-1} + f_i$$

we obtain

$$\begin{aligned} &M_k^{(i)} - m_k^{(i)} \le c_i + d_i(M_{k-1} - m_{k-1}) = \\ &= X_{i/n} - X_{(i-1)/n} - d_i(X_1 - X_0) + d_i(M_{k-1} - m_{k-1}) \\ &= X_{i/n} - X_{(i-1)/n} - \frac{X_{i/n} - X_{(i-1)/n}}{M_0 - m_0}(X_1 - X_0) + d_i(M_{k-1} - m_{k-1}) \\ &\le |X_{i/n} - X_{(i-1)/n}| + d_i(M_{k-1} - m_{k-1}). \end{aligned}$$

CASE 2. $X_{i/n} - X_{(i-1)/n} \ge 0$ and $X_1 - X_0 \le 0$.

Then $c_i \ge 0$, as above, and

$$\begin{aligned} &M_k^{(i)} - m_k^{(i)} \le c_i + d_i \cdot (M_{k-1} - m_{k-1}) = \\ &X_{i/n} - X_{(i-1)/n} - d_i(X_1 - X_0) + d_i \cdot (M_{k-1} - m_{k-1}) \le \\ &2|X_{i/n} - X_{(i-1)/n}| + d_i(M_{k-1} - m_{k-1}). \end{aligned}$$

CASE 3. $X_{i/n} - X_{(i-1)/n} < 0$ and $X_1 - X_0 < 0$. It is symmetric to Case 1.

CASE 4. $X_{i/n} - X_{(i-1)/n} < 0$ and $X_1 - X_0 > 0$. It is symmetric to case 2.

Next, we shall prove by induction that for all $k \ge 1$ and $1 \le i \le n$

$$M_k^{(i)} - m_k^{(i)} \le$$

$$3A + \frac{6A^2}{M_0 - m_0} + \cdots + \frac{3 \cdot 2^{k-1} A^k}{(M_0 - m_0)^{k-1}} = \frac{3A\left(1 - (2A/(M_0 - m_0))^k\right)}{1 - 2A/(M_0 - m_0)}$$

Let $k = 1$. By (1)

$$M_1^{(i)} - m_1^{(i)} \le 2|X_{i/n} - X_{(i-1)/n}| + d_i(M_0 - m_0) = 3|X_{i/n} - X_{(i-1)/n}| \le 3A,$$

as required.

Assume that for $k-1$ the inequality is satisfied. By the inductive assumption,

$$M_{k-1} \le M_0 + \frac{3A\left(1 - (2A/(M_0 - m_0)^{k-1}\right)}{1 - 2A/(M_0 - m_0)}$$

and

$$m_{k-1} \geq m_0 - \frac{3A\left(1-(2A/(M_0-m_0))^{k-1}\right)}{1-2A/(M_0-m_0)}.$$

Substracting the above inequalities and using once more (1) we obtain

$$M_k^{(i)} - m_k^{(i)} \leq$$

$$\leq 2|X_{i/n} - X_{(i-1)/n}| + d_i \cdot \left(M_0 - m_0 + \frac{6A\left(1-(2A/(M_0-m_0))^{k-1}\right)}{1-2A/(M_0-m_0)} \right)$$

$$= 3|X_{i/n} - X_{(i-1)/n}| + \frac{|X_{i/n} - X_{(i-1)/n}|}{M_0 - m_0} \frac{6A\left(1-(2A/(M_0-m_0))^{k-1}\right)}{1-2A/(M_0-m_0)}$$

$$\leq 3A + 2A/(M_0-m_0)\frac{3A\left(1-(2A/(M_0-m_0))^{k-1}\right)}{1-2A/(M_0-m_0)}$$

$$= \frac{3A\left(1-(2A/(M_0-m_0)^{k}\right)}{1-2A/(M_0-m_0)}.$$

Therefore, for every $i \in \{1, \ldots, n\}$

$$\sup_{(i-1)/n \leq t \leq i/n} F(t) - \inf_{(i-1)/n \leq t \leq i/n} F(t) = \lim_{k\to\infty} \frac{3A\left(1-(2A/(M_0-m_0))^{k}\right)}{1-2A/(M_0-m_0)} \leq$$

$$\leq \frac{3A}{1-2A/(M_0-m_0)}. \quad \blacksquare$$

Now we shall consider a stochastic process $\{X_t\}_{t\in[0,1]}$. We assume that $\{X_t\}_{t\in[0,1]}$ is nontrivial in the sense that

$$P\{\omega : X_0(\omega) = X_{1/n}(\omega) = \cdots X_{i/n}(\omega) = \cdots X_1(\omega)\} \overset{n\to\infty}{\longrightarrow} 0.$$

As an immediate consequence of Theorem 3.1 we obtain

3.2 Corollary. *Let F be a fractal interpolation of $(X_{i/n})_{0\leq i\leq n}$ with scaling factors $d_i = \frac{|X_{i/n}-X_{(i-1)/n}|}{\max_i\{X_{i/n}\}-\min_i\{X_{i/n}\}}$. If there exist constants H and c such that $\max_i\{|X_{i/n}|\} < H$ and $\max_i\{d_i\} < c < 1/2$ with probability 1, then F is a bounded process, i.e.*

$$\exists C\, P(\{\omega : \sup_{0\leq t\leq 1} |F(t,\omega)| \leq C) = 1.$$

Proof. Let $A = \max_i\{|X_{i/n} - X_{(i-1)/n}|\}$, $M_0 = \max_i\{X_{i/n}\}$, and $m_0 = \min_i\{X_{i/n}\}$. By Theorem 3.1 and the assumptions,

$$\sup_{(i-1)/n\le t\le i/n} F_t \le \max_i(X_{i/n}) + \frac{3A}{1-2A/(M_0-m_0)} \le H + \frac{3H}{1-c}$$

with probability 1. In the same way we prove that F is bounded from below.

■

3.3 Theorem. *Let $\{X_t\}_{t\in[0,1]}$ be a stochastic process that has continuous trajectories. Let F_n be the fractal interpolation of $(X_{i/n} : i = 0,1,\dots,n)$ with scaling factors $d_i^{(n)} = \frac{|X_{i/n}-X_{(i-1)/n}|}{\max_i\{X_{i/n}\}-\min_i\{X_{i/n}\}}$. If there is a constant c such that*

$$P\{\omega : \max_i d_i^{(n)}(\omega) \le c < 1/2\} \overset{n\to\infty}{\longrightarrow} 1$$

then

$$\sup_{t\in[0,1]} (|F_n(t) - X(t)|) \longrightarrow 0 \text{ in probability.}$$

Proof. Let $I = [0,1]$ and $I_i = [(i-1)/n, i/n]$, for $i = 1,\dots,n$. Let $\Omega_n = \{\omega \in \Omega : \max_i d_i^{(n)}(\omega) \le c\}$. By 1.1, $F_n(\omega)$ exists for $\omega \in \Omega_n$ and, by the assumption, $\lim_{n\to\infty} P(\Omega_n) = 1$. Clearly, for every $\omega \in \Omega_n$

$$\sup_{t\in I}(|F_n(t,\omega) - X(t,\omega)|) = \sup_{i\in\{1,2\dots,n\}} \sup_{t\in I_i}(|F_n(t,\omega) - X(t,\omega)|) \le$$
$$\sup_{i\in\{1,2\dots,n\}} \sup_{t\in I_i}\left(\left|F_n(t,\omega) - X\left(\frac{i-1}{n},\omega\right)\right|\right) +$$
$$\sup_{i\in\{1,2\dots,n\}} \sup_{t\in I_i}\left(\left|X\left(\frac{i-1}{n},\omega\right) - X(t,\omega)\right|\right).$$

We shall prove that both terms on the right side of the above inequality tend to 0 in probability i.e.,

(∗) $\forall\varepsilon > 0\ \ P(\omega \in \Omega : \sup_{i\in\{1,2\dots,n\}} \sup_{t\in I_i}(|X(\frac{i-1}{n},\omega) - X(t,\omega)| > \varepsilon) \overset{n\to\infty}{\longrightarrow} 0$

and

(∗∗) $\forall\varepsilon > 0\ \ P(\omega \in \Omega : \sup_{i\in\{1,2\dots,n\}} \sup_{t\in I_i}(|F_n(t,\omega) - X(\frac{i-1}{n},\omega)|) > \varepsilon) \overset{n\to\infty}{\longrightarrow} 0$.

Proof of (∗): With probability 1, the trajectories $X(\cdot,\omega)$ are uniformly continuous on $[0,1]$, hence we obtain easily that

$$P(\omega : \sup_{i\in\{1,2\dots,n\}} \sup_{t\in I_i}\left(\left|X\left(\frac{i-1}{n},\omega\right) - X(t,\omega)\right|\right) \overset{n\to\infty}{\longrightarrow} 0) = 1,$$

which implies convergence in probability.
Proof of (∗∗): Let

$A_n(\omega) = \max_i\{|X_{i/n}(\omega) - X_{(i-1)/n}(\omega)|\}$,

$M_0^{(n)}(\omega) = \max_i\{X_{i/n}(\omega)\}$,

and

$m_0^{(n)}(\omega) = \min_i\{X_{i/n}(\omega)\}$.

By Theorem 3.1 applied to F_n and the fact that $\lim_{n\to\infty} P(\Omega_n) = 1$ we infer that for all $\delta > 0$ and n large enough, the set of ω satisfying

$$\sup_{i\in\{1,2\ldots,n\}} \sup_{t\in I_i}(|F_n(t,\omega) - X((i-1)/n,\omega)|) \le$$

$$\sup_{i\in\{1,2\ldots,n\}} \sup_{(i-1)/n\le t\le i/n} F_n(t,\omega) - \inf_{(i-1)/n\le t\le i/n} F_n(t,\omega) \le$$

$$\frac{3A_n(\omega)}{1-2A_n(\omega)/(M_0^{(n)}(\omega) - m_0^{(n)}(\omega))} \le \frac{3A_n(\omega)}{1-c},$$

is of probability greater than $1-\delta$. Therefore, for every $\varepsilon > 0$ and $\delta > 0$

$$P\{\omega\colon \sup_{i\in\{1,2\ldots,n\}} \sup_{t\in I_i}(|F_n(t,\omega) - X((i-1)/n,\omega)| > \varepsilon\} \le P\Big\{\omega : \frac{3A_n(\omega)}{1-c} > \varepsilon\Big\} + \delta$$

for n large enough. Thus, by (*)

$$P\left\{\omega : \sup_{i\in\{1,2\ldots,n\}} \sup_{t\in I_i}(|F_n(t,\omega) - X\left(\frac{i-1}{n},\omega\right)| > \varepsilon\right\} \stackrel{n\to\infty}{\longrightarrow} 0. \qquad \blacksquare$$

3.4 Corrollary. *Let $\{X_t\}_{t\in[0,1]}$ be an α-fractional Brownian motion and let F_n be the fractal interpolation of $(X_{i/n} : i = 0,1,\ldots,n)$, for every $n \in N$. Then*

$$\sup_{t\in[0,1]} (|F_n(t) - X(t)|) \stackrel{n\to\infty}{\longrightarrow} 0$$

in probability.

Proof. We have to prove that scaling factors $d_i^{(n)}$ satisfy the condition

$$P\{\omega : \max_i\{d_i^{(n)}(\omega)\} \le c < 1/2\} \stackrel{n\to\infty}{\longrightarrow} 1$$

for some constant c.

By [17, Theorem 7] for α-fBm we have the following uniform Hölder condition

$$P\{\omega : \limsup_{|t-t'|=h\to 0, 0\le t,t'\le 1} \frac{|X_{t'}(\omega) - X_t(\omega)|}{h^\alpha\sqrt{\ln 1/h}} = 1\} = 1.$$

Therefore

$$P\{\omega : \exists n_0 \forall n \ge n_0 \max|X_{(i+1)/n} - X_{i/n}| < 2(1/n)^\alpha\sqrt{\ln n}\} = 1.$$

Let A_k be a subset of Ω satisfying the above condition for $n_0 = k$ and $k = 1,2\ldots$. Therefore $P(\bigcup_{k=1}^\infty A_k) = 1$ and since $A_1 \subset A_2 \subset \cdots \subset A_k \subset \cdots$

we get $\lim_{k\to\infty} P(A_k) = 1$. Let $B_k = \{\omega : \max_{0\le i\le k}\{|X_{(i+1)/k} - X_{i/k}|\} < 2(1/k)^\alpha\sqrt{\ln k}\}$. Since $A_k \subset B_k$ we obtain $\lim_{k\to\infty} P(B_k) = 1$.

Moreover, by [26], there is a positive constant c such that

$$\max_{0\le i\le k}\{X_{i/k}\}(\ln\ln k)^\alpha \longrightarrow c \text{ in probability.} \tag{2}$$

Let $C_k = \{\omega\colon c/2 \le \max_{0\le i\le k}\{X_{i/k}\}(\ln\ln k)^\alpha\}$. By (2), $\lim_{k\to\infty} P(C_k) = 1$.

If $\omega \in B_k \cap C_k$ then

$$\max_i\{d_i^{(k)}(\omega)\} \le 4(1/k)^\alpha\sqrt{\ln k}\ln\ln k)^\alpha/c.$$

Since the right side of the above inequality tends to 0 when $k \to \infty$ and $\lim_{k\to\infty} P(A_k \cap B_k) = 1$ we get the claim. ∎

4. Final remarks

As we have seen, fractal interpolation dimension can be used to identify a process in the class of fractional Brownian motions. Our results are even more general and they can be applied also to other classes of processes.

For a process for which we can experimentally verify that box and inerpolation dimensions are (almost) the same, the fractal interpolation determined by a set of data is likely to be a good approximation and can be used to extend the process beyond a given set of sample points. From computational point of view, the formula for fractal interpolation dimension is very easy to be handle with and is of linear complexity.

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