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C_0-SEMIGROUPS WITH WEAK SINGULARITY AND ITS APPLICATIONS

Abstract. In this paper we define the C_0 - semigroup with weak singularity and we give a characterization of its generator. We prove an analogue of the Hille-Yosida theorem.

1. Introduction

It is known that the theory of semigroups of linear operators in Banach space has applications in many branches of analysis. In the present paper we will restrict our attention to applications which are related to the solution of some initial value problems for partial differential equations. In the applications of the abstract theory, it is usually shown that a given differential operator A is the infinitesimal generator of a C_0-semigroup in a certain concrete Banach function space X (cf. [5]).

There are differential operators generating the semigroups which are not of class C_0. K. Taira in [7] has given an example of the boundary value problem for partial differential equation of the second order such that the associated with this problem operator A acting in the space $L_2(\Omega)$ generates a semigroup $U(z)$ on $L_2(\Omega)$ which is analytic in the sector

$$\Delta_\varepsilon = \{z : |\arg z| < \frac{\pi}{2} - \varepsilon, \quad z \neq 0\}$$

and

$$(1) \qquad \|U(z)\| \leq \frac{M_\varepsilon}{|z|^{1-\theta}}, M_\varepsilon > 0, \quad z \in \Delta_\varepsilon, 0 < \theta < 1.$$

In this example the singularity of the semigroup is connected with the boundary conditions (for details see, [6]).

On the other hand, it has been shown by Kielhöfer [4] and von Wahl [10], [11] that, in the space of Hölder continuous functions, an operator corresponding to a strongly elliptic differential operator with Dirichlet boundary conditions doesn't generate a C_0-semigroup (see also [12]).

2. C_0-semigroup with singularity

Let X be a Banach space and let $A : X \longrightarrow X$ be a closed linear operator with domain $D(A) \subset X$, where $\overline{D(A)} = X$.

Basing on the theory of analytic semigroups with weak singularity (cf. for example [9], [7]), we have the following definition.

DEFINITION 1. A family of bounded linear operators $U(t) : X \longrightarrow X$ depending on a parameter $t \in (0, +\infty)$ is said to be a semigroup with weak singularity for $t = 0$ if

$$U(t+s) = U(t)U(s) \quad \text{for } t, s \in (0, +\infty)$$

and

$$\|U(t)\| \le \frac{M}{t^{1-\theta}}, \quad M > 0,\ 0 < \theta < 1,\ t > 0. \tag{2}$$

DEFINITION 2. The semigroup $\{U(t),\ t > 0\}$ defined above is said to be the C_0-semigroup with weak singularity if the mapping $(0, +\infty) \ni t \longrightarrow U(t)x$ is continuous for $x \in X$.

DEFINITION 3. We say that an operator $A : X \longrightarrow X$ with domain $D(A)$ generates the C_0-semigroup with weak singularity if

(i) $D(A) := \{x \in X : dU(t)x/dt \text{ exists for } t > 0\}$,
(ii) $\frac{dU(t)x}{dt} = AU(t)x$, $t > 0$, $x \in D(A)$.

We shall prove the following

THEOREM 1. *A necessary and sufficient condition that a closed linear and densely defined in X operator A generates a C_0-semigroup with weak singularity is that the resolvent set $\rho(A) \supset (\omega, \infty)$, $\omega \ge 0$ and there exists a constant $M > 0$ such that*

$$\|R(\lambda, A)^n\| = \|(\lambda - A)^{-n}\| \le \frac{M}{(\lambda - \omega)^{n-1+\theta}} \frac{\Gamma(n-1+\theta)}{(n-1)!} \tag{3}$$

for $\lambda > \omega$, and all positive integers n.

Proof. (Sufficiency). First we assume that $\omega = 0$. We denote

$$V_n(t) := \left(1 - \frac{t}{n}A\right)^{-n}, \quad t > 0, \quad n = 1, 2, \ldots \tag{4}$$

(cf. [1,Ch. IX]).

Since $V_n(t) = (\frac{n}{t})^n R(\frac{n}{t}, A)^n$ and (3), we have

$$\|V_n(t)\| \le \left(\frac{n}{t}\right)^n \frac{M}{(\frac{n}{t})^{n-1+\theta}} \cdot \frac{\Gamma(n-1+\theta)}{(n-1)!}$$
$$= M\left(\frac{n}{t}\right)^{1-\theta} \frac{\Gamma(n-1+\theta)}{(n-1)!}$$
$$= \frac{M}{t^{1-\theta}} \cdot \frac{n^{1-\theta}}{(n-1)!}\Gamma(n-1+\theta) \le \frac{M}{t^{1-\theta}},$$

because $\frac{n^{1-\theta}}{(n-1)!}\Gamma(n-1+\theta) \le 1$. Hence

$$\|V_n(t)\| \le \frac{M}{t^{1-\theta}}, \quad t > 0. \tag{5}$$

From this follows that the sequence $\{V_n(t)\}$, defined by (4), is uniformly bounded in each interval $[a, b] \subset (0, \infty)$.

Analogously we get

$$\|(1 - \frac{t}{n}A)^{-n-1}\| = \left(\frac{n}{t}\right)^{n+1} \left\|R\left(\frac{n}{t}, A\right)^{n+1}\right\|$$
$$\le M\left(\frac{n}{t}\right)^{1-\theta} \frac{\Gamma(n+\theta)}{n!} = \frac{M}{t^{1-\theta}} \frac{\Gamma(n+\theta)}{n! n^{\theta-1}} \le \frac{M}{t^{1-\theta}}.$$

Therefore

$$\left\|\left(1 - \frac{t}{n}A\right)^{-n-1}\right\| \le \frac{M}{t^{1-\theta}}, \quad M > 0,\ t > 0,\ n \in N. \tag{6}$$

Furthermore, the mapping $(0, \infty) \ni t \to V_n(t)x$ is holomorphic, for $x \in X$, since the mapping $(0, \infty) \ni \lambda \to (\lambda - A)^{-1}x$ is holomorphic for $x \in X$.

In particular we have

$$\frac{dV_n(t)x}{dt} = A\left(1 - \frac{t}{n}A\right)^{-n-1} x, \quad t > 0\ x \in X. \tag{7}$$

From (6) and (7) we obtain

$$\left\|\frac{dV_n(t)x}{dt}\right\| = \left\|\left(1 - \frac{t}{n}A\right)^{-n-1} Ax\right\| \le \frac{M}{t^{1-\theta}}\|Ax\| \quad \text{for } x \in D(A). \tag{8}$$

By integration of the equality (7) and using inequality (8) we have

$$V_n(t)x - \lim_{s \to 0^+} V_n(s)x = \int_0^t \left(1 - \frac{s}{n}A\right)^{-n-1} Ax\, ds \quad \text{for } x \in D(A). \tag{9}$$

Because the improper Riemann integral on the right hand of (9) exists, then from (9) it follows that for $x \in D(A)$ $\lim_{t\to 0^+} V_n(t)x$ is an element of X.

We prove that

$$\lim_{t\to 0^+} V_n(t)x = x \quad \text{for } x \in D(A). \tag{10}$$

Indeed, we have

$$\begin{aligned}\left\|\left(1-\frac{t}{n}A\right)^{-1}x - x\right\| &= \left\|\left(1-\frac{t}{n}A\right)^{-1}\frac{t}{n}Ax\right\| \\ &\le \frac{M}{(\frac{n}{t})^{1-\theta}}\cdot\frac{t}{n}\cdot\|Ax\| \quad \text{for } x \in D(A).\end{aligned}$$

Hence, we get

$$\lim_{t\to 0^+}\left(1-\frac{t}{n}A\right)^{-1}x = x \quad \text{for } x \in D(A). \tag{11}$$

Similary, for $x \in D(A)$, and $k \in N$ we have

$$\begin{aligned}&\left\|\left(1-\frac{t}{n}A\right)^{-k}x - \left(1-\frac{t}{n}A\right)^{1-k}x\right\| \\ &= \left\|\left(1-\frac{t}{n}A\right)^{-k}\frac{t}{n}Ax\right\| \\ &\le \frac{M}{(\frac{t}{n})^{1-\theta}}\cdot\frac{t}{n}\|Ax\|\frac{\Gamma(k-1+\theta)}{(k-1)!} \\ &\le M\left(\frac{t}{n}\right)^{\theta}\|Ax\| \to 0 \quad \text{when } t\to 0^+.\end{aligned}$$

From this we obtain

$$\begin{aligned}\|V_n(t)x - x\| &= \left\|\left(1-\frac{t}{n}A\right)^{-n}x - x\right\| \\ &\le \left\|\left(1-\frac{t}{n}A\right)^{-n}x - \left(1-\frac{t}{n}A\right)^{1-n}x\right\| \\ &\quad + \left\|\left(1-\frac{t}{n}A\right)^{1-n}x - \left(1-\frac{t}{n}A\right)^{2-n}x\right\| + \dots \\ &\quad + \left\|\left(1-\frac{t}{n}A\right)^{-1}x - x\right\| \\ &\le nM\left(\frac{t}{n}\right)^{\theta}\|Ax\| = n^{1-\theta}M\|Ax\|t^{\theta} \to 0 \quad \text{when } t\to 0^+.\end{aligned}$$

Therefore by (9) and (10) we obtain

$$V_n(t)x - x = \int_0^t A\left(1 - \frac{s}{n}A\right)^{-n-1} x\,ds \quad \text{for } x \in D(A). \tag{12}$$

Now we shall prove that, for $x \in X$, the sequence $\{V_n(t)x\}$ converges uniformly with respect to t in any interval $[a,b] \subset (0,\infty)$ as $n \to \infty$. For this purpose let $x \in D(A)$ and let $t \in [a,b]$. We estime $V_n(t)x - V_m(t)x$.

For this we note, in virtue of (10), that

$$V_n(t)x - V_m(t)x = \int_0^t \frac{d}{ds}[V_m(t-s)V_n(s)x]ds. \tag{13}$$

To estime (13) we assume that $x \in D(A^2)$. Since the resolvent of A commutes with A, then (13) and (5) give

$$\begin{aligned}\|V_n(t)x - V_m(t)x\| &\le M^2\|A^2x\| \int_0^t \left(\frac{s}{n} + \frac{t-s}{m}\right) s^{\theta-1}(t-s)^{\theta-1}\,ds \\ &= M^2\|A^2x\| \left(\frac{1}{n} + \frac{1}{m}\right) \int_0^t (t-s)^{\theta-1}s^{\theta}\,ds.\end{aligned}$$

From this it follows that

$$\begin{aligned}&\|V_n(t)x - V_m(t)x\| \\ &\qquad \le M^2\|A^2x\|b^{2\theta}\frac{\Gamma(\theta+1)\Gamma(\theta)}{\Gamma(2\theta+1)}\left(\frac{1}{n} + \frac{1}{m}\right), \qquad t \in [a,b],\end{aligned} \tag{14}$$

which implies the Cauchy condition for uniform convergence in $[a,b]$ of the sequencce $V_n(t)x$, provided that $x \in D(A^2)$. Since $D(A^2)$ is dense in X and the sequence $\{V_n(t)x\}$ is norm bounded in $[a,b]$, then in view of the Banach-Steinhaus theorem, it follows that $\lim_{n\to\infty} V_n(t)x$ exists for $x \in X$.

We define

$$U(t)x := \lim_{n\to\infty} V_n(t)x \quad \text{for } x \in X \text{ and } t > 0. \tag{15}$$

Since $V_n(t)x \to U(t)x$ as $n \to \infty$ uniformly in t in any finite interval $[a,b] \subset (0,\infty)$ and the mapping $(0,\infty) \ni t \to V_n(t)x$ is continuous, then $U(t)x$ is continuous in $t \in (0,\infty)$. In other words, $U(t)$ is strongly continuous for $t > 0$. Furthermore

$$\|U(t)\| \le \frac{M}{t^{1-\theta}}, \qquad t > 0 \tag{16}$$

and

$$\lim_{t\to 0^+} U(t)x = x \quad \text{for } x \in D(A), \tag{17}$$

by (5) and (10).

Let us remark that

$$\int_0^t \left(1 - \frac{s}{n}A\right)^{-n-1} x\,ds = \int_0^t V_n(s)\left(1 - \frac{s}{n}A\right)^{-1} x\,ds \quad \text{for } x \in D(A). \tag{18}$$

It follows from (11) that $\lim_{n\to\infty}(1 - \frac{s}{n}A)^{-1}x = x$ for $x \in D(A)$ uniformly in s in $[a, b]$.

Thus by (18) we have

$$\int_0^t \left(1 - \frac{s}{n}A\right)^{-n-1} x\,ds \to \int_0^t U(s)x\,ds \quad \text{as } n \to \infty. \tag{19}$$

On the other hand from (12) for $x \in D(A)$ we get

$$A\int_0^t \left(1 - \frac{s}{n}A\right)^{-n-1} x\,ds = V_n(t)x - x \to U(t)x - x. \tag{20}$$

Because A is closed and because of (19) and (20) $\int_0^t U(s)xds \in D(A)$ for $x \in D(A)$. Moreover

$$U(t)x - x = A\int_0^t U(s)x\,ds. \tag{21}$$

Using again the closedness of A, from (21) we obtain

$$U(t)x - x = \int_0^t AU(s)x\,ds. \tag{22}$$

From (20) we have

$$\begin{aligned} V_n(t)x - x &= A\int_0^t \left(1 - \frac{s}{n}A\right)^{-n-1} x\,ds \\ &= \int_0^t \left(1 - \frac{s}{n}A\right)^{-n-1} Axds \quad \text{for } x \in D(A). \end{aligned} \tag{23}$$

It follows from (23), as $n \to \infty$, that

$$U(t)x - x = A\int_0^t U(s)x\,ds = \int_0^t AU(s)x\,ds = \int_0^t U(s)Ax\,ds \tag{24}$$

for $x \in D(A)$.

Since $U(s)Ax$ is continuous in s, then (24) shows that $U(t)x$ is differentiable in t for $x \in D(A)$ and

$$\frac{dU(t)x}{dt} = U(t)Ax = AU(t)x, \qquad t > 0, \qquad x \in D(A). \tag{25}$$

In order to prove that $\{U(t), t > 0\}$ is a semigroup we observe that, by (25), we have that

$$u(t) = U(t)u_0 \qquad \text{for } t > 0 \tag{26}$$

is a solution of the differential equation

$$\frac{du}{dt} = Au(t), \qquad t > 0 \tag{27}$$

with the initial condition

$$\lim_{t\to 0^+} u(t) = u_0, \tag{28}$$

provided the initial value $u_0 \in D(A)$. We prove that the problem (27), (28) has the unique solution. In fact, let u be a solution of (27), (28), where by solution of (27), (28) we mean a function $u \in C([0,+\infty)) \cap C^1(0,+\infty)$ such that $u(t) \in D(A)$ for $t > 0$ and (27), (28) hold true. Then

$$\begin{aligned} \frac{d}{ds}U(t-s)u(s) &= -U'(t-s)u(s) + U(t-s)u'(s) \\ &= -U(t-s)Au(s) + U(t-s)Au(s) = 0, \ 0 < s < t, \end{aligned} \tag{29}$$

because of (27). Thus the mapping $(0,t) \ni s \to U(t-s)u(s)$ is constant, for each $t > 0$. Since

$$\lim_{s\to t^-} U(t-s)u(s) = u(t) \quad \text{and} \quad \lim_{s\to 0^+} U(t-s)u(s) = U(t)x_0, \tag{30}$$

then, by (17), (28) and strong continuity of $U(t)$, for $t > 0$, we have

$$u(t) = U(t-s)u(s) = U(t)x_0 \qquad \text{for } 0 < s < t. \tag{31}$$

The equality (31) implies

$$U(t)x_0 = U(t-s)u(s) = U(t-s)U(s)x_0,$$

which is true for all $x_0 \in D(A)$. From this, by density $D(A)$ in X, we have $U(t) = U(t-s)U(s)$. This may be written as

$$U(t+s) = U(t)U(s), \qquad s,t > 0. \tag{32}$$

The latter means that $\{U(t),\ t > 0\}$ is a semigroup. The sufficiency is proved.

Finally we shall show the necessity. To do it we will use the following formula expressing the resolvent $R(\lambda, A)$ of the operator A in terms of $U(t)$. We have

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} U(t)x\,dt, \qquad \lambda > 0,\ x \in X, \tag{33}$$

which shows that the resolvent $R(\lambda, A)$ is the Laplace transform of the semigroup $U(t)$ with weak singularity. Note that the integral on the right hand side of (33) is an improper Riemann integral defined as the limit

$$\lim_{\tau\to\infty} \lim_{\varepsilon\to 0^+} \int_\varepsilon^\tau e^{-\lambda t} U(t)x\,dt. \tag{34}$$

The existence of this limit follows from inequality (2) and the continuity of the integrand function of $t \in (0, +\infty)$.

To prove (33) fix $x \in D(A)$. Then

$$dU(t)x/dt = U(t)Ax$$

and

$$(d/dt)e^{-\lambda t}U(t)x = -e^{-\lambda t}U(t)(\lambda - A)x.$$

Integration of this equality gives

$$\lim_{\varepsilon\to 0^+} e^{-\lambda\varepsilon}U(\varepsilon)x - \lim_{\tau\to\infty} e^{-\lambda\tau}U(\tau)x = \int_0^\infty e^{-\lambda t}U(t)(\lambda - A)x\,dt$$

and so

$$x = \int_0^\infty e^{-\lambda t}U(t)(\lambda - A)x\,dt \quad \text{for } x \in D(A). \tag{35}$$

Putting $v := (\lambda - A)x$ in (35) we get

$$(\lambda - A)^{-1}v = R(\lambda, A)v = \int_0^\infty e^{-\lambda t}U(t)v\,dt \quad \text{for } v \in X,$$

which proves the formula (33).

Since the resolvent $R(\lambda, A)x$ is a holomorphic function for $\lambda > 0$ then we get

$$\frac{d}{d\lambda}R(\lambda, A)x = \frac{d}{d\lambda}\int_0^\infty e^{-\lambda t}U(t)x\,dt = \int_0^\infty (-t)e^{-\lambda t}U(t)x\,dt.$$

Proceeding by induction we obtain

$$\frac{d^n}{d\lambda^n}R(\lambda, A)x = \int_0^\infty (-1)^n t^n e^{-\lambda t} U(t)x\, dt.$$

From this and by formula

$$\frac{d^n}{d\lambda^n}R(\lambda, A)x = (-1)^n n! R(\lambda, A)^{n+1}x,$$

we have

$$R(\lambda, A)^n x = \frac{1}{(n-1)!}\int_0^\infty t^{n-1} e^{-\lambda t} U(t)x\, dt. \tag{36}$$

Using the inequality (2), from (36) we obtain

$$\|R(\lambda, A)^n x\| \leq \frac{M}{(n-1)!}\int_0^\infty t^{n-2+\theta} e^{-\lambda t}\|x\|dt, \qquad M > 0,\ \lambda > 0. \tag{37}$$

The inequality (37) implies (3) with $\omega = 0$.

REMARK 1. Let be $\omega > 0$ in Theorem 1, i.e. let the semi-infinite interval $(\omega, \infty) \subset \varrho(A)$. Then the operator $A_1 := A - \omega I$ satisfies the assumptions of Theorem 1 with $\omega = 0$, so A_1 generates the semigroup $U_1(t),\ t > 0$ such that

$$\|U_1(t)\| \leq \frac{M}{t^{1-\theta}}.$$

If we set $U(t) := e^{\omega t}U_1(t)$, it can easily verified that $U(t)$ has all the properties stated in Theorem 1, with the following modification of (2)

$$\|U(t)\| \leq \frac{M}{t^{1-\theta}}e^{\omega t}, \qquad M > 0,\ 0 < \theta < 1,\ t > 0. \tag{38}$$

3. The abstract Cauchy problem

Let A be a linear operator satisfying the assumptions of Theorem 1 which generates a C_0-semigroup $\{U(t), t > 0\}$ with weak singularity for $t = 0$. We consider the following Cauchy problem

$$\frac{du}{dt} = Au + f, \qquad t \in (0, T], \tag{39}$$

$$u(0) = u_0, \tag{40}$$

where $f : [0, T] \to X$ is continuously differentiable and $u_0 \in D(A)$.

DEFINITION 4. A function $u : [0, T] \to X$ is said to be a (classical) solution of problem (39), (40) if:

(i) u is continuous in $[0, T]$,

(ii) u is of class C^1 in $(0,T]$,
(iii) u satisfies equation (39) in $(0,T]$ and $u(0) = u_0$.

We shall prove the following

THEOREM 2. *If the operator A, the function f and u_0 satisfy the above assumptions, then the problem* (39), (40) *has the unique classical solution u which is given by*

$$u(t) = U(t)u_0 + \int_0^t U(t-s)f(s)\,ds, \qquad t \in [0,T], \tag{41}$$

where

$$u_0 = u(0) := \lim_{t\to 0^+} u(t).$$

Proof. We know that the first term $U(t)x_0$ on the right hand side of (41) satisfies the homogeneous differential equation (39). i.e. $f = 0$, and the initial condition (40) (cf. (26)–(28)). Therefore it suffices to show that the second term of the right hand side of (41) satisfies (39) and has initial value zero. Denoting this term by $v(t)$, we have

$$v(t) = \int_0^t U(t-s)f(s)\,ds. \tag{42}$$

Since $f \in C^1([0,T])$ we get

$$f(t) = f(0) + \int_0^t f'(\tau)d\tau \qquad \text{for } t \in [0,T]. \tag{43}$$

By (42) and (43) we have

$$\begin{aligned} v(t) &= \int_0^t U(t-s)\Big[f(0) + \int_0^s f'(\tau)\,d\tau\Big]\,ds \\ &= \int_0^t U(t-s)f(0)\,ds + \int_0^t \Big[\int_\tau^t U(t-s)f'(\tau)\,ds\Big]\,d\tau \\ &= \int_0^t U(r)f(0)\,dr + \int_0^t \Big[\int_0^{t-\tau} U(r)f'(\tau)\,dr\Big]\,d\tau. \end{aligned} \tag{44}$$

But by (21)

$$A\int_0^t U(s)x\,ds = U(t)x - x \qquad \text{for } x \in D(A),\ t > 0.$$

This can be extended to an arbitrary $x \in X$ by choosing a sequence $\{x_n\} \subset D(A)$ such that $x_n \to x$ as $n \to \infty$ and going to the limit, for

$$\int_0^t U(s)x_n\,ds \to \int_0^t U(s)x\,ds$$

and

$$U(t)x_n - x_n \to U(t)x - x.$$

From this by the closedness of A we obtain $\int_0^t U(s)x\,ds \in D(A)$ and

$$A\int_0^t U(s)x\,ds = U(t)x - x, \quad \text{for } x \in X,\ t > 0. \tag{45}$$

It follows from (44) and (45) that $v(t) \in D(A)$ and

$$\begin{aligned} Av(t) &= U(t)f(0) - f(0) + \int_0^t \Big[U(t-\tau)f'(\tau) - f'(\tau)\Big]\,d\tau \\ &= U(t)f(0) - f(t) + \int_0^t U(t-\tau)f'(\tau)\,d\tau. \end{aligned} \tag{46}$$

On the other hand we have

$$v(t) = \int_0^t U(s)f(t-s)\,ds$$

and so

$$\begin{aligned} \frac{dv}{dt}(t) &= U(t)f(0) + \int_0^t U(s)f'(t-s)\,ds \\ &= U(t)f(0) + \int_0^t U(t-\tau)f'(\tau)\,d\tau. \end{aligned} \tag{47}$$

A comparison of (46) and (47) shows that

$$\frac{dv}{dt}(t) = Av(t) + f(t),$$

as we wished to prove. Also it is easy to show that

$$\lim_{t\to 0^+} v(t) = 0.$$

The continuity of dv/dt follows from (47) by continuity of $f' : [0,T] \to X$ and this completes the proof of Theorem 2.

EXAMPLE. Let $X := C^\alpha(\overline{\Omega})$ where Ω is a bounded region in R^n which the boundary $\partial\Omega$ is locally of class C^m.

Let us consider the following mixed problem

$$\tag{48} \begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) = \mathcal{A}(x,D)u(t,x) + f(t,x), \quad x \in \Omega,\ 0 < t \le T, \\ \dfrac{\partial^\alpha u}{\partial x^\alpha}(t,x) = 0,\ |\alpha| \le m/2 - 1,\ x \in \partial\Omega,\ 0 < t \le T, \\ u(0,x) = u_0(x),\ \dfrac{\partial u}{\partial t}(0,x) = u_1(x), \quad x \in \Omega, \end{cases}$$

where m is an even number, and

$$\mathcal{A}(x,D) := \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha,$$

is strongly elliptic differential operator.

Under assumptions that the coefficients a_α are the sufficiently regular the problem (48) may be consider as the abstract linear initial hyperbolic problem in the space X of the form

$$\tag{49} \begin{cases} \dfrac{d^2 u}{dt^2} = Au + f,\ 0 < t \le T, \\ u(0) = u_0,\ u'(0) = u_1, \end{cases}$$

where

$$A : D(A) \to X$$

$$D(A) := \left\{ u \in C^{m+\alpha}(\overline{\Omega}) : \frac{\partial^\alpha u}{\partial x^\alpha}(t,x)\Big|_{\partial\Omega} = 0,\ |\alpha| \le m/2 - 1 \right\}$$

and

$$(Au)(t,x) := \mathcal{A}(x,D)u(t,x),\ x \in \Omega,\ 0 < t \le T,$$

$$f : [0,T] \to X,\ f(t)(x) := f(t,x),\ x \in \overline{\Omega},\ 0 < t \le T,$$

$$u_0 := u_0(x),\ u_1 := u_1(x),\ x \in \Omega.$$

To solve the problem (49) we can reduce it to a first order Cauchy problem, of course under some assumptions on the operator A (for details see e. g. [2], [3], [8]). The problem (49) may be equivalent of the following first order Cauchy problem

$$\tag{50} \begin{cases} \dfrac{dV}{dt} = \mathcal{B}V + F \quad \text{in } (0,T], \\ V(0) = V_0, \end{cases}$$

where

$$\mathcal{B} := \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}, \quad V := \begin{bmatrix} u \\ v \end{bmatrix}, \quad F := \begin{bmatrix} 0 \\ f \end{bmatrix}, \quad V_0 := \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$$

$$\mathcal{B} : [D(B)] \times X \to [D(B)] \times X,$$

where $[D(B)]$ denotes the Banach space $D(B)$ with graph norm $\|x\|_B := \|x\| + \|Bx\|$, and operator $B : X \to X$ is such that $B^2 = A$; $D(\mathcal{B}) := D(A) \times D(B)$.

Using the results of the papers [4], [10], [11] we obtain

$$\|R(\lambda, A)\| \leq \frac{M}{\lambda^{1-\frac{\alpha}{m}}}, \qquad \text{for } \lambda > 0, \tag{51}$$

in norm of the space $X = C^\alpha(\overline{\Omega})$.

From (51) similary as in [8], we can deduce that

$$\|R(\lambda, \mathcal{B})\| = \|(\lambda - \mathcal{B})^{-1}\| \leq \frac{M}{|\lambda|^\theta} \tag{52}$$

which $\theta = 1 - \frac{\alpha}{m}$, and so $0 < \theta < 1$.

From the inequality (52) it follows that the mixed problem (48) can be reduced to the abstract Cauchy problem with weak singularity, i.e. to the problem (39), (40).

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